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THE EIGENVALUES OF COMPLEMENTARY PRINCIPAL SUBMATRICES OF A POSITIVE DEFINITE MATRIX

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1. Introduction. Let C be an n-square Hermitian matrix, presented in partitioned form as

$$C = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix},$$

where A is a-square and B is b-square. Let $\gamma_1 \geqq \ldots \geqq \gamma_n$, $\alpha_1 \geqq \ldots \geqq \alpha_a$, $\beta_1 \geqq \ldots \geqq \beta_b$ denote the eigenvalues of C, A, B, respectively. In a recent paper [**10**] the following inequality was established:

$$\sum_{s=1}^{m} \gamma_{i_s+j_s-s} + \sum_{s=1}^{m} \gamma_{n-m+s} \leqq \sum_{s=1}^{m} \alpha_{i_s} + \sum_{s=1}^{m} \beta_{j_s}, \tag{1.1}$$

if

$$1 \leqq i_1 < \ldots < i_m \leqq a, \quad 1 \leqq j_1 < \ldots < j_m \leqq b. \tag{1.2}$$

This inequality is a simplification and a sharpening of an inequality established earlier in [**6**], and is a wide generalization of an inequality of Aronszajn [**4**]. This earlier inequality proved in [**6**] was modelled on the Amir-Moez inequalities for the eigenvalues of a sum of Hermitian matrices [**1**] and it took the form

$$\sum_{s=1}^{2m} \gamma_{k_s''} \leqq \sum_{s=1}^{m} \alpha_{j_s''} + \sum_{s=1}^{m} \beta_{i_s''}, \tag{2}$$

where i_s'', j_s'', k_s'' are certain somewhat complicated subscripts; an exact description of these subscripts may be found in [**1**] or in [**2**]. Recently Amir-Moez and Perry [**3**] have shown that if C is positive definite, then an exactly analogous multiplicative version of (2) is valid for the eigenvalues of C, A, B, namely,

$$\prod_{s=1}^{2m} \gamma_{k_s''} \leqq \prod_{s=1}^{m} \alpha_{i_s''} \prod_{s=1}^{m} \beta_{j_s''}. \tag{3}$$

Since (1) is simpler and sharper than (2), it is natural to ask if the following multiplicative version of (1) is valid when C is positive definite: if the subscripts satisfy (1.2), then

$$\prod_{s=1}^{m} \gamma_{i_s+j_s-s} \prod_{s=1}^{m} \gamma_{n-m+s} \leqq \prod_{s=1}^{m} \alpha_{i_s} \prod_{s=1}^{m} \beta_{j_s}. \tag{4}$$

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The inequality (4), if it were true, would be simpler and sharper than (3) just as (1) is simpler and sharper than (2).

It is the purpose of this paper to establish two classes of inequalities, one of which will contain (4) as a special case. Our two classes of inequalities may each be regarded as a generalization of the Fischer determinantal inequality.

It is worth noting that the inequality (3) proved by Amir-Moez and Perry has the blemish that their subscripts k_s'' in the left-hand side of (3) are not always distinct. (This blemish is also present in the inequalities of [**6**].) However this defect does not occur in (1) or in any of the inequalities to be proved in this paper.

In this paper we shall draw upon techniques developed in several recent papers studying the eigenvalues of matrix sums and products. At the core of our proofs is a construction due to J. Hersch and B. P. Zwahlen [**5**; **11**] of a subspace satisfying a certain very tight set of dimensionality restrictions. Without this extremely useful construction of Hersch and Zwahlen the proofs given below would not have been found.

This paper is the twenty-fifth in a series of papers studying the eigenvalues of minors, sums, and products of matrices. This series of papers and a second series of number theoretical papers (totalling five so far) were begun when the senior author was a member of the Summer Research Institute of the Canadian Mathematical Congress, in Kingston, Ontario, 1961. The senior author wishes to express his appreciation to the Canadian Mathematical Congress for providing him ten years ago with an opportunity to begin the pursuit of the ideas leading to these papers.

2. Preliminary lemmas. The following somewhat combinatorial Lemma 1 is of independent interest and is useful in situations other than those arising in this paper. The symbol ${}^\perp L$ denotes the orthogonal complement of a subspace L in a unitary space.

LEMMA 1. *Let $U_0, \ldots, U_n$ be subspaces of a unitary vector space V_n, each space having dimension equal to its subscript. Suppose*

$$U_0 \subset U_1 \subset \ldots \subset U_n.$$

Let $p_1, \ldots, p_m$ be integers satisfying $1 \leqq p_1 < \ldots < p_m \leqq n$, and let $p_1', \ldots, p_{n-m}'$ denote the integers $1, \ldots, n$ complementary to $p_1, \ldots, p_m$, numbered such that $1 \leqq p_1' < \ldots < p_{n-m}' \leqq n$. Let L_m be an m-dimensional subspace of V_n.

(i) *Suppose that*

$$\text{(5)} \qquad \dim(L_m \cap U_{p_s}) \geqq s \quad (s = 1, \ldots, m).$$

Then

$$\text{(6)} \qquad \dim({}^\perp L_m \cap {}^\perp U_{p_s'-1}) \geqq n - m + 1 - s \quad (s = 1, \ldots, n - m).$$

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(ii) *Suppose that*

$$\dim(L_m \cap {}^{\perp}U_{p_s-1}) \geqq m + 1 - s \quad (s = 1, \ldots, m). \tag{7}$$

Then

$$\dim({}^{\perp}L_m \cap U_{p_s'}) \geqq s \quad (s = 1, \ldots, n - m). \tag{8}$$

Proof. (i) Let us arrange the integers $p_1, \ldots, p_m$ into strings of consecutive integers, as follows:

$$\begin{array}{c} g_1 + 1, \ldots, g_1 + e_1; \\ g_2 + 1, \ldots, g_2 + e_2; \\ \ldots \\ g_r + 1, \ldots, g_r + e_r. \end{array} \tag{9}$$

Here $g_1 + e_1 < g_2, g_2 + e_2 < g_3, \ldots, g_{r-1} + e_{r-1} < g_r, 0 \leqq g_1$, and $g_r + e_r \leqq n$. For convenience let $e_0 = 0 = g_0$ and $g_{r+1} = n$.

We shall use the following easily derived identity: for any subspaces L and U in V_n we have

$$\dim({}^{\perp}L \cap {}^{\perp}U) = n - \dim L - \dim U + \dim(L \cap U). \tag{10}$$

What are the integers $p_1', \ldots, p_{n-m}'$ when expressed in terms of the g_t and e_t? Arranged in strings of consecutive integers they are:

$$\begin{array}{c} 1, \ldots, g_1; \\ g_1 + e_1 + 1, \ldots, g_2; \\ g_2 + e_2 + 1, \ldots, g_3; \\ \ldots \\ g_{r-1} + e_{r-1} + 1, \ldots, g_r; \\ g_r + e_r + 1, \ldots, n. \end{array} \tag{11}$$

It is straightforward to check that if $1 \leqq t \leqq g_{s+1} - g_s - e_s$, then the integer $g_s + e_s + t$ in the list (11) occupies position

$$g_s + e_s + t - \sum_{\rho=0}^{s} e_\rho$$

in this list. To verify (6) thus amounts to verifying

$$\dim({}^{\perp}L_m \cap {}^{\perp}U_{g_s+e_s+t-1}) \geqq n - m + 1 - g_s - e_s - t + \sum_{\rho=0}^{s} e_\rho.$$

But, by (10), we have

$$\begin{aligned} \dim({}^{\perp}L_m \cap {}^{\perp}U_{g_s+e_s+t-1}) &= n - m - g_s - e_s - t + 1 + \dim(L_m \cap U_{g_s+e_s+t-1}) \\ &\geqq n - m - g_s - e_s - t + 1 + \dim(L_m \cap U_{g_s+e_s}) \\ &\geqq n - m - g_s - e_s - t + 1 + \sum_{\rho=0}^{s} e_\rho. \end{aligned}$$

The first inequality here follows from the nested property of the spaces U_t, and the second inequality follows from (5) and the fact that $g_s + e_s$ occupies position $e_0 + e_1 + \ldots + e_s$ in the list (9). This completes the proof of (i).

(ii) Let $W_s = {}^{\perp}U_{n-s}$, for $s = 0, \ldots, n$. Apply (i) to the nested subspaces W_s and the integers $n + 1 - p_{m+1-s}$, $s = 1, \ldots, m$. The complementary integers in $1, \ldots, n$ to these integers are the integers $n + 1 - p_{n-m+1-s}'$, $s = 1, \ldots, n - m$. Now (ii) follows from (i).

LEMMA 2. *Let H be an n-square positive semidefinite Hermitian linear transformation on unitary space V_n. Let $h_1 \geqq \ldots \geqq h_n$ be the eigenvalues of H, and let $u_1, \ldots, u_n$ be an associated orthonormal system of eigenvectors. Let $U_s = \langle u_1, \ldots, u_s\rangle$, $s = 1, \ldots, n$, where $\langle\ \rangle$ indicates the linear span of the enclosed vectors. Let integers $p_1, \ldots, p_m$ satisfy $1 \leqq p_1 < \ldots < p_m \leqq n$ and suppose that m-dimensional subspace L_m satisfies (5). Let $x_1, \ldots, x_m$ be any orthonormal basis for L_m. Then*

$$\det((Hx_i, x_j))_{1\leqq i,j\leqq m} \geqq h_{p_1} \ldots h_{p_m}.$$

If, on the other hand, L_m satisfies (7) then

$$\det((Hx_i, x_j))_{1\leqq i,j\leqq m} \leqq h_{p_1} \ldots h_{p_m}.$$

Here (,) denotes the inner product in V_n.

Proof. Lemma 2 is proved as [**9**, Lemmas 1 and 2].

3. The first main result.

THEOREM 1. *Let*

$$C = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$$

be an n-square positive definite Hermitian matrix with eigenvalues $\gamma_1 \geqq \ldots \geqq \gamma_n$. Let A be a-square and B be b-square with eigenvalues $\alpha_1 \geqq \ldots \geqq \alpha_a$, $\beta_1 \geqq \ldots \geqq \beta_b$, respectively. Let $0 \leqq \mu \leqq a$, $0 \leqq \nu \leqq b$ and let integers $i_1, \ldots, i_\mu, j_1, \ldots, j_\nu$ be given such that

$$1 \leqq i_1 < \ldots < i_\mu \leqq a,\ 1 \leqq j_1 < \ldots < j_\nu \leqq b. \tag{12}$$

Define

$$i_s = a + s - \mu \text{ for } s > \mu, \tag{12.1}$$

$$j_s = b + s - \nu \text{ for } s > \nu. \tag{12.2}$$

Then

$$\prod_{s=1}^{\mu+\nu} \gamma_{k_s} \leqq \prod_{s=1}^{\mu} \alpha_{i_s} \prod_{s=1}^{\nu} \beta_{j_s}, \tag{13}$$

where $k_s = i_s + j_s - s$ for $s = 1, \ldots, \mu + \nu$.

Remark. If we set $\mu = \nu = m$, inequality (13) reduces to (4).

Proof. Let $g_1, \ldots, g_n$ be an orthonormal system of column n-tuple eigenvectors of C associated respectively with the eigenvalues $\gamma_1, \ldots, \gamma_n$. Let $e_1, \ldots, e_a$ be an orthonormal system of column a-tuple eigenvectors of A associated respectively with $\alpha_1, \ldots, \alpha_a$ and let $f_1, \ldots, f_b$ be an orthonormal system of column b-tuple eigenvectors of B associated respectively with $\beta_1, \ldots, \beta_b$. Let

$$(14.1) \qquad E_s = \begin{bmatrix} e_s \\ 0 \end{bmatrix} (s = 1, \ldots, a),\ E_{s+a} = \begin{bmatrix} 0 \\ f_s \end{bmatrix} (s = 1, \ldots, b),$$

be column n-tuples and let

$$(14.2) \qquad F_s = \begin{bmatrix} 0 \\ f_s \end{bmatrix} (s = 1, \ldots, b),\ F_{s+b} = \begin{bmatrix} e_s \\ 0 \end{bmatrix} (s = 1, \ldots, a),$$

also be column n-tuples. Set $m = \mu + \nu$.

Let $z_s = i_s - s$, $w_s = j_s - s$, for $s = 1, \ldots, m$. Denote by

$$i_1' < \ldots < i_{n-m}', \quad j_1' < \ldots < j_{n-m}', \quad k_1' < \ldots < k_{n-m}'$$

the integers $1, \ldots, n$ complementary to $i_1 < \ldots < i_m$, $j_1 < \ldots < j_m$, $k_1 < \ldots < k_m$, respectively. By the Lemma of [**8**] we know that

$$\begin{aligned} i_s' &= s + \delta_{z_1}(s) + \ldots + \delta_{z_m}(s), \\ j_s' &= s + \delta_{w_1}(s) + \ldots + \delta_{w_m}(s), \\ k_s' &= s + \delta_{z_1+w_1}(s) + \ldots + \delta_{z_m+w_m}(s) \quad (s = 1, \ldots, n-m). \end{aligned}$$

Here $\delta_x(y)$ is a jump function defined by $\delta_x(y) = 0$ if $y \leqq x$, $=1$ if $y > x$. Since $z_m + w_m \leqq n - m$, we know from [**9**, Lemma 4] that an $(n-m)$-dimensional subspace L_{n-m} of column n-space exists such that

$$(15.1) \quad \dim(L_{n-m} \cap \langle E_1, \ldots, E_{i_s'}\rangle) \geqq s,$$

$$(15.2) \quad \dim(L_{n-m} \cap \langle F_1, \ldots, F_{j_s'}\rangle) \geqq s,$$

$$(15.3) \quad \dim(L_{n-m} \cap \langle g_{k_s'}, \ldots, g_n\rangle) \geqq n - m + 1 - s \ (s = 1, \ldots, n-m).$$

Let $L_m = {}^{\perp}L_{n-m}$. The integers complementary to $i_1', \ldots, i_{n-m}'$ are $i_1, \ldots, i_m$; those complementary to $j_1', \ldots, j_{n-m}'$ are $j_1, \ldots, j_m$; and those complementary to $k_1', \ldots, k_{n-m}'$ are $k_1, \ldots, k_m$, where $k_s = i_s + j_s - s$, for $s = 1, \ldots, m$. By Lemma 1 above, applied to (15.1), (15.2), (15.3) we find that

$$(16.1) \quad \dim(L_m \cap \langle E_{i_s}, \ldots, E_n\rangle) \geqq m + 1 - s,$$

$$(16.2) \quad \dim(L_m \cap \langle F_{j_s}, \ldots, F_n\rangle) \geqq m + 1 - s,$$

$$(16.3) \quad \dim(L_m \cap \langle g_1, \ldots, g_{k_s}\rangle) \geqq s \qquad (s = 1, \ldots, m).$$

In particular, by setting $s = \mu + 1$ in (16.1) we get

$$\dim(L_m \cap \langle F_1, \ldots, F_b\rangle) \geqq \nu$$

and by setting $s = \nu + 1$ in (16.2) we get

$$\dim(L_m \cap \langle E_1, \ldots, E_a\rangle) \geqq \mu.$$

For $s \leqq \mu$ we now have

$$\begin{aligned}
&\dim(L_m \cap \langle E_{i_s}, \ldots, E_a\rangle)\\
&\quad = \dim(L_m \cap \langle E_{i_s}, \ldots, E_n\rangle \cap L_m \cap \langle E_1, \ldots, E_a\rangle)\\
&\quad = \dim(L_m \cap \langle E_{i_s}, \ldots, E_n\rangle) + \dim(L_m \cap \langle E_1, \ldots, E_a\rangle)\\
&\qquad\qquad - \dim(L_m \cap \langle E_{i_s}, \ldots, E_n\rangle + L_m \cap \langle E_1, \ldots, E_a\rangle)\\
&\quad \geqq m + 1 - s + \mu - m = \mu + 1 - s.
\end{aligned}$$

Thus

$$\dim(L_m \cap \langle E_{i_s}, \ldots, E_a\rangle) \geqq \mu + 1 - s \quad (s = 1, \ldots, \mu), \tag{17.1}$$

and similarly

$$\dim(L_m \cap \langle F_{j_s}, \ldots, F_b\rangle) \geqq \nu + 1 - s \quad (s = 1, \ldots, \nu). \tag{17.2}$$

By virtue of (17.1) we may find orthonormal vectors $X_1, \ldots, X_\mu$ such that

$$X_s \in L_m \cap \langle E_{i_s}, \ldots, E_a\rangle \quad (s = 1, \ldots, \mu)$$

and by virtue of (17.2) we may find orthonormal vectors $X_{\mu+1}, \ldots, X_{\mu+\nu}$ such that

$$X_{\mu+s} \in L_m \cap \langle F_{j_s}, \ldots, F_b\rangle \quad (s = 1, \ldots, \nu).$$

Since the spaces $\langle E_1, \ldots, E_a\rangle$ and $\langle F_1, \ldots, F_b\rangle$ are orthogonal, it follows that $X_1, \ldots, X_\mu, X_{\mu+1}, \ldots, X_{\mu+\nu}$ form an orthonormal basis of L_m. Furthermore because of (14) we see that

$$X_s = \begin{bmatrix} x_s \\ 0 \end{bmatrix} \ (s = 1, \ldots, \mu),\ X_{\mu+s} = \begin{bmatrix} 0 \\ x_{\mu+s} \end{bmatrix} \ (s = 1, \ldots, \nu), \tag{18}$$

and hence that

$$\dim(\langle x_1, \ldots, x_\mu\rangle \cap \langle e_{i_s}, \ldots, e_a\rangle) \geqq \mu + 1 - s \quad (s = 1, \ldots, \mu), \tag{19.1}$$

$$\dim(\langle x_{\mu+1}, \ldots, x_{\mu+\nu}\rangle \cap \langle f_{j_s}, \ldots, f_b\rangle \geqq \nu + 1 - s \quad (s = 1, \ldots, \nu). \tag{19.2}$$

By (16.3) and Lemma 2, we see that

$$\gamma_{k_1} \ldots \gamma_{k_m} \leqq \det((CX_i, X_j)_{1 \leqq i, j \leqq m} \tag{20}$$

Applying the Fischer inequality to (20) and using (18) we get

$$\gamma_{k_1} \ldots \gamma_{k_m} \leqq \det((Ax_i, x_j))_{1 \leqq i, j \leqq \mu} \cdot \det((Bx_i, x_j))_{\mu < i, j \leqq m}.$$

Finally, applying Lemma 2 to each of the factors on the right-hand side obtained here and making use of (19.1) and (19.2) we get

$$\det((Ax_i, x_j))_{1 \leqq i, j \leqq \mu} \leqq \alpha_{i_1} \ldots \alpha_{i_\mu},$$
$$\det((Bx_i, x_j))_{\mu < 1, j \leqq m} \leqq \beta_{j_1} \ldots \beta_{j_\nu}.$$

The proof of (13) is now complete.

4. The second main result.

THEOREM 2. *Assume the hypotheses of Theorem* 1. *Let* $\delta_x(y)$ *denote a jump function, defined by* $\delta_x(y) = 0$ *if* $y \leqq x$, $\delta_x(y) = 1$ *if* $y > x$. *Let integers* $Z_1, \ldots, Z_{a-\mu}, W_1, \ldots, W_{b-\nu}$ *satisfy*

$$\mu \geqq Z_1 \geqq Z_2 \geqq \ldots \geqq Z_{a-\mu} \geqq 0, \tag{21.1}$$

$$\nu \geqq W_1 \geqq W_2 \geqq \ldots \geqq W_{b-\nu} \geqq 0. \tag{21.2}$$

Set

$$Z_s = 0 \ (s > a - \mu),\ W_s = 0 \ (s > b - \nu). \tag{21.3}$$

Define integers I_s, J_s, K_s *by*

$$I_s = s + \delta_{Z_1}(s) + \ldots + \delta_{Z_{a-\mu}}(s) \qquad (s = 1, \ldots, \mu), \tag{22.1}$$

$$J_s = s + \delta_{W_1}(s) + \ldots + \delta_{W_{b-\nu}}(s) \qquad (s = 1, \ldots, \nu), \tag{22.2}$$

$$K_s = s + \delta_{Z_1+W_1}(s) + \ldots + \delta_{Z_{n-m}+W_{n-m}}(s) \qquad (s = 1, \ldots, \mu + \nu). \tag{22.3}$$

Then

$$\prod_{s=1}^{\mu+\nu} \gamma_{K_s} \leqq \prod_{s=1}^{\mu} \alpha_{I_s} \prod_{s=1}^{\nu} \beta_{J_s}. \tag{23}$$

Proof. Define z_s, w_s by $z_s = \mu - Z_s$, $w_s = \nu - W_s$ for all $s = 1, 2, \ldots$. Then $0 \leqq z_1 \leqq \ldots \leqq z_{n-m} \leqq \mu$, $0 \leqq w_1 \leqq \ldots \leqq w_{n-m} \leqq \nu$, and $z_{n-m} + w_{n-m} \leqq m$, where $m = \mu + \nu$. Set

$$i_s' = s + \delta_{z_1}(s) + \ldots + \delta_{z_{n-m}}(s), \tag{24.1}$$

$$j_s' = s + \delta_{w_1}(s) + \ldots + \delta_{w_{n-m}}(s), \tag{24.2}$$

$$k_s' = s + \delta_{z_1+w_1}(s) + \ldots + \delta_{z_{n-m}+w_{n-m}}(s) \quad (s = 1, \ldots, m). \tag{24.3}$$

Observe that for $s \leqq \mu$ we have

$$i_s' = s + \delta_{z_1}(s) + \ldots + \delta_{z_{a-\mu}}(s) \tag{25.1}$$

since $z_{a-\mu+1} = \ldots = z_{n-m} = \mu$. Also observe that for $s \leqq \nu$ we have

$$j_s' = s + \delta_{w_1}(s) + \ldots + \delta_{w_{b-\nu}}(s), \tag{25.2}$$

since $w_{b-\nu+1} = \ldots = w_{n-m} = \nu$.

Let $E_1, \ldots, E_n, F_1, \ldots, F_n$ be as in § 3, and set

$$\begin{aligned} \hat{E}_1 = E_a = \begin{bmatrix} e_a \\ 0 \end{bmatrix}, \ldots, \hat{E}_a = E_1 = \begin{bmatrix} e_1 \\ 0 \end{bmatrix}, \\ \hat{E}_{a+1} = E_{a+1} = F_1, \ldots, \hat{E}_n = E_n = F_b, \\ \hat{F}_1 = F_b = \begin{bmatrix} 0 \\ f_b \end{bmatrix}, \ldots, \hat{F}_b = F_1 = \begin{bmatrix} 0 \\ f_1 \end{bmatrix}, \\ \hat{F}_{b+1} = F_{b+1} = E_1, \ldots, \hat{F}_n = F_n = E_a, \\ \hat{g}_1 = g_n, \ldots, \hat{g}_n = g_1. \end{aligned} \tag{26}$$

By (24) and [**9**, Lemma 4], we may find subspace L_m such that

(27.1) $\dim(L_m \cap \langle \hat{E}_1, \ldots, \hat{E}_{i_s'}\rangle) \geqq s$

(27.2) $\dim(L_m \cap \langle \hat{F}_1, \ldots, \hat{F}_{j_s'}\rangle) \geqq s$

(27.3) $\dim(L_m \cap \langle \hat{g}_{k_s'}, \ldots, \hat{g}_n\rangle) \geqq m + 1 - s \quad (s = 1, \ldots, m).$

For $s \leqq \mu$ we obtain from (27.1) that

$$\dim(L_m \cap \langle E_{a+1-i_s'}, \ldots, E_a\rangle) \geqq s. \tag{28}$$

Now by (25.1), for $s \leqq \mu$ we have

$$\begin{aligned} a + 1 - i_{\mu+1-s}' &= s + \sum_{t=1}^{a-\mu} (1 - \delta_{z_t}(\mu + 1 - s)) \\ &= s + \sum_{t=1}^{a-\mu} \delta_{\mu - z_t}(s) \\ &= s + \sum_{t=1}^{a-\mu} \delta_{Z_t}(s) = I_s. \end{aligned}$$

Thus (27.1) states that

(29.1) $\dim(L_m \cap \langle E_{I_s}, \ldots, E_a\rangle \geqq \mu + 1 - s \quad (s = 1, \ldots, \mu).$

Similarly from (27.2) and (25.2) we get

(29.2) $\dim(L_m \cap \langle F_{J_s}, \ldots, F_b\rangle) \geqq \nu + 1 - s \quad (s = 1, \ldots, \nu).$

We also obtain from (27.3) and (24.3) the inequality

$$\dim(L_m \cap \langle g_1, \ldots, g_{K_s}\rangle \geqq s \quad (s = 1, \ldots, m). \tag{29.3}$$

Using (29.1) in succession for $s = \mu, \mu - 1, \ldots,$ we may construct an L_μ in L_m and in $\langle E_1, \ldots, E_a\rangle$ such that

$$\dim(L_\mu \cap \langle E_{I_s}, \ldots, E_a\rangle \geqq \mu + 1 - s \quad (s = 1, \ldots, \mu).$$

And we may also use (29.2) to find an L_ν inside L_m and $\langle F_1, \ldots, F_b\rangle$ such that

$$\dim(L_\nu \cap \langle F_{J_s}, \ldots, F_b\rangle) \geqq \nu + 1 - s \quad (s = 1, \ldots, \nu).$$

Then by (26) we see that $L_m = L_\mu \perp L_\nu$, and so we may find an orthonormal basis

$$X_1 = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \ldots, X_\mu = \begin{bmatrix} x_\mu \\ 0 \end{bmatrix}, X_{\mu+1} = \begin{bmatrix} 0 \\ x_{\mu+1} \end{bmatrix}, \ldots, X_m = \begin{bmatrix} 0 \\ x_m \end{bmatrix}$$

of L_m such that

(30.1) $\dim(\langle x_1, \ldots, x_\mu\rangle \cap \langle e_{I_s}, \ldots, e_a\rangle) \geqq \mu + 1 - s \quad (s = 1, \ldots, \mu),$

(30.2) $\dim(\langle x_{\mu+1}, \ldots, x_m\rangle \cap \langle f_{J_s}, \ldots, f_b\rangle) \geqq \nu + 1 - s \quad (s = 1, \ldots, \nu).$

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Using the Fischer inequality, Lemma 2, (30.1), (30.2), and (29.3), we obtain

$$\begin{aligned}\gamma_{K_1}\ldots\gamma_{K_m} &\leqq \det((CX_i, X_j))_{1\leqq i,j\leqq m}\\ &\leqq \det((CX_i, X_j))_{1\leqq i,j\leqq \mu}\det((CX_i, X_j))_{\mu<i,j\leqq m}\\ &= \det((Ax_i, x_j))_{1\leqq i,j\leqq \mu}\det((Bx_i, x_j))_{\mu<i,j\leqq m}\\ &\leqq \alpha_{I_1}\ldots\alpha_{I_\mu}\beta_{J_1}\ldots\beta_{J_\nu}.\end{aligned}$$

This completes the proof of Theorem 2.

5. Additional remarks. It is interesting to compare the inequalities (13) and (23). In the inequalities of Theorem 1 we set $i_s = I_s$ $(s \leqq \mu)$ and $j_s = J_s$ $(s \leqq \nu)$. Thus

(31.1) $$i_s = s + \delta_{Z_1}(s) + \ldots + \delta_{Z_{a-\mu}}(s) \quad (s = 1, \ldots, \mu),$$

(31.2) $$j_s = s + \delta_{W_1}(s) + \ldots + \delta_{W_{b-\nu}}(s) \quad (s = 1, \ldots, \nu).$$

Because of (21.1) and (12.1), the relation (31.1) is valid for $s > \mu$ and because of (21.2) and (12.2), the equality (31.2) is valid for $s > \nu$. We thus obtain the following formulas showing both similarities and contrasts between the k_s and the K_s:

(32.1) $$k_s = s + \delta_{Z_1}(s) + \ldots + \delta_{Z_{a-\mu}}(s) + \delta_{W_1}(s) + \ldots + \delta_{W_{b-\nu}}(s),$$

(32.2) $$K_s = s + \delta_{Z_1+W_1}(s) + \ldots + \delta_{Z_{(a-\mu)+(b-\nu)}+W_{(a-\mu)+(b-\nu)}}(s) \quad (s = 1, \ldots, \mu+\nu).$$

In (32) the Z_t, W_t are constrained by the conditions (21). One may ask whether $k_s \geqq K_s$ for all s or whether $K_s \geqq k_s$ for all s. Simple numerical examples show that neither of these possibilities can hold in general. For example, if $a = b = 5$, $\mu = \nu = 3$, $Z_1 = W_1 = 3$, $Z_2 = W_2 = 2$, then $k_1 = 1$, $k_2 = 2$, $k_3 = 5$, $k_4 = 8$, $k_5 = 9$, $k_6 = 10$, whereas $K_1 = 3$, $K_2 = 4$, $K_3 = 5$, $K_4 = 6$, $K_5 = 8$, $K_6 = 9$.

We will not give here the proof of the claim that (4) is both simpler than and sharper than (3), since this proof exactly parallels the proof in [**10**, § 4] and is basically a restatement of the last part of [**7**]. The inequalities of Theorems 1 and 2 may be extended by an easy induction on k to obtain inequalities comparing the eigenvalues of a positive definite

$$C = (A_{st})_{1\leqq s,t\leqq k}$$

partitioned into a $k \times k$ block matrix with the eigenvalues of the main diagonal blocks $A_{11}, \ldots, A_{kk}$. We do not state this result or give its proof. The form of these results are directly analogous to the form of the corresponding additive inequalities in [**10**, Theorems 3 and 4].

References

1. A. R. Amir-Moez, *Extreme properties of eigenvalues of a Hermitian transformation and singular values of the sum and product of linear transformations*, Duke Math. J. *23* (1956), 463–476.

2. A. R. Amir-Moez and C. Perry, *Remarks on Thompson, Freede theorems*, Bull. Australian Math. Soc. *5* (1971), 221–226.

3. ——— *Positive transformations restricted to subspaces and inequalities among their proper values*, Proc. Amer. Math. Soc. *32* (1972), 363–367.

4. N. Aronszajn, *Rayleigh-Ritz and A. Weinstein methods for approximation of eigenvalues. I. Operators in a Hilbert space.* Proc. Nat. Acad. Sci. U.S.A. *34* (1948), 474–480.

5. J. Hersch and B. P. Zwahlen, *Evaluations par defaut pour une somme quelconque de valeurs propres γ_k d'un opérateur $C = A + B$ à l'aide de valeurs α_i de A et β_j de B*, C. R. Acad. Sci. Paris *254* (1962), 1559–1561.

6. R. C. Thompson and L. J. Freede, *Eigenvalues of partitioned Hermitian matrices*, Bull. Australian Math. Soc. *3* (1970), 23–37.

7. ——— *On the eigenvalues of sums of Hermitian matrices*, Linear Algebra and Appl. *4* (1971), 369–376.

8. ——— *On the eigenvalues of sums of Hermitian matrices* II, Aequationes Math. *5* (1970), 103–115.

9. R. C. Thompson and S. Therianos, *The singular values of a matrix product.* I (to appear in Scripta Math.).

10. ———*Inequalities connecting the eigenvalues of a Hermitian matrix with the eigenvalues of complementary principal submatrices*, Bull. Australian Math. Soc. *6* (1972), 117–132.

11. B. P. Zwahlen, *Über die Eigenwerte der Summe zweier selbstadjungierter Operatoren*, Comment. Math. Helv. *40* (1966), 81–116.

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