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A ONE-PAGE PROOF OF A THEOREM OF BELEZNAY

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Abstract. We give a short proof of a theorem of Beleznay asserting that the set $L2$ of reals coding linear orders of the form $I + I$ is complete analytic.

In studying functions of the form $g \circ g$, Humke and Laczkovich [4] proved that $L2$ is not Borel and Beleznay [2] improved this to the result in the abstract. Combined with [4, Lemma 3], this solved a problem of Becker [1, p. 4] and Kechris [5, p. 215]. We give a one-page proof. Let $A \subset \mathbb{R}$ be Σ^1_1. Thus to each $x \in \mathbb{R}$ one can effectively associate a tree T_x on $\mathbb{N}$ such that T_x is well-founded if and only if $x \notin A$. Uniformly in x (see, e.g., [7, Section VI.1.1]), one can find an ill-founded tree K_x recursive in x with no infinite branch in $\Delta^1_1(x)$. Let S_x be the tree of pairs $(l, m) \in T_x \times K_x$ of the same length ordered by $(l, m) <_{S_x} (l', m')$ if $l <_{T_x} l'$ and $m <_{K_x} m'$.

Observe that S_x is ill-founded if T_x is. Conversely, from any branch through S_x we can compute branches through both T_x and K_x. Let $L_x = \omega^{\mathsf{KB}(S_x)}$, where $\mathsf{KB}(S_x)$ denotes the Kleene–Brouwer ordering on S_x and $\omega^{\mathsf{KB}(S_x)}$ is the natural order on formal Cantor normal forms $\omega^{x_1} \cdot m_1 + \omega^{x_2} \cdot m_2 + \cdots + \omega^{x_k} \cdot m_k$, where $x_1 > \cdots > x_k$ are elements of $\mathsf{KB}(S_x)$. Note that if $x \notin A$, then L_x is well-ordered and additively indecomposable, so $L_x \notin L2$.

In contrast, if $x \in A$, then S_x is ill-founded but has no branch which is $\Delta^1_1(x)$, so $\mathsf{KB}(S_x)$ has no $\Delta^1_1(x)$ infinite descending sequence; neither does $L_x = \omega^{\mathsf{KB}(S_x)}$, by a result of Girard and Hirst (see [6, Theorem 1.3]). By a result of Harrison [3], $\mathsf{KB}(S_x)$ and L_x are respectively of the form $\omega_1^x \cdot (1 + \mathbb{Q}) + \alpha$ and $\omega_1^x \cdot (1 + \mathbb{Q}) + \alpha_0$ for some $\alpha, \alpha_0 < \omega_1^x$ (smallest non-x-recursive ordinal). However, $\omega^{\omega_1^x \cdot (1+\mathbb{Q}) + \alpha}$ contains arbitrarily large copies of $\mathbb{Q}$ and thus $\alpha_0 = 0$. Since $\mathbb{Q} \cong \mathbb{Q} + 1 + \mathbb{Q}$, we have $L_x = \omega_1^x \cdot (1 + \mathbb{Q}) \cong \omega_1^x \cdot (1 + \mathbb{Q}) \cdot 2 \in L2$.

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