π-DOMAINS, OVERRINGS, AND DIVISORIAL IDEALS

by D. D. ANDERSON

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1. Introduction. In this paper we study several generalizations of the concept of unique factorization domain. An integral domain is called a *π-domain* if every principal ideal is a product of prime ideals. Theorem 1 shows that the class of π-domains forms a rather natural subclass of the class of Krull domains. In Section 3 we consider overrings of π-domains. In Section 4 generalized GCD-domains are introduced: these form an interesting class of domains containing all Prüfer domains and all π-domains.

In this paper all rings are commutative with identity. We use the letter D to represent an integral domain with quotient field K. If A is a (fractional) ideal of D, we denote $(A^{-1})^{-1}$ by A_v. We call A *divisorial* if $A = A_v$. The group of invertible ideals of D modulo the subgroup of principal ideals is called the *class group* of D and will be denoted by $C(D)$. If D is completely integrally closed, the divisorial ideals form a group. The group of divisorial ideals modulo the subgroup of principal ideals is called the *divisor class group* of D and will be denoted by $\mathrm{Cl}(D)$. Notice that $C(D)$ is a subgroup of $\mathrm{Cl}(D)$.

Our general references are Gilmer [**6**] and Kaplansky [**7**]. For the theory of Krull domains, the reader is referred to Bourbaki [**2**] and Fossum [**5**].

2. Characterizations of π-domains. Theorem 1 shows that the class of π-domains forms a rather natural subclass of the class of Krull domains.

THEOREM 1. *The following statements are equivalent for an integral domain D:*

(1) *D is a π-domain,*

(2) *D is a Krull domain that is locally a UFD,*

(3) *D is a Krull domain and the prime ideals of rank one are invertible,*

(4) *D is a Krull domain and every divisorial ideal of D is invertible,*

(5) *D is a Krull domain and $C(D) = \mathrm{Cl}(D)$,*

(6) *D is a Krull domain and each product of divisorial ideals of D is divisorial,*

(7) *D is a Krull domain and the intersection of any two non-zero principal ideals of D is invertible,*

(8) *D is a Krull domain and the intersection of any two invertible ideals of D is invertible.*

Proof. The equivalence of (1) and (3) follows from [**9**, Theorem 1.2].

The implication (4) $\Rightarrow$ (2) follows from [**5**, Proposition 9.2].

We show (2)$\Leftrightarrow$(1). Let x be a non-zero non-unit of D. We must show that xD is a product of prime ideals. Since D is a Krull domain, $xD = P_1^{(n_1)} \cap \ldots \cap P_s^{(n_s)}$, where $P_1, \ldots, P_s$ are the prime ideals of rank one containing x. We show that $xD = P_1^{n_1} \ldots P_s^{n_s}$. Let M be a fixed maximal ideal of D. If $P_i \nsubseteq M$, then $P_{iM}^{n_i} = D_M = P_{iM}^{(n_i)}$. If $P_i \subseteq M$, then P_{iM} is a prime ideal of rank one in the UFD D_M and hence is principal. Thus $P_{iM}^{n_i}$ is primary

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and hence $P_{iM}^{n_i} = P_{iM}^{n_i}$. Moreover, since the prime ideals P_{iM} are principal, $xD_M = P_{1M}^{(n_1)} \cap \ldots \cap P_{sM}^{(n_s)} = P_{1M}^{n_1} \cap \ldots \cap P_{sM}^{n_s} = P_{1M}^{n_1} \ldots P_{sM}^{n_s} = (P_1^{n_1} \ldots P_s^{n_s})_M$. Thus xD and $P_1^{n_1} \ldots P_s^{n_s}$ are equal locally and hence globally.

To prove the equivalence of (3) and (4), we note that every divisorial ideal of a Krull domain has the form $P_1^{(n_1)} \cap \ldots \cap P_s^{(n_s)}$, where $P_1, \ldots, P_s$ are minimal non-zero prime ideals. If every divisorial ideal is invertible, then every prime ideal of rank one is invertible. If P is an invertible prime ideal, then P^n is P-primary, and hence $P^{(n)} = P^n$. Moreover, $P_1^{(n_1)} \cap \ldots \cap P_s^{(n_s)} = P_1^{n_1} \cap \ldots \cap P_s^{n_s} = P_1^{n_1} \ldots P_s^{n_s}$ is invertible.

It is immediate that (4) and (5) are equivalent and that (4) implies (6).

We next prove that (6) implies (1). Suppose that D is a Krull domain with the property that every product of divisorial ideals is divisorial. Let x be a non-zero non-unit of D. Then $(x) = P_1^{(n_1)} \cap \ldots \cap P_s^{(n_s)}$, where $P_1, \ldots, P_s$ are minimal prime ideals of D and hence are divisorial ideals. But $(x) = P_1^{(n_1)} \cap \ldots \cap P_s^{(n_s)} = (P_1^{n_1} \ldots P_s^{n_s})_v = P_1^{n_1} \ldots P_s^{n_s}$ because $P_1^{n_1} \ldots P_s^{n_s}$ is a product of divisorial ideals and hence is divisorial. Thus (x) is a product of prime ideals.

(4)$\Rightarrow$(8). Let I and J be invertible ideals. Then $I \cap J$ is divisorial and hence invertible.

Since it is obvious that (8) implies (7), it only remains to prove that (7) implies (2). Let M be a maximal ideal of D. Then D_M is a Krull domain in which the intersection of any two principal ideals is principal. Thus D_M is a GCD domain. But D_M, being a Krull domain, satisfies the ascending chain condition on principal ideals. By [**6**, Proposition 16.4], D_M is a UFD.

The equivalence of (2) and (4) in a weaker form appears in Bourbaki [**2**, p. 503], Fossum [**5**, p. 40] and Gilmer [**6**, p. 558]. Condition (6) is considered by Krull [**8**].

We end this section by giving some examples of π-domains. A UFD is a π-domain. For a domain D and set of indeterminates $\{X_\alpha\}$, $D[\{X_\alpha\}]$ is a π-domain if and only if D is a π-domain. Indeed, suppose that D is a π-domain. Then D and $D[\{X_\alpha\}]$ are Krull domains. Let M be a maximal ideal of $D[\{X_\alpha\}]$ and let $P = M \cap D$. Then $D[\{X_\alpha\}]_M = (D_P[\{X_\alpha\}])_{MD_P[\{X_\alpha\}]}$. Now D_P is a UFD and hence $D[\{X_\alpha\}]_M$ is also. Thus $D[\{X_\alpha\}]$ is a Krull domain and is locally a UFD, so by Theorem 1, $D[\{X_\alpha\}]$ is a π-domain. Conversely, if $D[\{X_\alpha\}]$ is a π-domain, it is easily verified that D is a π-domain. In the next section we show that if D is a π-domain and S is a multiplicatively closed set of ideals, then D_S is a π-domain. A Dedekind domain is a π-domain; more generally, a regular domain is locally a UFD, and hence a π-domain. See Claborn [**4**] for some remarks concerning Noetherian π-domains—that is, Noetherian domains that are locally unique factorization domains.

Let G be an abelian group and n an "integer" with $1 \leq n \leq \infty$. Then there exists a π-domain of Krull dimension n with divisor class group G. Claborn [**3**] has shown that there exists a Dedekind domain D with $\mathrm{Cl}(D) \cong G$. If $n = 1$, then D is the desired example. For $1 < n < \infty$, both $D[X_1, \ldots, X_{n-1}]$ and $D[[X_1, \ldots, X_{n-1}]]$ have the required properties. If $n = \infty$, we may take $D[X_1, X_2, \ldots]$ as the desired example.

3. Overrings and generalized quotient rings. Let D be a domain with quotient field K and let S be a multiplicatively closed collection of ideals of D. Then $D_S = \{x \in K \mid xA \subseteq D$ for some $A \in S\}$ is an overring of D which is called the *S-transform* of D or the *generalized quotient ring* of D with respect to S. For an account of generalized quotient rings, the reader is referred to Arnold and Brewer [**1**]. If every ideal of S is invertible, then D_S is called an *invertible generalized quotient* ring of D. It follows from [**1**, Theorem 1.3] that an invertible generalized quotient ring is a flat overring. If D is a Krull domain with defining family $\{D_P\}$, where P ranges over $X = X(D)$, the set of prime ideals of D of rank one, then $D' = \bigcap_{Q \in Y} D_Q$, where $Y \subseteq X$, is called a *subintersection* of D. Any quotient ring of a Krull domain is a subintersection; more generally, any flat overring is a subintersection [**5**, Corollary 6.6]. Fossum has shown that a subintersection need not be a flat overring [**5**, p. 32].

Claborn [**3**] showed that a Krull domain has a torsion divisor class group if and only if every subintersection is a quotient ring. This result also appears in Storch [**11**], where such rings are said to be *almost factorial*. Storch gives several equivalent characterizations of almost factorial domains. We generalize these results in Theorem 4.

Arnold and Brewer [**1**] showed that for a Krull domain D and S a multiplicatively closed collection of ideals, D_S is again a Krull domain. It is implicit in their result that for Krull domains, subintersections and generalized quotient rings are one and the same object. This is stated as our Theorem 2. They further showed that if D is a UFD and S is a multiplicatively closed set of ideals, then D_S is a UFD; in fact, D_S is a quotient overring of D.

THEOREM 2 (Arnold and Brewer [**1**, Theorem 2.2]). *Let $\{D_P\}$ be the defining family for a Krull domain D. If S is a multiplicatively closed set of ideals of D, then $D_S = \cap\{D_P \mid P \not\supseteq A$ for each $A \in S\}$ and so D_S is a subintersection. If $D' = \bigcap_{Q \in Y} D_Q$, where $Y \subseteq X(D)$, is a subintersection, then $D' = D_S$, where S is generated by the elements of $X(D) - Y$ and so every subintersection is a generalized quotient ring.*

Let D be a domain and S a multiplicatively closed set of ideals of D generated by a set $\{A_\alpha\}$ of ideals of D. Let S^* be the multiplicatively closed set generated by the set $\{(A_\alpha^{n_\alpha})_v\}$, where n_α is a positive integer depending upon A_α. It is not difficult to prove that $D_S = D_{S^*}$. Thus every S-transform is what we might call a divisorial generalized quotient ring of D.

THEOREM 3. *Let D be a π-domain and D_1 an overring of D. Then the following statements are equivalent:*

(1) *D_1 is a flat overring,*
(2) *D_1 is a generalized quotient ring of D,*
(3) *D_1 is an invertible generalized quotient ring of D,*
(4) *D_1 is a subintersection.*

If D_1 satisfies any of the above conditions, then D_1 is a π-domain.

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Proof. That (1) implies (2) is true for any ring [**1**, Theorem 1.3].

The equivalence of (2) and (4) follows from Theorem 2.

(2) $\Rightarrow$ (3). Let $D_1 = D_S$, where S is a multiplicatively closed set of ideals. Let S^* be the multiplicatively closed set of ideals generated by A_v for $A \in S$. Since D is a π-domain, each A_v is invertible, and hence D_{S^*} is an invertible generalized quotient ring. But by the remarks preceding Theorem 3 we have $D_S = D_{S^*}$.

The implication (3) $\Rightarrow$ (1) holds for any ring.

It follows from [**9**, Theorem 1.3] that a flat overring of a π-domain is a π-domain.

That every flat overring of a π-domain need not be a quotient ring follows from the existence of Dedekind domains whose class groups are not torsion [**5**, Proposition 6.8].

We generalize Storch's notion of an almost factorial ring. We call a Krull domain D an *almost π-domain* if $\mathrm{Cl}(D)/C(D)$ is torsion. Note that $\mathrm{Cl}(D)/C(D)$ is isomorphic to the group of divisorial ideals modulo the subgroup of invertible ideals. We remark that $\mathrm{Cl}(D)/C(D)$ has functorial properties similar to those of $\mathrm{Cl}(D)$.

The proof of the next theorem is similar to that of [**5**, Proposition 6.8] and will therefore be omitted.

THEOREM 4. *For a Krull domain D, the following statements are equivalent:*

(1) *D is an almost π-domain,*

(2) *every subintersection is an invertible generalized quotient ring,*

(3) *for each prime ideal P of D of rank one, $P^{(n)}$ is invertible for some $n > 0$,*

(4) *some power of every invertible ideal is a product of invertible primary ideals,*

(5) *if I and J are invertible ideals, then there exists an integer $n > 0$ such that $I^n \cap J^n$ is invertible.*

4. Generalized GCD-domains. A domain D is called a *GCD domain* if each pair of nonzero elements of D has a greatest common divisor. This is equivalent to the condition that each pair of nonzero elements of D has a least common multiple or to the condition that the intersection of any two principal ideals is principal. A GCD domain is a UFD if and only if it satisfies the ascending chain condition on principal ideals. An invertible ideal of a GCD domain is principal. For facts about GCD domains see Gilmer [**6**], Kaplansky [**7**], and Sheldon [**10**].

We call a domain D a *generalized GCD-domain* if the intersection of any two invertible ideals of D is invertible. It follows from Theorem 1 that a π-domain is a generalized GCD-domain. In fact, a generalized GCD-domain is a π-domain if and only if it satisfies the ascending chain condition on invertible ideals. This follows from the easily proved fact that a product-irreducible invertible ideal in a generalized GCD-domain is prime. Also a Prüfer domain is a generalized GCD-domain [**6**, 25.4]. Thus generalized GCD-domains provide a nice class of domains containing both Prüfer domains and π-domains.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MISSOURI-COLUMBIA
COLUMBIA, MISSOURI 65201

Present address:
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF IOWA
IOWA CITY, IOWA 52242

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