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PURE SUBGROUPS OF LCA GROUPS

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This paper originated with our interest in the open question "If every pure subgroup of an LCA group G is closed, must G be discrete ?" that was raised by Armacost. The answer was surprisingly easy, but led to some interesting questions. We attempted to characterise those LCA groups that contain a proper pure dense subgroup, and found that every non-discrete torsion-free LCA group contains a proper pure dense subgroup; so does every non-discrete infinite self-dual torsion LCA group. We also give a necessary and sufficient condition for a torsion LCA group to contain a proper pure dense subgroup.

All groups in this paper are assumed to be locally compact Abelian (abbreviated as LCA) Hausdorff groups. Notations and terminologies used in this paper can be found in [1] or [4]. The notation $\simeq$ is used to indicate topological isomorphism between LCA groups, while $\cong$ means algebraic isomorphism between (discrete) groups.

1. CHARACTERISATION OF DISCRETE ABELIAN GROUPS

It is well-known that if every subgroup of an LCA group G is closed, then G must be discrete. In [**1**, 7.24] (see also [**2**]) Armacost asked whether the answer is still true if the condition is replaced by "every pure subgroup of an LCA group is closed". To our surprise, the answer to the question is yes.

THEOREM 1.1. *Let G be an LCA group. Then the following are equivalent:*

(1) *Every subgroup of G is closed;*
(2) *Every pure subgroup of G is closed;*
(3) *G is discrete.*

PROOF: The equivalence of (1) and (3) was stated in [**1**, 1.26]. (3) $\Rightarrow$ (2) is automatic. It remains to prove that (2) $\Rightarrow$ (3).

First of all, by [**1**, 7.18(b)] we know that G must be totally disconnected. Let K be an open and compact subgroup of G. We claim that K must be finite. If K is not finite, let C be a countably infinite subset of K. Let $< C >$ be the subgroup of K that

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is generated by C; then $< C >$ is countable. So by [3, 26.2], there exists a countable pure subgroup H containing $< C >$. By (2), H is closed. Now by [4, 4.26], H must be discrete. So $H \cap K$ must be compact and discrete, hence finite. But $C \subseteq H \cap K$, a contradiction. Therefore K is finite and so G is discrete. ∎

Because of Theorem 1.1, one knows that any non-discrete LCA group contains at least one proper pure subgroup that is not closed. Naturally, one wonders what kind of LCA groups contain a proper pure dense subgroup. This motivates our study in the next section.

2. LCA groups that contain a proper pure dense subgroup

Since any LCA group that is not totally disconnected contains a proper pure dense subgroup, the study of LCA groups that contain a proper pure dense subgroup is the study of totally disconnected ones. In the following we shall show that any non-discrete torsion-free LCA group and any infinite self-dual torsion LCA group contain a proper pure dense subgroup (see Proposition 2.3 and Proposition 2.7). Since it is not always true that a torsion LCA group contains a proper pure dense subgroup (see [1, 2.26]), it is desirable to obtain a necessary and sufficient condition for a torsion LCA group G to contain such a subgroup, which shall be realised by the proof of Theorem 2.5.

Lemma 2.1. *Let G be an LCA group. If G contains a dense subset S with $|S| < |G|$, then G contains a proper pure dense subgroup.*

Proof: By [3, Theorem 26.2], G contains a pure subgroup P containing S with cardinality $|S|$; P is clearly dense and proper. ∎

Lemma 2.2. *Any non-discrete divisible torsion-free LCA group G contains a proper pure dense subgroup.*

Proof: First of all, by using Lemma 2.1 it is very straightforward to show that if G contains Ω_p as a direct summand then G contains a proper pure dense subgroup. Now by the structure theorem [5, Theorem 6.18] we know that

$$G \simeq R^n \times Q^{a*} \times \widehat{Q}^b \times \sum_{p \in \wp} \left(B_p(\mu_p) : \Delta_p^{\mu_p}\right).$$

If $n \neq 0$ or $b \neq 0$, then G is not totally disconnected. So [1, 7.18(b)] shows that G contains a proper pure dense subgroup. Otherwise, we must have that $\wp \neq \emptyset$ since G is not discrete. Therefore, G has some Ω_p as a direct summand. Hence G contains a pure dense proper subgroup. ∎

Proposition 2.3. *Any non-discrete torsion-free LCA group G contains a proper pure dense subgroup.*

PROOF: When G is divisible, Lemma 2.2 implies that G contains a pure dense proper subgroup. When G is not divisible, let P be a proper pure dense subgroup of the minimal divisible extension E of G. We claim that $P \cap G$ is a pure dense proper subgroup of G. Indeed, the fact that G is open in E implies that $P \cap G \neq G$, that is, $P \cap G$ is proper in G. Now for any neighbourhood U of any $x \in G$, the openness of $U \cap G$ in E implies that $(U \cap G) \cap P \neq \emptyset$. Therefore $P \cap G$ is dense in G. It remains to show that $P \cap G$ is pure in G. Assume that $nx \in P \cap G$ for some non-negative integer n and $x \in G$; then $nx \in P$ and $nx \in G$. Since P is pure, there is some $y \in P$ such that $nx = ny$, which implies that $n(x - y) = 0$. But E is torsion-free, so we must have $x - y = 0$, that is, $x = y$. Hence $x \in P \cap G$. This shows that $P \cap G$ is pure in G. □

DEFINITION 2.4: Let H be a subgroup of an abelian group G. H is said to be weak splitting if for any subgroup B of G containing H, $B/H \cong Z(p^n)$ for some prime p and non-negative integer n implies that H is a direct summand of B.

Unlike torsion-free LCA groups, it is not always true that a non-discrete torsion group contains a proper pure dense subgroup. The following theorem characterises non-discrete torsion groups that contain a proper pure dense subgroup.

THEOREM 2.5. *Let G be an LCA group that is a torsion group. Then G contains a proper pure dense subgroup if and only if G contains a proper dense weak splitting subgroup H such that G/H is algebraically isomorphic to one of the groups $Z(p^n)$, or $Z(p^\infty)$.*

PROOF: Assume that G contains a proper pure dense subgroup N. Then there is some prime number p such that $G/N \cong A_p \oplus G_1$, where $A_p \neq \{0\}$ and $(G_1)_p = \{0\}$. Let $P = \pi^{-1}(G_1)$. Since N is pure in G and $P/N \cong G_1$ is pure in G/N (direct summands are pure), by [**3**, Theorem 26.1] we know that P is a pure dense subgroup of G. Now for convenience, denote $A = G/P$. Then A is a torsion p-group. Hence by [**3**, Theorem 32.3], there exists a p-basic group B such that A/B is divisible.

(1) If $B = A$, then A is a direct sum of cyclic p-groups, that is, $A = \sum_{i \in I} Z(p^{n_i})$, where $n_i \geqslant 0$ for all $i \in I$. It is clear that we can write $A \cong Z(p^n) \oplus B_1$. Let $H = \pi^{-1}(B_1)$. Since $P \subseteq H \subseteq G$, P is pure in G and $H/P \cong B_1$ is pure in G/P (every direct summand is a pure subgroup), by [**3**, Theorem 26.1] H is pure in G and $G/H = Z(p^n)$.

(2) If $B \neq A$, then A/B is divisible. Let $P_1 = \pi^{-1}(B)$. Then P_1 is pure in G (since any p-basic subgroup is pure in a p-group, that is, $P_1/P = B$ is pure in G/P) and $G/P_1 \cong A/B \cong \sum_{j \in J} Z^j(p^\infty)$. Therefore, we can write $G/P_1 \cong Z(p^\infty) \oplus B_1$. Let $H = \pi^{-1}(B_1)$. Then H is pure in G and $G/H \cong Z(p^\infty)$.

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The subgroup constructed above is obviously dense in G since it contains the dense subgroup P.

Let B be a subgroup of G containing H and $B/H \cong Z(p^k)$. Since H is pure in G, it is definitely pure in B. Hence by [**3**, Theorem 28.2], H is a direct summand of B. So H is weak splitting in G. Therefore, H is proper dense weak splitting in G.

Conversely, assume that H is a dense weak splitting subgroup of G such that $G/H \cong Z(p^n)$. Then by our assumption, H is a direct summand of G. So H is pure in G (since a direct summand is pure). Now assume that $G/H \cong Z(p^\infty)$. Note that for any $n > 0$, $n^{-1}H/H$ is a torsion subgroup of G/H with bounded order. Hence we must have $n^{-1}H/H \cong Z(p^k)$ for some $k \geqslant 0$. So by our assumption, H is a direct summand of $n^{-1}H/H$. Hence by [**3**, Theorem 28.4] we know that H is pure in G. Therefore, H is a pure dense proper subgroup of G. ∎

Theorem 2.5 would sound more interesting if one could omit the weak splitting assumption of H. Unfortunately, example 1 shows that the fact that G/H is a group of the form $Z(p^\infty)$ doesn't imply that H is dense in G.

EXAMPLE 1. Let G be the minimal divisible extension of $\prod_{i=1}^{\infty} Z^i(p)$. Pick a proper pure subgroup H of G so that $G/H \neq \{0\}$. Then by a process similar to that in the proof of the above theorem, we can actually find a subgroup P of G for which $G/P \cong Z(p^\infty)$. This P must not be dense in G; otherwise P would be a pure dense subgroup of G, which is impossible since it is well-known (see [**1**, 2.26]) that the above group G does not contain any proper dense subgroup.

LEMMA 2.6. *Let G be a non-discrete torsion LCA group. Then any two open and compact subgroups of G have the same cardinality.*

PROOF: Let K and K_1 be open and compact subgroups of G. Then $H = K \cap K_1$ is also an open and compact subgroup of G. Since G is not discrete, we must have $|H| \geqslant \aleph_0$. But K/H and K_1/H are both compact and discrete, hence must be finite. Therefore $|K| = |H| = |K_1|$, and the lemma is proved. ∎

PROPOSITION 2.7. *Let G be a non-discrete torsion group such that G and $\widehat{G}$ contain open and compact subgroups K and K_1, respectively, with the same cardinality; then G contains a proper pure dense subgroup. In particular, any infinite torsion self-dual LCA group contains a pure dense proper subgroup.*

PROOF: Let K and K_1 be open and compact subgroups of G and $\widehat{G}$, respectively, with the same cardinality. Then by [**4**, 25.9], $K = \prod_{i \in I} C_i$, where each C_i is a finite cyclic group and there is a finite upper bound on the set $\{|C_i| : i \in I\}$. By our assumptions that G is not discrete, we must have $|I| \geqslant \aleph_0$. Therefore, $|K| = 2^{|I|}$.

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Since $G/K = \sum_{j \in J} D_j$, where each D_j is a finite cyclic group, $A(\widehat{G}, K) = \prod_{j \in J} D_j$ is open and compact in $\widehat{G}$. If $|J| < \aleph_0$, then G/K is finite, which implies that G is compact. Again by [4, 25.9], we can write $G = \prod_{l \in L} A_l$, where each A_l is cyclic with $|L| \geqslant \aleph_0$. In this case, it easy to see that $\sum_{i \in L} A_l$ is a proper pure dense subgroup of G. If $|J| \geqslant \aleph_0$, then $|G/K| = |J| < \left|A(\widehat{G}, K)\right| = 2^{|J|}$. But by Lemma 2.6 and our assumption, $|K| = |K_1| = \left|A(\widehat{G}, K)\right| = 2^{|J|}$, which implies that $|J| = |I|$. Now K contains a dense subset of cardinality $|I|$ and G/K has cardinality $|J| = |I|$, so by [4, 5.38(f)] we know that G contains a dense subset S with $|S| \leqslant |I|\,|I| < 2^{|I|} \leqslant |G|$. Therefore, by Lemma 2.1 G contains a proper pure dense subgroup. The last statement of the proposition follows automatically from the above proof. $\square$

Now by using the information we have for LCA groups that contain a proper pure dense subgroup we summarise necessary conditions for an LCA group not to contain any proper pure dense subgroup in the following proposition, which also gives a partial answer to the question "Which LCA groups G have no proper pure dense subgroup?" that was asked by Armacost in [1, 7.18(c)].

PROPOSITION 2.8. *Let G be an LCA group that is not discrete. If G does not have any proper pure dense subgroup, then G must be totally disconnected and for any dense subset S of G, $|S| = |G|$; and if G is not torsion, then*

(1) *G does not contain any group Ω_p as a direct summand,*
(2) *G does not contain any group Δ_p as a direct summand,*
(3) *no G_p is torsion-free or no G_p contains any proper pure dense subgroup,*
(4) *G is not torsion-free,*
(5) *the torsion-free subgroup of G is not a direct summand of G, and*
(6) *the torsion subgroup of G is not a direct summand of G;*
if G is torsion, then
(7) *G is not compact,*
(8) *G is not self-dual,*
(9) *for any open and compact subgroup K of G, $|G/K| = |G|$, and*
(10) *for any open and compact subgroup K of G and K_1 of $\widehat{G}$, $|K| \neq |K_1|$.*

PROOF: Assume that G is a non-discrete LCA group that does not contain any proper pure dense subgroup. Then by [1, 7.18(b)], G must be totally disconnected and Lemma 2.1 implies that any dense subset of G has cardinality $|G|$.

Now if G is not torsion, then Proposition 2.3 implies that G satisfies (4). Since if $G = G_1 \times G_2$ and G_1 contains a proper pure dense subgroup P, then $P \times G_2$ will be proper pure dense subgroup of G, and Ω_p and Δ_p both contain a dense subset S with

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cardinality $|S| = \aleph_0 < 2^{\aleph_0}$. It is clear that G also satisfies (1), (2), (3), (5) and (6).

If G is torsion, then Proposition 2.7 shows that G must satisfy (8) and (10). To show that G also satisfies (7) and (9), let us first assume that G is compact. Then by the proof of Proposition 2.7 we can see that G contains a proper pure dense subgroup, a contradiction. Also if G contains an open and compact subgroup K such that $G/K < |G|$, then it is not difficult to see that K contains a proper dense subset with cardinality less than $|K|$ and so by [4, 5.38(f)], G contains a dense subset S with cardinality $|S| < |G|$. But then by Lemma 2.1, G contains a proper pure dense subgroup, a contradiction. Therefore, the proposition is proved. $\square$

References

[1] D.L. Armacost, *The structure of locally compact abelian groups* (Marcel Dekker Inc., New York, Basel, 1981).

[2] W.W. Comfort, 'Recent topological developments on topological groups', *Proceedings of the 1992 Prague Topological Symposium* (to appear).

[3] L. Fuch, *Infinite Abelian groups* (Academic Press, New York and London, 1970).

[4] E. Hewitt and K.A. Ross, *Abstract harmonic analysis I* (Springer-Verlag, Berlin, Heidelberg, New York, 1979).

[5] S.L. Wu, 'Classification of self-dual torsion-free LCA Groups', *Fund. Math.* **140** (1992).

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