CHANGING THE ORDER OF INTEGRATION

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Repeated improper Riemann integrals arise in a variety of contexts, and the validity of changing the order of integration is often in question. Fubini's theorem ensures the equality of two repeated Lebesgue integrals when one of them is absolutely convergent. For many years I have assumed that an analogous test is applicable to repeated improper R-integrals, since they will be absolutely convergent and therefore in agreement with the corresponding L-integrals.

There is a fallacy in this reasoning. The assumed absolute convergence of *one* of the repeated improper R-integrals may not ensure absolute convergence of the other. It certainly ensures absolute convergence and equality of both the corresponding repeated L-integrals; but there may be no connection between these and the *other* repeated improper R-integral if this is only conditionally convergent.

The first purpose of this note is to demonstrate by an example that this possibility must be taken seriously. Example 1 shows that reversal of order in an absolutely convergent repeated improper R-integral can destroy absolute convergence while preserving convergence; and this even for a continuous integrand.

The other purpose is to give theorems justifying change of order in repeated improper R-integrals. In Theorems 1, 2, 3 respectively it is assumed that both, one, neither repeated integral is absolutely convergent. Theorem 2 justifies my former erring ways, after Example 1 has shown that absolute convergence of both repeated integrals cannot be proved from that of one. Theorem 3 reduces the requirements further, after Example 2 has shown that both repeated improper R-integrals may be conditionally convergent although the corresponding repeated L-integrals are absolutely convergent. The test provided by Theorem 3 is even moderately practicable.

DEFINITIONS. It is sufficient to consider real-valued functions defined on the open half-line $H = (0, \infty)$ or on the open quadrant $H \times H$.

The improper R-integral of f on $(0, \infty)$ is defined as

$$\text{(1)} \qquad \int_0^\infty f(x)dx = \sum_{i=1}^{k} \left(\lim_{\alpha_i \to x_{i-1}+0} \int_{\alpha_i}^{\xi_i} f(x)dx + \lim_{\beta_i \to x_i - 0} \int_{\xi_i}^{\beta_i} f(x)dx \right)$$

provided that there is a subdivision (finite set of points)

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(2) $$0 = x_0 < \xi_1 < x_1 < \xi_2 < x_2 < \dots < x_{k-1} < \xi_k < x_k = \infty$$

such that f is R-integrable on each closed sub-interval $[\alpha_i, \beta_i]$ of (x_{i-1}, x_i) $(i = 1, 2, \dots, k)$ and the above limits exist.

Consistency theorem. If the improper R-integral exists, then it is absolutely convergent if and only if f is L-integrable on $(0, \infty)$. If so, the two integrals are equal:

(3) $$\int_{(0,\infty)} f(x)dx = \int_0^\infty f(x)dx.$$

The symbol on the left denotes the L-integral, that on the right denotes the improper R-integral; we use this notation consistently (with $(0, \infty)$ often replaced by H for brevity).

THEOREM 1. *If $f(x, y)$ is measurable (in particular continuous) on the open quadrant $H \times H$, and the repeated improper R-integrals*

(4) $$\int_0^\infty dx \int_0^\infty f(x, y)dy \quad \text{and} \quad \int_0^\infty dy \int_0^\infty f(x, y)dx$$

both exist and are both absolutely convergent, then these integrals are equal.

PROOF. Since the repeated integral on the left exists and is absolutely convergent, its inner integral, $g(x)$ say, exists and is absolutely convergent for all but a finite set of $x \in H$. So for these x

(5) $$g(x) = \int_0^\infty f(x, y)dy = \int_H f(x, y)dy,$$

by the consistency theorem. Similarly with f replaced by $|f|$, so that

(6) $$|g(x)| \leqq \int_0^\infty |f(x, y)|dy = \int_H |f(x, y)|dy.$$

Since g has an improper R-integral there is a subdivision (2) such that g is R-integrable on each closed sub-interval of each (x_{i-1}, x_i). Thus $|g|$ is also R-integrable on these sub-intervals, and in accordance with (1) it has an improper R-integral finite or infinite. This satisfies, using (6),

$$\int_0^\infty |g(x)|dx \leqq \int_0^\infty dx \int_0^\infty |f(x, y)|dy < \infty.$$

Thus g has an absolutely convergent improper R-integral, and so by the consistency theorem g is L-integrable on H and

$$\int_0^\infty g(x)dx = \int_H g(x)dx;$$

whence, using (5),

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$$\int_0^\infty dx \int_0^\infty f(x, y)dy = \int_H dx \int_H f(x, y)dy. \tag{7}$$

Our hypotheses with f replaced by $|f|$ also hold, as an immediate consequence of the hypotheses themselves. Hence (7) holds with f replaced by $|f|$; that is,

$$\int_0^\infty dx \int_0^\infty |f(x, y)|dy = \int_H dx \int_H |f(x, y)|dy. \tag{8}$$

The equations corresponding to (7) and (8) for integrals in the reverse order are also true, by the symmetry in x and y of the hypotheses. So it remains only to prove that the order of integration in the repeated L-integral on the right of (7) can be reversed.

The left side of (8) is finite, by data, hence so is the right. Since $f(x, y)$ is measurable on $H \times H$, Fubini's theorem applies, giving the reversibility of the order of integration on the right of (7), and hence also on the left.

REMARK. It should be noticed that the existence of the repeated L-integral on the right of (7) does not *of itself* ensure its absolute convergence. It only ensures the finiteness of

$$\int_H |f(x, y)|dy \quad \text{(for almost all } x\text{), and of} \quad \int_H dx \left| \int_H f(x, y)dy \right|.$$

This is evident from the example

$$f(x, y) = \frac{\cos xy}{1+y^2} \tag{9}$$

for which

$$\int_H dx \int_H f(x, y)dy = \int_H \tfrac{1}{2}\pi e^{-x} dx = \tfrac{1}{2}\pi;$$

for

$$\int_0^\infty |f(x, y)|dy = \int_0^\infty \frac{|\cos u|}{x^2+u^2}\, x\, du \geqq \int_0^{\pi/3} \tfrac{1}{2}\, \frac{x}{x^2+u^2}\, du = \tfrac{1}{2} \operatorname{artan} \frac{\pi}{3x},$$

and so

$$\int_H dx \int_H |f(x, y)|dy \geqq \int_0^\infty \tfrac{1}{2} \operatorname{artan} \frac{\pi}{3x}\, dx = \infty.$$

EXAMPLE 1. *There is a function $f(x, y)$, continuous in $H \times H$, for which the repeated improper R-integrals* (4) *both exist, one being absolutely convergent but the other not.*

This example does not claim that Theorem 1 is wrong if only one of the integrals is absolutely convergent. It only answers the natural question whether absolute convergence of one of the integrals ensures absolute convergence of the other. Specifically it shows that, for the four repeated improper R-integrals

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$$(10)\quad \int_0^\infty dx \int_0^\infty f(x, y)dy \qquad (12)\quad \int_0^\infty dx \int_0^\infty |f(x, y)|dy$$

$$(11)\quad \int_0^\infty dy \int_0^\infty f(x, y)dx \qquad (13)\quad \int_0^\infty dy \int_0^\infty |f(x, y)|dx,$$

existence of (10), (11) and (12) does not imply existence of (13), even if f and the inner integrals of (10), (11) and (12) are continuous.

Let a_n and b_n be positive constants, b_n all different, such that

$$(14)\qquad \sum_{n=2}^{\infty} a_n = A < \infty, \qquad 0 < b_n < 1;$$

and let

$$(15)\qquad f(x, y) = \sum_{n=2}^{\infty} a_n f_n(x, y), \qquad f_n(x, y) = e^{-|y-b_n|x}\frac{\sin x}{\sqrt{x}},$$

supplemented by $f_n(0, y) = 0$.

The above contentions will be justified if $a_n = 1/n^3$ and b_n are the terminating binary decimals in (0, 1) arranged as in (30). However most of the proof requires only the simpler information (14).

We suppose throughout that x and y are in $\bar{H}$, the closed half-line $[0, \infty)$, except where otherwise stated. Always n is an integer greater than 1.

(a) *Proof that f is continuous in $\bar{H} \times \bar{H}$.* Since $x^{-\frac{1}{2}} \sin x$ tends to 0 as $x \to 0$ and as $x \to \infty$, its maximum modulus occurs at one of its stationary points. At these points $\tan x = 2x$, and so

$$(16)\qquad \left|\frac{\sin x}{\sqrt{x}}\right| = \sqrt{\frac{4x}{1+4x^2}} \leqq 1.$$

Thus $|f_n(x, y)| \leqq 1$. So, by (15), $f(x, y)$ is the sum of a uniformly convergent series of continuous functions in $\bar{H} \times \bar{H}$, and so it is continuous.

(b) *Proof that* (12) *exists.* By (15), for $x \in H$,

$$\int_0^\infty |f_n(x, y)|dy = \frac{|\sin x|}{\sqrt{x}}\left(\int_0^{b_n} e^{x(y-b_n)}dy + \int_{b_n}^\infty e^{-x(y-b_n)}dy\right) = \frac{|\sin x|}{x^{\frac{3}{2}}}(2-e^{-b_n x}).$$

The sign of $f_n(x, y)$ is independent of n and y. So, by a Lebesgue theorem and the consistency theorem,

$$(17)\qquad \begin{aligned} \int_H |f(x, y)|dy &= \sum_{n=2}^{\infty} a_n \int_H |f_n(x, y)|dy \\ &= \sum_{n=2}^{\infty} a_n \int_0^\infty |f_n(x, y)|dy = \frac{|\sin x|}{x^{\frac{3}{2}}} \sum_{n=2}^{\infty} a_n(2-e^{-b_n x}) \end{aligned}$$

showing that the integral on the left is finite. By the consistency theorem again,

$$(18)\qquad \int_0^\infty |f(x, y)|dy = \frac{|\sin x|}{x^{\frac{3}{2}}}\chi(x), \quad \text{where} \quad \chi(x) = \sum_{n=2}^{\infty} a_n(2-e^{-b_n x}).$$

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Now χ is continuous, positive and bounded above (by $2A$). So the inner integral of (12) is continuous in H, and the repeated integral (12) exists.

(c) *Proof that* (10) *exists.* The preceding argument with the modulus signs removed holds as far as (17); the term-by-term integration being valid since (17) (unaltered) shows that f is L-integrable with respect to y on H. Also the consistency theorem again applies, giving

$$\int_0^\infty f(x, y)dy = \frac{\sin x}{x^{\frac{1}{2}}}\chi(x). \tag{19}$$

As for (18), this is continuous in H and the repeated integral (10) exists.

(d) *Proof of the following term by term integration, for* $y \in \bar{H}$:

$$\int_0^\infty f(x, y)dx = \sum_{n=2}^\infty a_n \int_0^\infty f_n(x, y)dx. \tag{20}$$

Suppose $\lambda \geqq 0$ and $\eta \geqq 0$, and let F and C be defined such that

$$F(x) = \int_0^x \frac{\sin t}{\sqrt{t}}dt, \qquad |F(x)| \leqq C. \tag{21}$$

Integrating by parts, and supposing for the moment that $\eta > 0$,

$$\int_\lambda^\infty e^{-\eta x}\frac{\sin x}{\sqrt{x}}dx = -e^{-\eta\lambda}F(\lambda)+\int_\lambda^\infty \eta e^{-\eta x}F(x)dx; \tag{22}$$

this gives

$$\left|\int_\lambda^\infty e^{-\eta x}\frac{\sin x}{\sqrt{x}}dx\right| \leqq 2Ce^{-\eta\lambda}; \tag{23}$$

and this inequality holds when $\eta = 0$ as well as when $\eta > 0$.

Given $y \in \bar{H}$ and $\varepsilon > 0$, choose p such that $b_n \neq y$ for all $n \geqq p$. Thus p may be chosen as 2 except if y happens to be one of the b_n; and the choice is independent of ε. Also choose $q > p$ such that

$$\sum_{n=q}^\infty a_n < \frac{\varepsilon}{4C},$$

and define

$$\delta = \delta(\varepsilon, y) = \min_{n=p}^{q-1} |b_n - y| > 0.$$

Then, using (23) with $\eta = |b_n - y|$,

$$\left|\sum_{n=p}^\infty a_n \int_\lambda^\infty f_n(x, y)dx\right| \leqq 2C\sum_{n=p}^\infty a_n e^{-|b_n-y|\lambda}$$

$$\leqq 2C\sum_{n=p}^{q-1} a_n e^{-\delta\lambda} + 2C\sum_{n=q}^\infty a_n < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon$$

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provided that $\lambda > \delta^{-1} \log (4CA\varepsilon^{-1})$. This proves that

$$\sum_{n=p}^{\infty} a_n \int_0^{\lambda} f_n(x, y)dx \to \sum_{n=p}^{\infty} a_n \int_0^{\infty} f_n(x, y)dx \quad \text{as } \lambda \to \infty. \tag{24}$$

This conclusion also holds with p replaced by 2, by adding a fixed number of terms to each side.

By the uniform convergence proved in (a) we have, for each $\lambda > 0$,

$$\int_0^{\lambda} \sum_{n=2}^{\infty} a_n f_n(x, y)dx = \sum_{n=2}^{\infty} a_n \int_0^{\lambda} f_n(x, y)dx.$$

Using (24) with p replaced by 2, this gives

$$\int_0^{\lambda} f(x, y)dx \to \sum_{n=2}^{\infty} a_n \int_0^{\infty} f_n(x, y)dx \quad \text{as } \lambda \to \infty. \tag{25}$$

Thus the left side of (20) exists and is equal to the right side.

(e) *Proof that all the integrals in* (20) *are continuous in* $\bar{H}$. The integral on the left of (22), with $\lambda = 0$, is a Laplace transform; and its continuity as a function of η in $\eta \geqq 0$ is analogous to the conclusion of Abel's continuity theorem for power series. We can prove this by a minor modification to (22) and (23). Writing

$$\Phi(x) = F(\infty) - F(x) = \int_x^{\infty} \frac{\sin t}{\sqrt{t}}\, dt$$

(compare (21)), there is $\Delta(\varepsilon) > 0$ such that

$$|\Phi(x)| < \varepsilon \quad \text{whenever} \quad \varepsilon > 0 \quad \text{and} \quad x > \Delta(\varepsilon).$$

Putting (22) in terms of Φ,

$$\left| \int_{\lambda}^{\infty} e^{-\eta x} \frac{\sin x}{\sqrt{x}}\, dx \right| = \left| e^{-\eta\lambda}\Phi(\lambda) - \int_{\lambda}^{\infty} \eta e^{-\eta x}\Phi(x)dx \right| < e^{-\eta\lambda} . \tfrac{1}{2}\varepsilon + e^{-\eta\lambda} . \tfrac{1}{2}\varepsilon \leqq \varepsilon$$

whenever $\varepsilon > 0$, $\lambda > \Delta(\frac{1}{2}\varepsilon)$ and $\eta \geqq 0$.

Thus the integral

$$\int_0^{\infty} e^{-\eta x} \frac{\sin x}{\sqrt{x}}\, dx$$

is uniformly convergent on $\eta \geqq 0$. So it represents a continuous function of η in $\eta \geqq 0$, since its integrand has the requisite continuity.

From (15) it now follows that the integrals on the right of (20) are continuous functions of y in $\bar{H}$. They are also uniformly bounded, by (23) with $\lambda = 0$. So the series in (20) is a uniformly convergent series of continuous functions, whence the left side of (20) is also continuous.

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(f) *Proof that* (11) *exists.* Writing $|y-b_n| = \eta_n$ and

$$g_n(y) = \int_0^\infty f_n(x, y)dx = \int_0^\infty e^{-\eta_n x} \frac{\sin x}{\sqrt{x}} dx, \tag{26}$$

$$|g_n(y)| \leqq \int_0^\infty e^{-\eta_n x} \frac{|\sin x|}{\sqrt{x}} dx \leqq \int_0^\infty e^{-\eta_n x} x^{\frac{1}{2}} dx = \frac{\Gamma(\frac{3}{2})}{\eta_n^{\frac{3}{2}}} < \frac{1}{(y-1)^{\frac{3}{2}}} \tag{27}$$

for $y > 1$ and all n; here we have used (14). Also by (23) with $\lambda = 0$, we have $|g_n(y)| \leqq 2C$ for all y and all n.

By these two inequalities, there is $\omega \in L(0, \infty)$ such that

$$|g_n(y)| \leqq \omega(y) \quad \text{for all } y \text{ and all } n.$$

Further

$$|\sum_{n=2}^{\infty} a_n g_n(y)| \leqq \sum_{n=2}^{\infty} a_n |g_n(y)| \leqq A\omega(y) \qquad \text{for all } y, \tag{28}$$

establishing the L-integrability on H of the function on the left. So, by a corollary of the dominated convergence theorem,

$$\int_{(0,\lambda)} \sum_{n=2}^{\infty} a_n g_n(y)dy \to \int_H \sum_{n=2}^{\infty} a_n g_n(y)dy \qquad \text{as } \lambda \to \infty.$$

Also (28) ensures absolute convergence of the expression on the right, so by another corollary of the dominated convergence theorem, this may be written

$$\int_{(0,\lambda)} \sum_{n=2}^{\infty} a_n g_n(y)dy \to \sum_{n=2}^{\infty} a_n \int_H g_n(y)dy \qquad \text{as } \lambda \to \infty,$$

the existence of the right side being assured by that corollary. The integrands on both sides are continuous by (e), so the integrals can be written as R-integrals, proper on the left and improper on the right. Observing (20) and (26), this gives

$$\int_0^\lambda dy \int_0^\infty f(x, y)dx \to \sum_{n=2}^{\infty} a_n \int_0^\infty dy \int_0^\infty f_n(x, y)dx \qquad \text{as } \lambda \to \infty. \tag{29}$$

This proves the existence of the repeated improper R-integral (11). We already know from (e) that its inner integral is continuous.

(g) *Proof that* (13) *does not exist* when $a_n = 1/n^3$ and b_n are the terminating binary decimals in (0, 1), as follows:

$$\begin{array}{cccc} & b_2 = \cdot 1, & & \\ & b_3 = \cdot 01, & b_4 = \cdot 11, & \\ b_5 = \cdot 001, & b_6 = \cdot 011, & b_7 = \cdot 101, & b_8 = \cdot 111, \end{array} \tag{30}$$

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and so on. We prove that the inner integral of (13) is divergent for $y = b_n$, for all n.

Let $y = b_k$, where b_k has $h+1$ binary digits; thus $2^h < k \leqq 2^{h+1}$. By the two middle steps of (27),

$$\sum_{n=k+1}^{\infty} a_n \int_0^{\infty} |f_n(x, y)| dx < \sum_{\substack{n=2^h+1 \\ n \neq k}}^{\infty} \frac{a_n}{|y-b_n|^{\frac{3}{2}}}$$

$$< \sum_{n=2^h+1}^{2^{h+1}} \frac{a_n}{(2/2^{h+1})^{\frac{3}{2}}} + \sum_{n=2^{h+1}+1}^{2^{h+2}} \frac{a_n}{(1/2^{h+2})^{\frac{3}{2}}} + \sum_{n=2^{h+2}+1}^{2^{h+3}} \frac{a_n}{(1/2^{h+3})^{\frac{3}{2}}} + \cdots$$

(31) $$< \frac{2^{\frac{3}{2}(h+1)}}{2(2^h)^2} + \frac{2^{\frac{3}{2}(h+2)}}{2(2^{h+1})^2} + \frac{2^{\frac{3}{2}(h+3)}}{2(2^{h+2})^2} + \cdots$$

using the inequality

$$\sum_{n=m+1}^{2m} a_n < \sum_{n=m+1}^{\infty} \frac{1}{n^3} < \int_m^{\infty} \frac{1}{x^3} dx = \frac{1}{2m^2}.$$

The geometric series (31) is convergent, with sum $2(\sqrt{2}+1)/2^{\frac{1}{2}h}$.

For any $\lambda > 0$, using (a) and (15),

$$\int_0^{\lambda} |f(x, y)| dx \geqq \int_0^{\lambda} |a_k f_k(x, y)| dx - \int_0^{\lambda} |a_k f_k(x, y) - f(x, y)| dx$$

$$\geqq a_k \int_0^{\lambda} |f_k(x, y)| dx - \int_0^{\lambda} \sum_{n \neq k} a_n |f_n(x, y)| dx$$

$$= \frac{1}{k^3} \int_0^{\lambda} \left| \frac{\sin x}{\sqrt{x}} \right| dx - \sum_{n \neq k} a_n \int_0^{\lambda} |f_n(x, y)| dx$$

(32) $$\geqq \frac{1}{k^3} \int_0^{\lambda} \left| \frac{\sin x}{\sqrt{x}} \right| dx - \sum_{n \neq k} a_n \int_0^{\infty} |f_n(x, y)| dx.$$

The integrals in the summation in (32) are all convergent when $y = b_k$, since $b_n \neq b_k$ whenever $n \neq k$. So the series is convergent, as proved above using (31). Also its sum is independent of λ. So the whole expression (32) tends to infinity with λ. Thus the improper R-integral

$$\int_0^{\infty} |f(x, y)| dx$$

is divergent for $y = b_k$, where b_k is any of the terminating binary decimals in (0, 1). That is, it is divergent in a dense set of y-values in (0, 1). Consequently it has no improper R-integral, and (13) does not exist.

THEOREM 2. *If $f(x, y)$ is measurable on the open quadrant $H \times H$, and the repeated improper R-integrals*

(33) $$\int_0^{\infty} dx \int_0^{\infty} f(x, y) dy \quad \text{and} \quad \int_0^{\infty} dy \int_0^{\infty} f(x, y) dx$$

both exist and one is absolutely convergent, then they are equal.

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Thus the requirement in Theorem 1 that both integrals be absolutely convergent can be reduced. Example 1 shows that this reduction is significant, and Theorem 2 shows that the integrals in Example 1 are equal.

PROOF. Suppose the absolutely convergent integral is that on the left. The proof of Theorem 1 omitting the last two paragraphs still applies, establishing (7) and (8).

The left side of (8) is finite, by the absolute convergence hypothesis; hence so is the right side. This, with the measurability of $f(x, y)$ on $H \times H$, gives the finiteness and equality of

$$\int_H dx \int_H |f(x, y)| dy = \int_H dy \int_H |f(x, y)| dx < \infty \tag{34}$$

by Fubini's theorem; and it follows that

$$\psi \in L(0, \infty), \qquad \text{where} \qquad \psi(y) = \int_H |f(x, y)| dx. \tag{35}$$

Since the expression on the right in (33) exists, its inner integral exists for all but a finite set of y in H. Also by (35) the corresponding L-integral exists for almost all y in H. So, by the consistency theorem, the improper R-integral is absolutely convergent for almost all y in H, and the integrals are equal:

$$\int_0^\infty f(x, y) dx = \int_H f(x, y) dx \qquad \text{for almost all } y \in H. \tag{36}$$

Again since the expression on the right in (33) exists, its outer integral exists and there is a subdivision (2) such that it is equal to

$$\lim_{n \to \infty} \sum_{i=1}^{k} \left(\int_{x_{i-1}+1/n}^{\xi_i} + \int_{\xi_i}^{x_i - 1/n} \right) dy \int_0^\infty f(x, y) dx;$$

here the meaningless upper terminal $x_k - 1/n$ is to be read as $\xi_k + n$. These several outer integrals are proper R-integrals, and so equal to the corresponding L-integrals. Abbreviating the union of their intervals of integration thus

$$I_n = \bigcup_{i=1}^{k} \{(x_{i-1}+1/n, \xi_i) \cup (\xi_i, x_i - 1/n)\},$$

and using an L-integral on I_n, the integral on the right in (33) is

$$\begin{aligned} \int_0^\infty dy \int_0^\infty f(x, y) dx &= \lim_{n \to \infty} \int_{I_n} dy \int_0^\infty f(x, y) dx \\ &= \lim_{n \to \infty} \int_{I_n} dy \int_H f(x, y) dx \qquad \text{using (36),} \\ &= \int_H dy \int_H f(x, y) dx \qquad (37) \end{aligned}$$

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by a corollary of the dominated convergence theorem, since the inner integral is L-integrable by (34).

Finally, the right sides of (7) and (37) are equal by Fubini's theorem and (34), hence the left sides are equal as required.

EXAMPLE 2. *There is a function* $f(x, y)$, *continuous in* $H \times H$, *for which the repeated improper R-integrals* (33) *both exist but neither is absolutely convergent, and yet the corresponding repeated L-integrals are absolutely convergent and equal.*

Let $g(x, y)$ be the function discussed in Example 1, defined in (15) and (30) and labelled $f(x, y)$; and let $f(x, y)$ now be the continuous function

$$f(x, y) = g(x, y)+2g(y, x).$$

The existence of (10) and (11) and the linearity of the integrals ensure the existence of both integrals (33).

Now

$$|f(x, y)| \geqq |g(x, y)|-2|g(y, x)|,$$

$$\int_0^\infty |g(x, y)|dx \qquad \text{is divergent at a dense set of } y \text{ in } (0, 1),$$

$$\int_0^\infty |g(y, x)|dx \qquad \text{is convergent for all } y \text{ by (18);}$$

from these we infer that

$$\int_0^\infty |f(x, y)|dx \qquad \text{is divergent at a dense set of } y \text{ in } (0, 1).$$

Thus the integral on the right in (33) is not absolutely convergent. Neither is that on the left, by a similar argument.

Since g fulfils the hypotheses of Theorem 2, (34) gives

$$\int_H dx \int_H |f(x, y)|dy \leqq \int_H dx \int_H |g(x, y)|dy+2\int_H dx \int_H |g(y, x)|dy$$
$$= 3\int_H dx \int_H |g(x, y)|dy < \infty;$$

so the two repeated L-integrals of f are absolutely convergent and therefore equal.

THEOREM 3. *If* $f(x, y)$ *is measurable on the open quadrant* $H \times H$, *the repeated improper R-integrals* (33) *both exist, and there is* $\phi \in L(0, \infty)$ *such that*

$$\text{(38)} \qquad \int_0^\infty |f(x, y)|dy \leqq \phi(x) \qquad \text{for almost all } x \in H,$$

then the integrals (33) *are equal.*

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Although it looks like a re-statement of Fubini's theorem, this theorem is significant in that *neither* of the integrals (33) need be absolutely convergent. It would prove the equality of the integrals of Example 2, via (18), if this were not already a consequence of the application of Theorem 2 to Example 1.

PROOF. By (38) and the consistency theorem, $f(x, y)$ is L-integrable on H with respect to y for almost all $x \in H$, and

$$\int_0^\infty f(x, y)dy = \int_H f(x, y)dy \qquad \text{for almost all } x \in H. \tag{39}$$

The same equation holds with f replaced by $|f|$, and this with (38) shows that the integral on the left of (34) is finite; whence (34) holds.

It follows from (34) that $f(x, y)$ is L-integrable on H with respect to x for almost all $y \in H$. Then by the consistency theorem

$$\int_0^\infty f(x, y)dx = \int_H f(x, y)dx \qquad \text{for almost all } y \in H. \tag{40}$$

The long paragraph in the proof of Theorem 2 now applies, the role of (36) being supplied by (40); the result is (37). The same paragraph applies again with repeated integrals in the reverse order, the role of (36) being supplied by (39); the result is

$$\int_0^\infty dx \int_0^\infty f(x, y)dy = \int_H dx \int_H f(x, y)dy.$$

Since the right sides of this equation and (37) are equal, by (34) and Fubini's theorem, the left sides are equal; this proves the theorem.

Acknowledgment

Mr. J. J. Koliha has pointed out to me that the theorems in this paper are related to a Fubini-type theorem on Perron-Stieltjes integrals proved by Mařík [1, p. 125-7]. This theorem states the equality of repeated Perron-Stieltjes integrals on compact intervals X and Y (the inner integrals being upper or lower) when the corresponding dimetric Perron-Stieltjes integral on $X \times Y$ exists. From this we can deduce the cases of our theorems in which the half-line H is replaced by a compact interval I, as follows.

The foregoing proofs show that in each of Theorems 1, 2 and 3 $f(x, y)$ is L-integrable on $H \times H$, and hence on $I \times I$. Consequently it is P-integrable on $I \times I$ and the above theorem is applicable. Now improper R-integration on I is included in P-integration [1, p. 132-3]. Thus our repeated improper R-integrals are repeated P-integrals, and so are equal by the above theorem.

Another related result is a Fubini-type theorem formally similar to Mařík's, given for variational integrals by Henstock [2, p. 109].

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However no references have been found to consider the primary question discussed in this paper, namely whether absolute convergence of a repeated improper R-integral implies absolute convergence of the reversed repeated integral. The stimulus for this came in part from the practical importance of improper R-integrals, since these are so widely used. But in fact a truncated form of Example 1 also answers the corresponding question about absolute convergence of repeated Perron integrals.

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