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RINGS OVER WHICH EVERY SIMPLE MODULE IS RATIONALLY COMPLETE

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1. Introduction. In 1958, G. D. Findlay and J. Lambek defined a relationship between three R-modules, $A \leqq B(C)$, to mean that $A \subseteq B$ and every R-homomorphism from A into C can be uniquely extended to an irreducible partial homomorphism from B into C. If $A \leqq B(B)$, then B is called a rational extension of A and in [**5**] it is shown that every module has a maximal rational extension in its injective hull which is unique up to isomorphism. A module is called rationally complete provided it has no proper rational extension.

In 1969, R. C. Courter [**1**] completely characterized the rings for which every module is rationally complete. Furthermore, the rings for which every right R-module is rationally complete are described in the review of [**1**] which appeared in Zbl. *182*, p. 55.

This paper is concerned with the related problem of describing the rings for which every simple right module is rationally complete. It is shown that the collection of rings which has this property contains all finite direct sums of matrix rings over duo rings. It is also shown that the ring

$$\begin{bmatrix} Q & Q \\ 0 & Q \end{bmatrix}$$

does not belong to this collection. Results are also obtained concerning rings for which every simple right module is injective.

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2. Preliminaries. Throughout this paper, R will denote an associative ring with unity 1 and each module M will be a unitary right R-module. $E(M)$ will denote the injective envelope of M.

2.1 *Definition.* If A and B are two R-modules and $C \subseteq B$, then $f \in \mathrm{Hom}_R(C, A)$ is called a partial homomorphism of B into A. The set of all $f \in \mathrm{Hom}_R(C, A)$ for any $C \subseteq B$ will be denoted by $\mathrm{Par}_R(B, A)$. If $f \in \mathrm{Par}_R(B, A)$ and f cannot be extended to another partial homomorphism of B into A whose domain properly contains the domain of f, then f will be called an irreducible partial homomorphism of B into A.

2.2 *Definition.* In [**5**] Findlay and Lambek have defined the following relationship between A, B and C: If $A \subseteq B$, then $A \leqq B(C)$ means that for

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any R-homomorphism $f \in \mathrm{Par}_R(B, C)$, $f(A) = 0$ if and only if f is a zero mapping. In general, if $A \cong D$ and $D \leqq B(C)$, we identify A with D and write $A \leqq B(C)$.

The following four propositions are taken from [**5**] and the reader is referred to that paper for their proofs.

2.3 PROPOSITION. *If $C \leqq B(A)$ and $C \subseteq D \subseteq B$ then $C \leqq D(A)$ and $D \leqq B(A)$.*

2.4 PROPOSITION. *If $C \leqq B(A)$ and ψ is any homomorphism into B, then $\psi^{-1}(C) \leqq \psi^{-1}(B)(A)$.*

2.5 *Definition.* B is said to be a rational extension of A provided that $A \leqq B(B)$. A module is called rationally complete if and only if it has no proper rational extension.

2.6 *Definition.* A submodule N of M is called large in M (written $N \subseteq' M$) and M is called an essential extension of N provided that $N \cap K \neq 0$ for every nonzero submodule K of M.

2.7 PROPOSITION.
(i) *$A \leqq B(B)$ implies $A \subseteq' B$.*
(ii) *If $A \subseteq' A^*$, then $C \leqq B(A)$ if and only if $C \leqq B(A^*)$.*

2.8 PROPOSITION. *$A \leqq B(B)$ if and only if for all b_1, $0 \neq b_2 \in B$, there exists $r \in R$ such that $b_1 r \in A$ and $b_2 r \neq 0$.*

2.9 PROPOSITION. *Let M be any module, let $E = E(M)$, let $\Lambda = \mathrm{Hom}_R(E, E)$ and let $M^\Lambda = \{\lambda \in \Lambda | \lambda(M) = 0\}$. Then $C(M) = \bigcap \{\ker \lambda | \lambda \in M^\Lambda\}$ is a maximal rational extension of M which is contained in E. If G is any other maximal rational extension of M, then the identity mapping of M into $C(M)$ can be extended to an isomorphism of G onto $C(M)$.*

Proof. See [**3**, p. 60].

2.10 *Notation.* Let A and B be two R-modules so that $A \subseteq B$. If $b \in B$,

$$(A : b)_R = \{r \in R | br \in A\} \quad \text{and} \quad (b)^R = \{r \in R | br = 0\}.$$

2.11 PROPOSITION. *Let A be any module, $E = E(A)$ and $C = C(A)$* (as in 2.9). *Then*

$$C = \{x \in E | (A : x)_R \leqq R(A)\}.$$

Proof. Let $x \in E$ be such that $(A : x)_R \leqq R(A)$ and let $\lambda \in \mathrm{Hom}_R(E, E)$ be such that $\lambda(A) = 0$. Then $(A : x)_R \subseteq (\lambda(x))^R$ so that by 2.3, $(\lambda(x))^R \leqq R(A)$ and hence $(\lambda(x))^R \leqq R(E)$ by 2.7 (ii). Define $f \in \mathrm{Hom}_R(R, E)$ by $f(r) = \lambda(x)r$ for all $r \in R$. Then $f((\lambda(x))^R) = 0$. Therefore image $f = 0$ so that $f(1) = \lambda(x)1 = 0$. Thus $\lambda(x) = 0$. Hence $x \in \ker \lambda$ for all $\lambda \in \mathrm{Hom}_R(E, E)$ such that $\lambda(A) = 0$ and thus $x \in C$.

If $x \in C$, then $A \leqq C(C)$ so that $A \leqq (A + xR)\ (A)$. Define $\psi : R \rightarrow A + xR$ by $\psi(r) = xr$ for all $r \in R$. Then $\psi^{-1}(A) = (A : x)_R$ and $\psi^{-1}(A + xR) = R$. From 2.4, it follows that $(A : x)_R \leqq R(A)$.

2.12 *Definition.* Let M be any module. Then $Z_R(M) = \{m \in M | (m)^R \subseteq' R\}$ is called the singular submodule of M.

2.13 PROPOSITION. *If $Z_R(M) = 0$, then $M \leqq N(N)$ if and only if $M \subseteq' N$.*

Proof. See [**3**, Chapter 7].

2.14 *Remark.* $M \subseteq C(M) \subseteq E(M)$ and it follows from 2.13 that if $Z_R(M) = 0$, then $C(M) = E(M)$.

3. α-Rings.

3.1 *Definition.* A module will be called a simple module provided it contains no proper submodules.

3.2 *Remark.* As an immediate consequence of the definition, if A is a simple module, then:
(i) If $0 \neq x \in E(A)$ then $A \subseteq' xR$.
(ii) $A \cong R/M$ where M is a maximal right ideal in R.

The following question is currently an unsolved problem: What are the rings for which every simple module is injective? In [**8**], Rosenberg and Zelinsky credit Kaplansky with the partial solution below:

3.3 THEOREM. *If R is a commutative ring with 1, then every simple R-module A is injective if and only if for every $x \in R$ there is a $y \in R$ so that $x^2y = x$.*

In this paper this result is extended to a larger class of rings which includes the commutative rings with 1.

Since every injective module is rationally complete, it seems natural to consider the related question: What are the rings for which every simple module is rationally complete?

3.4 *Definition.* If R is a ring such that every simple module is rationally complete, then R will be called an α-ring. The symbol $\mathscr{A}$ will be used to denote the collection of all α-rings.

3.5 *Example.* Let D be any division ring and let $R = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$ and $M = \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}$. Then for $0 \neq m \in M$, $m = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}$ for some $d \in D$, $d \neq 0$. Thus

$$mR = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D & D \\ 0 & D \end{bmatrix} = M$$

so that M is a simple module. If $0 \neq m = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix} \in M$, then $\begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & dc \\ 0 & 0 \end{bmatrix}$ and hence $(m)^R = \begin{bmatrix} D & D \\ 0 & 0 \end{bmatrix}$. Now $\begin{bmatrix} D & D \\ 0 & 0 \end{bmatrix} \cap \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} = 0$ so that (m^R) is not large in R and hence $Z_R(M) = 0$.

Let $N = \begin{bmatrix} D & D \\ 0 & 0 \end{bmatrix}$. Then if

$$x = \begin{bmatrix} d_1 & d_2 \\ 0 & 0 \end{bmatrix}, \quad x\begin{bmatrix} 0 & u \\ 0 & v \end{bmatrix} = \begin{bmatrix} 0 & d_1u + d_2v \\ 0 & 0 \end{bmatrix}$$

so if $x \neq 0$, then either d_1 or $d_2 \neq 0$ so u and v can be chosen so that $d_1u + d_2v \neq 0$ and hence $M \cap xR \neq 0$. Thus $M \subseteq' N$. Then by 2.14, since $Z_R(M) = 0$, $M \leqq N(N)$. Therefore M is a simple R-module which is not rationally complete.

3.6 *Remark*. Example 3.5 shows that there exists at least one non-trivial example of a ring which is not an α-ring. Since every division ring is a member of $\mathscr{A}$, $\mathscr{A}$ is not an empty collection.

3.7 PROPOSITION. *If $R \in \mathscr{A}$ and if f is any ring homomorphism on R, then $f(R) \in \mathscr{A}$.*

Proof. Let f be a ring homomorphism and let $T = f(R)$. If A_T is any right T module, then by defining $a \cdot r = af(r)$ for all $r \in R$, $a \in A$, A becomes a right R-module and if A_T is simple, so is A_R. Now if B is any right T-module so that $A \leqq B(B_T)$, then by 2.8, if $b_1, b_2 \in B$ and $b_2 \neq 0$, there is a $t \in T$ so that $b_1t \in A$ and $b_2t \neq 0$. But $t = f(r)$ for some $r \in R$, so we have $A \leqq B(B_R)$ and since $R \in \mathscr{A}$, this implies that $B = A$.

3.8 PROPOSITION. *Let $R = \Pi R_\lambda$ be a direct product of rings over some index set Λ. Then $R \in \mathscr{A}$ if and only if each $R_\lambda \in \mathscr{A}$.*

Proof. If $R \in \mathscr{A}$, then for each $\lambda \in \Lambda$, R_λ is the image of the canonical epimorphism $r \rightarrow r(\lambda)$ and hence by 3.7, each $R_\lambda \in \mathscr{A}$.

Conversely suppose $R_\lambda \in \mathscr{A}$ for every $\lambda \in \Lambda$ and let M be any simple R-module. Then there is one $\lambda' \in \Lambda$ such that $M = MR_{\lambda'}$ and $MR_\lambda = 0$ if $\lambda \neq \lambda'$. In fact, for $\lambda \neq \lambda'$, if $x \in E(M)$ and $x \neq 0$, then $M \subseteq xR_{\lambda'}$ and $xR_\lambda = 0$. Hence for any $x \in E(M)$ and for any $r \in R$, $xr = xr(\lambda')$. Suppose $M \leqq N(N_R)$. Then $M \subseteq' N$ by 2.7(i). Let $\psi \in \mathrm{Par}_{R_{\lambda'}}(N, N)$ such that $\psi(M) = 0$. Then if $x \in \mathrm{dom}\,\psi$, $\psi(xr) = \psi(xr(\lambda')) = \psi(x)r(\lambda') = \psi(x)r$. Hence $\psi \in \mathrm{Par}_R(N, N)$ and thus $\psi = 0$. Therefore $M \leqq N(N_{R_{\lambda'}})$. But by hypothesis $R_{\lambda'} \in \mathscr{A}$ and so M is rationally complete as an $R_{\lambda'}$-module and therefore $M = N$. It follows that M has no proper rational extension as an R-module and is therefore rationally complete. Thus $R \in \mathscr{A}$.

3.9 PROPOSITION. *Let R be any ring with 1 and let n be any positive integer. Then $R \in \mathscr{A}$ if and only if $R_n \in \mathscr{A}$.*

Proof. This is an immediate consequence of the equivalence between the category of right R-modules and the category of right R_n-modules (see [**6**]).

3.10 *Remark.* From 3.8 and 3.9, a sufficient condition that a ring belongs to $\mathscr{A}$ is that it be a direct product of matrix rings over rings which are themselves α-rings. Furthermore, it follows from 3.7 that any α-ring is a subdirect product of subdirectly irreducible α-rings.

3.11 *Definition.* A ring is called a right duo ring provided that every right ideal is a left ideal. This definition is due to Feller [**4**].

3.12 *Example.* Let $R = D_1 \oplus D_2$ where D_1 and D_2 are non-commutative division rings. Then R is not a commutative ring but R is a right duo ring.

3.13 *Remark.* It can be shown that every right duo ring is an α-ring. Furthermore, rings which have a unique (up to isomorphism) simple module can be shown to belong to $\mathscr{A}$. In this paper we consider both right duo rings and unique-simple-module rings as members of a larger class of rings possessing "property P" defined as follows:

3.14 *Definition.* A ring will be said to have property P provided that for each maximal right ideal M, any right ideal I and any mapping $\psi : R/M \rightarrow R/I$ so that $\psi(R/M) \subseteq' R/I$, there exists a right ideal N so that $I \subseteq N$ and $R/M \cong R/N$.

3.15 PROPOSITION. *Every right duo ring and every ring having (up to isomorphism) a unique simple module has property P.*

Proof. Let $\psi : R/M \rightarrow R/I$ be given such that $\psi(RM) \subseteq' R/I$. Since $1 \in R$, I is contained in some maximal right ideal N. If R has only one simple module, then $R/M \cong R/N$.

If R is right duo, then I is an ideal and hence if $i \in I$ and $\psi(1 + M) = x + I$, then $\psi(i + M) = \psi(1 + M)i = xi + I = 0 + I$. But $\ker \psi = M$ so that $I \subseteq M$ and hence R has property P.

3.16 LEMMA. *Let A be any simple R-module and let*

$$S(A) = \{(x)^R | 0 \neq x \in E(A)\}.$$

Then (i) $0 \neq x \in C(A)$ *if and only if* (ii) $(x)^R$ *is maximal in* $S(A)$.

Proof. If $0 \neq x \in C(A)$ and $(x)^R$ is not maximal in $S(A)$, then there is a $0 \neq y \in E(A)$ such that $(x)^R \subseteq (y)^R$ and there is an $r' \in R$ such that $xr' \neq 0$ but $yr' = 0$. Define $\psi : xR \rightarrow yR$ by $\psi(xr) = yr$ for all $r \in R$. Then ψ can be extended to $\bar{\psi} \in \mathrm{Hom}_R(E(A), E(A))$ and $\bar{\psi}(xr') = \psi(xr') = yr' = 0$. Thus $0 \neq xr' \in \ker \bar{\psi}$ and therefore $\ker \bar{\psi} \neq 0$. Then since A is simple and $A \cap \ker \bar{\psi} \neq 0$, $\bar{\psi}(A) = 0$. Thus $C(A) \subseteq \ker \bar{\psi}$ and in particular $\bar{\psi}(x) = 0$. Hence $yR = \psi(xR) = \psi(x)R = 0$. Thus it follows that $y = 0$ which contradicts the original assumption. Thus $(x)^R$ is maximal in $S(A)$ and hence (i) implies (ii).

If $0 \neq x \in E(A)$ and $(x)^R$ is maximal in $S(A)$, let

$$\lambda \in \Lambda = \mathrm{Hom}_R(E(A), E(A))$$

such that $\lambda(A) = 0$. Since $xR \neq 0$, there is an $r' \in R$ such that $0 \neq xr'$ and $xr' \in A$ and hence $\lambda(x)r' = \lambda(xr') \neq 0$. Now $(x)^R \subseteq (\lambda(x))^R$ and $r' \notin (x)^R$. Since this contradicts the maximality of $(x)^R$ in $S(A)$, we conclude that if $\lambda \in \Lambda$ and $\lambda(A) = 0$, then $\lambda(x) = 0$ and thus $x \in C(A) = \bigcap \{\ker \lambda | \lambda(A) = 0\}$. Therefore (ii) implies (i).

3.17 THEOREM. *If the ring R has property P, then $R \in \mathscr{A}$.*

Proof. Let A be a simple R-module and let $x \in C(A)$. For $0 \neq a_0 \in A$, $A = a_0 R \subseteq' xR$. Let $M = (a_0)^R$, $I = (x)^R$ and let θ be the mapping from R/M into R/I by the composition:

$$R/M \to A \to xR \to R/I$$

(where $A \to xR$ is the inclusion mapping). Then $\theta(R/M) \subseteq' R/I$. Therefore by property P, there exists a maximal right ideal N such that $I \subseteq N$ and $R/M \cong R/N$. Now $N = (1 + N)^R$ so that $N = (a)^R$ for some $a \in A$ and hence $(x)^R \subseteq (a)^R$. But $x \in C(A)$ so $(x)^R$ is maximal in $S(A)$ (see 3.16) and hence $(x)^R = (a)^R$. Then, since $(x)^R$ is a maximal right ideal, xR is a simple module and thus $xR = A$.

Since for any $x \in C(A)$, $xR = A$, it follows that $C(A) = A$ and hence $R \in \mathscr{A}$.

3.18 THEOREM. *Let R be a right Noetherian ring. Then $R \in \mathscr{A}$ if and only if R has property P.*

Proof. If R has property P then $R \in \mathscr{A}$ by 3.17. Conversely, suppose $R \in \mathscr{A}$ and let M, I and ψ be as in 3.14 so that $\psi(R/M) \subseteq' R/I$. Then $I = (1 + I)^R$ so $I = (x)^R$ for some $x \in E(R/M)$. If R is right Noetherian, then the set

$$S' = \{(y)^R | (x)^R \subseteq (y)^R \text{ and } y \in E(R/M)\}$$

has a maximal element $(y')^R$ and hence $(y')^R$ is maximal in $S(R/M)$ (see 3.16) so $y' \in C(R/M)$ and since $R \in \mathscr{A}$, $y' \in R/M$. Thus $(y')^R$ is a maximal right ideal and $R/M \cong R/(y')^R$ with $I \subseteq (y')^R$ and hence R has property P.

3.20 *Remark.* Let

$$R = \begin{bmatrix} Z & Z \\ Z & Z \end{bmatrix}.$$

Since R is a matrix ring over a commutative ring, $R \in \mathscr{A}$. Now R is right Noetherian so by 3.19, R has property P. Since R is not right duo and has more than one simple module, R is a counterexample to the converse of 3.15.

3.21 *Remark.* If R is a ring such that every maximal right ideal is also a left ideal, then R is called a right quasi-duo ring. The ring

$$R = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$$

of example 3.5 has exactly two maximal right ideals:

$$M = \begin{bmatrix} D & D \\ 0 & 0 \end{bmatrix} \text{ and } N = \begin{bmatrix} 0 & D \\ 0 & D \end{bmatrix}.$$

Moreover, these maximal right ideals are also left ideals and R has the following properties:

(1) R is a right quasi-duo ring.

(2) R has exactly two distinct simple right modules: R/M and R/N.

(3) R is not an α-ring (see 3.5).

(4) R is not a duo ring since

$$I = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$$

is a right ideal but not a left ideal.

Hence R is a counterexample to the following conjectured generalizations of Proposition 3.15.

3.15A *Conjecture.* Every right quasi-duo ring has property P and is hence an α-ring. This is false by (1) and (3).

3.15B *Conjecture.* If R has only a finite number of distinct simple modules, then $R \in \mathscr{A}$. This is false by (2) and (3).

4. α-rings and V-rings.

4.1 *Definition.* A ring will be called a V-ring provided that every simple right module is injective.

4.2 THEOREM (Villamayor). *R is a V-ring if and only if every right ideal is the intersection of maximal right ideals of R* [**3**, p. 130, No. 17].

4.3 *Definition.* A ring R is called a regular ring provided that for each $r \in R$, there is some $x \in R$ such that $rxr = r$. If, to each $r \in R$, there corresponds an element x so that $r = r^2x$ or $r = xr^2$, then R is called strongly regular. It is known [**9**, Theorem 7] that R is a strongly regular ring if and only if R is a regular duo ring.

4.4 *Definition.* Let R be any ring. For each maximal right ideal M, let $O_M = \{r \in R | (rx)^R \not\subseteq M \text{ for every } x \in R\}$.

In [**7**], Koh has shown that $O_M = (E(R/M))^R$ for any M in a large class of right ideals called "almost maximal" right ideals provided R contains an element c such that $(c)^R = 0$. In particular, if M is a maximal right ideal and $1 \in R$ then O_M is the annihilator of the injective hull of R/M.

4.5 PROPOSITION. *Let O_M be as in* 4.4. *Then if M is a maximal right ideal, $O_M \subseteq M$.*

Proof. See [**7**, Lemma 2].

4.6 PROPOSITION. *Let R be a right duo regular ring. Then for each maximal right ideal $M \subseteq R$, $O_M = M$.*

Proof. Let M be a maximal right ideal. By 4.5, $O_M \subseteq M$. If $M \not\subseteq O_M$, let $m' \in M$ be such that $m' \notin O_M$. Then there exists $x \in R$ such that $(m'x)^R \subseteq M$. Let $m = m'x$ so that both m and $(m)^R \subseteq M$. Since R is regular, there is a $u \in R$ such that $mum = m$. Then if $K = \{r - umr | r \in R\}$, $K \subseteq (m)^R \subseteq M$. Since R is right duo, $um \in M$ and also $umR \subseteq M$ so $K + umR \subseteq M$. But then if $r \in R$, $r = (r - umr) + umr$ which is an element of $K + umR$. Since $K + umR \subseteq M$, it follows that $R = M$. This contradiction arises from the assumption that $M \not\subseteq O_M$ so we conclude that $M \subseteq O_M$ and hence $M = O_M$.

4.7 PROPOSITION. *If R is a regular right duo ring, then for each simple module A, $C(A) = E(A)$.*

Proof. If A is simple, then $A \cong R/M$ for some maximal right ideal M. By 4.6, $M = O_M = (E(A))^R$. Hence if $x \in E(A)$, $(x)^R = M$ so that $(x)^R$ is maximal in $S(A)$ (see 3.16). Thus $x \in C(A)$ and hence $E(A) = C(A)$.

4.8 THEOREM. *Let R be a right duo ring. Then R is a V-ring if and only if R is a regular ring.*

Proof. Suppose R is a V-ring and let $a \in R$. Then $a^2 \in aR$ so $a^2R \subseteq aR$. Now $a^2R = \bigcap M_\alpha (\alpha \in A)$ where $\{M_\alpha | \alpha \in A\}$ is the collection of maximal right ideals which contain a^2R. If $a^2R \neq aR$, then $a \notin a^2R$ so that $a \notin \bigcap M_\alpha(\alpha \in A)$. Hence there is some M_{α_0} so that $a \notin M_{\alpha_0}$. Since $R = M_{\alpha_0} + aR$, $1 = m + ar$ for some $m \in M_{\alpha_0}$, $r \in R$ and hence $a = am + a^2r$. Then since M_{α_0} is a left ideal, $am \in M_{\alpha_0}$ and $a^2r \in a^2R \subseteq M_{\alpha_0}$ and hence $a \in M$. This is a contradiction based on the assumption that $a^2R \neq aR$. Therefore for each $a \in R$, $a^2R = aR$ so that $a = a^2x$ for some $x \in R$. This condition is called strongly regular and it is known that each strongly regular ring is a regular ring.

Conversely, if R is a right duo regular ring, then by 3.15 and 3.17 if A is simple then $A = C(A)$ and by 4.7 $C(A) = E(A)$ and therefore $A = E(A)$ for each simple module A and consequently R is a V-ring.

4.9 *Remark.* The conjecture that the collection of V rings is precisely the collection of regular α-rings was shown to be false in a paper by Cozzens [**2**] in which a V-ring was constructed which is not a regular ring.

4.10 *Remark.* Suppose R is a right quasi-duo ring. Then from 4.2, if R is a V-ring, it follows that each right ideal of R is the intersection of two-sided ideals and thus R is a right duo ring. Then Theorem 4.8 shows that R is strongly regular. Thus we have the following generalization of Theorem 4.8:

4.11 THEOREM. *Let R be a right quasi-duo ring. Then the following are equivalent:*

(1) *R is a V-ring.*
(2) *R is a strongly regular ring.*
(3) *R is a regular duo ring.*

Proof. (1) $\Rightarrow$ (2) follows from 4.10, while (2) $\Rightarrow$ (3) is shown in Theorem 7 of [**9**]. Finally, (3) $\Rightarrow$ (1) is shown in 4.8.

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