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HALF-SILVERED MIRRORS AND WYTHOFF'S GAME

BY

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ABSTRACT. We propose the following problem. Given an array of vertical mirrors that simultaneously transmit and reflect, and a single incoming ray of light, describe the configuration of all light rays that are generated. We solve the problem here for a certain infinite configuration of mirrors; the solution involves the winning positions $(a(n), b(n))$ of Wythoff's game.

1. **Introduction.** We propose the following problem. Given an array of vertical mirrors that simultaneously transmit and reflect, and a single incoming ray of light, describe the configuration of all light rays that are generated. We solve the problem here for a certain infinite configuration of mirrors; the solution involves two infinite sequences of integers, the $a(n)$, $b(n)$ of Section 2, that occur in the theory of Wythoff's game ([4], [9], [10], [11], [13]). The pairs $(a(n), b(n))$ are the winning positions of Wythoff's game. These sequences also occur in quasicrystallography ([6], [7], [21], [25]); see [33] for their history, [28] for a connection with affine transformations, and [27] for a connection with a certain dynamical system.

For only two straight line mirrors in the plane the problem is easily solved, whether or not they intersect or are parallel. As the number of mirrors increases, so does the complexity of the problem. The case of mirrors that only reflect is usually found under the heading of "billiards". Here the standard reference seems to be [12], Chapter 6, and this should be supplemented by [34], Chapters 15–18.

A simplified version of the apparatus used in the famous Michelson–Morley experiment incorporates a half-silvered mirror [1] (see [20], pp. 465–467, for a somewhat more technical account). Some further optical analysis of plates that both transmit and reflect is found in [20], pp. 200–204. However all of what the authors have found connected with the above problem-circle seems essentially one dimensional in nature. Much of it grew out of a problem posed by Moser and Wyman (in [22]). For a good sampling of results in this direction consult [18] and the references therein. However, our problem is two-dimensional; "vertical" and "horizontal" will mean parallel to the y and x axis, respectively.

2. **The main result.** Place straight half-silvered mirrors on the positive x and y

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axes of a coordinate plane and on the line segments defined by $x + y = 1, 2, 4, 7, \ldots, F_{n+2} - 1, \ldots, x \geqq 0, y \geqq 0$, where F_n are the Fibonacci numbers.

We shall always assume that the incoming ray is vertical with entry point at $(n, 0)$ or horizontal with entry point at $(0, m)$, where n, m are positive integers. Rays that leave the first quadrant are considered to have escaped forever.

Next define a partition $Z^+ = A \cup B$ of the set Z^+ of positive integers into two disjoint classes A, B, as follows. Let $\phi = (1 + \sqrt{5})/2$ be the golden mean, and set $a(n) = [n\phi]$, and $b(n) = [n\phi^2] = a(n) + n$ (here we use $[x]$ to denote the integral part of x). Then let $A = \{a(n)\}_{n=1}^{\infty}$, and $B = \{b(n)\}_{n=1}^{\infty}$. That the "Beatty sequences" A and B constitute such a partition is well known [23], [33], [34]. Also, $a(F_{2n-1}) = F_{2n}$ and $a(F_{2n}) = F_{2n+1} - 1$.

THEOREM. *For the above system of half-silvered mirrors with a vertical incoming ray starting at $(n, 0)$, there are only two possible final configurations. If $n \in A$, the final configuration consists of all vertical rays $x = v$, $y \geqq 0$ and all horizontal rays $y = h$, $x \geqq 0$, where $v, h \in A$. If $n \in B$, the same conclusion holds but now with $v, h \in B$.*

3. **Preliminary results.** Since the denominators of the convergents of the continued fraction for ϕ are the Fibonacci numbers, the distance from $m\phi$ to the nearest integer has its local extrema exactly when m is a Fibonacci number. Further, the convergents to a continued fraction are alternatively above and below their limit, and therefore the fractional part $\{m\phi\}$ is closest to 1 when $m = F_{2s}$ and closest to zero when $m = F_{2s+1}$. For example, when $1 \leqq m < F_{2s+2}$, the quantity $m\phi$ is closest to 1 when $m = F_{2s}$. In fact, the elementary theory of continued fractions (see, e.g., [24], Chapter 7) yields the following slightly more detailed statement:

LEMMA.

(i) $$\max_{1 \leqq m < F_{2s+2}} \{m\phi\} = \{F_{2s}\phi\} = 1 - \phi^{-2s}$$

(ii) $$\min_{1 \leqq m < F_{2s+3}} \{m\phi\} = \{F_{2s+1}\phi\} = \phi^{-(2s+1)}$$

(iii) $$\min_{-F_{2s+2} < m \leqq -1} \{m\phi\} = \{-F_{2s}\phi\} = \phi^{-2s}$$

(iv) $$\max_{-F_{2s+3} < m \leqq -1} \{m\phi\} = \{-F_{2s+1}\phi\} = 1 - \phi^{-(2s+1)}$$

Next we prove:

PROPOSITION 1. *If n is a positive even integer, then*

(1.*i*) $$a(k + F_n) = a(k) + F_{n+1}, \textit{ for } -F_{n+2} < k < F_{n+1},\ k \neq 0,$$

and

(1.*ii*) $$a(k + F_{n+1}) = a(k) + F_{n+2}, \textit{ for } -F_{n+1} < k < F_{n+2}.$$

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PROOF. Let k satisfy $k \neq 0$ and $-F_{n+2} < k < F_{n+1}$. If $k > 0$ then part (ii) in the Lemma applies (with $n = 2s + 2$) to give $\{k\phi\} \geqq \phi^{-(n-1)}$, and if $k < 0$ then part (iii) in the Lemma applies (with $n = 2s$) to give $\{k\phi\} \geqq \phi^{-n}$. In either case we get $\{k\phi\} \geqq \phi^{-n} = F_{n+1} - F_n\phi$, so $F_{n+1} - F_n\phi \leqq \{k\phi\} = k\phi - a(k) < 1 < 1 + F_{n+1} - F_n\phi$. Hence $(k+F_n)\phi - 1 < a(k) + F_{n+1} \leqq (k+F_n)\phi$, and therefore $a(k+F_n) = [(k+F_n)\phi] = a(k) + F_{n+1}$. This proves (1.i). Suppose next that $-F_{n+1} < k < F_{n+2}$. By applying parts (i) and (iv) of the Lemma we obtain in a similar way $\{k\phi\} \leqq 1 - \phi^{-n} < 1 - \phi^{-(n+1)} = 1 + F_{n+2} - F_{n+1}\phi$, so $F_{n+2} - F_{n+1}\phi < 0 < k\phi - a(k) < 1 + F_{n+2} - F_{n+1}\phi$. Hence $(F_{n+1} + k)\phi - 1 < a(k) + F_{n+2} < (F_{n+1} + k)\phi$ and therefore $a(k + F_{n+1}) = [(k + F_{n+1})\phi] = a(k) + F_{n+2}$. This completes the proof of Proposition 1. □

Note that the bounds on k are best possible since

$$\begin{aligned} a(F_{n+1} + F_n) &= a(F_{n+2}) = F_{n+3} - 1 \\ &\neq a(F_{n+1}) + F_{n+1} = F_{n+2} + F_{n+1} = F_{n+3}, \\ a(-F_{n+2} + F_n) &= a(-F_{n+1}) = -F_{n+2} - 1 \\ &\neq a(-F_{n+2}) + F_{n+1} = -F_{n+3} + F_{n+1} = -F_{n+2}, \end{aligned}$$

and

$$\begin{aligned} a(F_{n+2} + F_{n+1}) &= a(F_{n+3}) = F_{n+4} \\ &\neq a(F_{n+2}) + F_{n+2} = F_{n+4} - 1, \\ a(-F_{n+1} + F_{n+1}) &= 0 \\ &\neq a(-F_{n+1}) + F_{n+2} = -F_{n+2} - 1 + F_{n+2} = -1. \end{aligned}$$

Next we show that the function $a(n)$ is "almost linear" with suitable constraints on its arguments.

PROPOSITION 2. *If* $1 \leqq j, l$ *and* $j + l = F_n$, *then*

$$a(j) + a(l) = F_{n+1} - 1 \tag{2.i}$$

and

$$b(j) + b(l) = F_{n+2} - 1. \tag{2.ii}$$

PROOF. By Proposition 1 (note that $-F_n < -l$) and the identity $[-x] = -[x] - 1$ for non-integral $x \neq 0$, we have $a(j) = a(-l + F_n) = a(-l) + F_{n+1} = -a(l) - 1 + F_{n+1}$. This proves equation (2.i). To obtain (2.ii) simply add (2.i) and the equation $j + l = F_n$.

PROPOSITION 3. *The diophantine equation*

$$a(p) + b(q) = r, \ r \geqq 3,$$

has a solution $p, q \geqq 1$ *if and only if* r *is not of the form* $r = F_n - 1$.

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PROOF. First suppose that r is of the form $r = F_n - 1$. If $a(p)+b(q) = r$ with $p, q \geqq 1$, then necessarily $p < F_{n-1}$ and by Proposition 2 we have $a(p) + a(F_{n-1} - p) = r$. This leads to the contradiction $a(F_{n-1} - p) = b(q)$.

To prove the converse we will use the Zeckendorf number system (see [4] for example). Every integer $m \geqq F_2 = 1$ may be written in a unique way as a sum of Fibonacci numbers by means of the greedy algorithm: let $m = F_n + m'$ where F_n is maximal subject to $1 \leqq F_n \leqq m$, and apply the same process to the remainder m', etc., until some remainder is zero. We may then represent m by a string of zeros and ones with the leftmost character a one and no two ones consecutive. For example, 10000101 is $F_2+F_4+F_9 = 1+3+34 = 38$; the "place values" are Fibonacci numbers rather than powers of ten. This is the Zeckendorf- or Fibonacci-number system, and is quite ubiquitous in the Fibonacci literature. Moreover, m belongs to A to B depending upon whether its representation as a string of ones and zeros ends in an even or an odd number of zeros, respectively. Here an excellent reference is [14], especially pp. 339–340 for the quadratic case with $a = 1$ in the notation of [14].

Next, call a zero (or block of zeros) in a Zeckendorf representation *internal* if there is at least one more one to its right. Corresponding to every internal zero of m there is a simple minded decomposition $m = m_1 + m_2$, with $m_1, m_2 \geqq 1$. For example, if $m = 101000010001$, then corresponding to the sixth zero from the right (or any other zero in its block) we have $m = 101000000000 + 10001$.

For the proof we first assume that r ends with a one. We have three cases. First suppose that the representation of r has a block with an even number of zeros. It is easy to see that to the right of the rightmost such block, call it B_R, there must be an odd number of characters (if an even number of odd blocks of zeros occur to the right of B_R, there are an odd number of separating ones). Hence the decomposition of r corresponding to any zero in B_R gives $r = r_1 + r_2$ where the number of zeros at the end of r_1 and r_2 differ in parity. For example, if $r = 101000010001$ then $r = 101000000000 + 10001$.

Secondly, say r has no even internal blocks, but does have an internal block of at least 3 zeros. Call the rightmost such block B_R, call the ones to its left and right 1_L and 1_R respectively, and let $0', 0''$ be the first and second zeros of B_R. Replace $1_L 0' 0''$ by 011. This is no longer a valid Zeckendorf representation, but since $F_{n-2} + F_{n-1} = F_n$, it also represents r. Now decompose r by splitting it between the two new 1's. For example, $r = 10101000101 = 10100110101 = 10100100000 + 10101 = r_1 + r_2$. Since every "block" of zeros to the right of B_R has only one zero, there are an odd number of ones to the right of B_R if the number of zeros to the right is even, and vice versa. Hence the number of terminal zeros of r_1 and r_2 differ in parity.

If the above cases do not occur, we have the third case in which the representation of r consists of alternating zeros and ones. Upon adding one to it and using $2F_2 = F_3$ followed by repeated applications of $F_{n-1} + F_n = F_{n+1}$, we find that $r+1$ is a Fibonacci number. This completes the proof for r ending in 1.

If r has terminal zeros, let $r' < r$ be the number obtained by removing them. If r' belongs to case one or two above, decompose it accordingly, and restore the zeros to each term to obtain the desired decomposition of r. Otherwise r is $1010 \cdots 101$

followed by one or more zeros. If one zero, then r in an $F_n - 1$. If more than one zero, use the construction of case two to complete the argument. This ends the proof of Proposition 3.

4. **Proof of the theorem.** Suppose the incoming ray is vertical with entry point at $(n_0, 0)$. For every $F_n - 1 > n_0$ the ray gets reflected to produce a horizontal ray at height $F_n - 1 - n_0$. This ray goes left, gets reflected back at the y-axis to re-enter the first quadrant, and produces in turn vertical rays (by going initially downward, then bouncing back up on the x-axis). The resulting vertical rays sit at the x-values $F_m - 1 - (F_n - 1 - n_0)$ for each $F_m - 1 > F_n - 1 - n_0$. The process continues to produce vertical rays at $x = v_1$, $x = v_2$, $x = v_3$, ... and horizontal rays at $y = h_1$, $y = h_2$, $y = h_3, \ldots$. By renumbering, we may assume $v_i < v_{i-1}$ and $h_i < h_{i-1}$. Observe that a vertical ray entering at any v_i (or created at any point with abscissa v_i) must generate the original ray, since by reflexions on the axes, all rays retrace themselves in both directions.

When $n_0 \in A$, it follows from Proposition 2 that all rays are placed at A-positions, *e.g.*, $v_i \in A$ and $h_i \in A$ for each $i = 1, 2, 3, \ldots$. Similarly, for $n_0 \in B$, we get $v_i, h_i \in B$ for all $i = 1, 2, 3, \ldots$. Hence the final configuration of rays is, at any rate, a subset of what the theorem states.

We now show that $v_1 = h_1 = 1$ or $v_1 = h_1 = 2$. First, if $n_0 \geqq 3$ then we can find n with $n_0 + 2 \leqq F_n \leqq 2n_0 + 1$. Hence $n_1 = F_n - 1 - n_0 < n_0$ and therefore there is a horizontal ray at $y = n_1 < n_0$. If $n_1 \geqq 3$, the same argument shows that there is a vertical ray at $x = n_2 < n_1$. This continues until we reach 1 or 2, either as horizontal or verti cal positions. But the mirror $x + y = 2 = F_4 - 1$ (resp. $x + y = 4 = F_5 - 1$) produces vertical and horizontal reflexions of 1 (resp. 2) into itself, and our claim is proved.

In view of these remarks, it suffices to analyse the cases $n_0 = 1$, and $n_0 = 2$. Assume $n_0 = 1$. Then $v_1 = h_1 = 1$ and by symmetry $v_i = h_i$ for all $i = 1, 2, \ldots,$ since the configuration can be generated either by the vertical ray entering at (1, 0) or by its reflexion at $x + y = 2$, the horizontal ray entering at (0, 1). Easy calculations reveal the values 3, 4, 6, and 8 among the v_i. We prove that $\{v_i\}_{i=1}^{\infty} \supseteq A$ as follows. Suppose by contradiction that $a(j)$ is the smallest member of A not listed in $\{v_i\}_{i=1}^{\infty}$, and let F_n satisfy $F_n - 1 \leqq a(j) < F_{n+1} - 1$. Set $l = F_n - j$. Observe that $a(l) < a(j)$. In fact, from Proposition 2, $a(l) = F_{n+1} - 1 - a(j)$ so the reverse inequality $a(l) \geqq a(j)$ would imply $F_{n+1} - 1 \geqq 2a(j)$, whence $2(F_n - 1) \leqq F_{n+1} - 1$ and this fails for $n \geqq 5$. Therefore $a(l)$ is in $\{v_i\}_{i=1}^{\infty}$, by the minimality of $a(j)$. However this implies that $a(l) = F_{n+1} - 1 - a(j)$ is an h_i, hence a v_i, and a contradiction follows. The proof for $v_1 = 2$ is similar. And by symmetry, the case of a horizontal incoming ray also follows. This proves the theorem.

If any mirror $x + y = r$ with $r \geqq 3$, but $r + 1$ not a Fibonacci number, is added to our system of mirrors, then the final configuration of light rays consists of all vertical half lines at $x = v$ and all horizontal half lines at $y = h$, where $v, h \in Z^+$ are arbitrary. This follows immediately from Proposition 3.

To determine what happens when mirrors are removed seems considerably more involved. Here we shall restrict our attention to the case in which the ray entering the first quadrant is $x = 1$, the mirror $x + y = 2$ is retained (this insures symmetry), and only a finite number of mirrors are removed. We show that the conclusion of the theorem remains valid.

Let K denote the set of all integers k such that $x + y = k$ is a mirror in the array under consideration, and call $(1, m_1, m_2, \ldots, m_j)$ a K-mirror sequence if $1 + m_1, m_1 + m_2, \ldots, m_{j-1} + m_j$ all belong to K. Then the set of all m such that the ray $x = m$ (resp. $y = m$) is part of the final configuration is exactly the set of all elements of all K-mirror sequences. We show that if m occurs in a mirror sequence σ that uses the $x + y = F_{s-1} - 1$ mirror, it also occurs in a sequence that uses only larger mirrors. Say σ has consecutive elements m_i, m_{i+1} with $m_i + m_{i+1} = F_{s-1} - 1$. If in σ we replace m_i, m_{i+1} by m_i, $F_{s+1} - 1 - m_i$, $F_{s+1} - 1 - m_{i+1}$, m_{i+1} we are rid of this occurrence of the $x + y = F_{s-1} - 1$ mirror. We still have a K-mirror sequence since the sum of the two additional elements is

$$F_{s+1} + F_s + F_{s-1} - 2 - (m_1 + m_{i+1}) = F_{s+2} - 1.$$

The claim follows upon sufficiently many such replacements.

5. **Remarks.** An attempt at providing a reasonably complete bibliography of the mathematics of Beatty sequences such as A and B was made in [33], and this was soon updated by [15]. However, these sources do not mention Lord Rayleigh, whose treatise [30] contains the first formulation of the $\alpha^{-1}+\beta^{-1} = 1$ result on complementary sequences (see pages 119–123). The authors learned of Rayleigh's work from [34], Chapter 4. In recent times the literature has become exceedingly large, so we shall not attempt a full updating of [33]. The published proceedings of a recent conference on quasicrystallography [16] has over 500 pages of often tersely written notes. Moreover, it is quite possible [5] that much of the voluminous literature on almost periodic functions is bound up with the subject. Hence we shall for now refer the reader to the following papers for connections of our subject with other topics: [17], [21], [25] for quasicrystallography and Penrose tilings, [35] for automatic sequences, [4] for deterministic games, [30] for mechanical vibrations (here Chapter 5 of Arnold's book [2], which involves the Courant–Fisher characterization of eigenvalues (see also [3], [19]), helps to clarify Lord Rayleigh's approach), [26], [27], [28], [29] for semigroups, and to the present paper for the connection with elementary optics. The bibliography of [33] did refer to the connection with geodesics on certain 2-manifolds via two papers of H. Cohn, but missed the interesting nineteenth century precursors [8] and [32]. For the details and more references on this topic see [31].

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