CORRESPONDENCE.

ON THE VALUE OF OPTIONS IN CERTAIN CONTRACTS.

To the Editor.

SIR,—I propose, in the present letter, to give the mathematical solution of the question which Mr. Stephenson ennunciated in the April Number of the *Magazine* (p. 178), namely, to find the single premium P_x which a person aged x should pay to secure a deferred annuity of £1 to be entered upon at the age $x+n$ and to be continued for the remainder of life, on the understanding that P_x is to be returned at death, if that should happen during the n years, and that the individual is to have the option during the same period of withdrawing the sum P_x at any moment, and so putting an end to the contract.

Before giving the method of determining P_x, it may be as well to point out the reason why the formula which Mr. Stephenson has obtained is not the answer to the question proposed. I found it difficult, at first, to understand how that gentleman could have been led to believe that he *had* solved the question, seeing that throughout his solution the contingency of withdrawal is nowhere taken into consideration—but his mistake clearly lies in his imagining that because the *interest only* of P_x is used during the n years, therefore P_x is not *sunk* during that period, and consequently that it must be free to be withdrawn at any moment during the n years, or to be returned at death if that should happen in the same interval. An examination of his solution, however, will easily show that such an idea, as far as withdrawal is concerned, is quite erroneous. He says, if $p_{x\overline{n|}}$ be the annual premium payable at the end of the year for assuring to x a deferred annuity of £1 after n years, then $P_x\cdot\dfrac{i}{p_{x\overline{n|}}}$ will be the amount of the deferred annuity which can be assured by the conversion of the annual interest into an annual premium; but when he afterwards writes $\dfrac{a_{x\overline{n|}}}{a_x-a_{x\overline{n|}}}$ for $p_{x\overline{n|}}$, the assumption is involved that nothing but *death* will cause the payment of $p_{x\overline{n|}}$ to cease, and consequently $P_x\cdot\dfrac{i}{p_{x\overline{n|}}}$ can only represent the amount of the deferred annuity upon *that supposition.* This proves most conclusively that the contingency of withdrawal is not included in his solution. The reason why his formula gives a somewhat larger value for P_x than the ordinary expression, which provides merely for a return of the premium at the end of the year of death, is quite obvious, for his method of solution supposes the interest of P_x to be invested only at the *end* of each year, and therefore it leaves unappropriated the fractional part of the year's interest to the day of death, if death should happen at any time between the beginning and the end of one of the n years. His formula, therefore, is sufficient to provide for a return of P_x at death + the fractional part of the current year's interest to the day of death, and therefore if these sums were left with the Office till the end of the year of death the amount returnable would be $P_x(1+i)$, as Mr. Makeham has shown by a different mode of reasoning.

The mathematical solution of the question, which I will now give, is not without interest, and I will take the problem precisely as it is proposed, the premium being returnable at the moment of death, or withdrawable at any time in the n years, on demand.

Suppose it has been found from actual experience that in a large number, β', of such policies, the withdrawals in the first, second, third, &c., years have been w', w'', w''', &c., respectively, then, for any other number, l_x, of cases, the withdrawals would be $\frac{l_x}{\beta'}w'$, $\frac{l_x}{\beta'}w''$, &c., which we will denote by w_1, w_2, &c., and for the moment suppose £1 to be the single premium. Imagine l_x policies to be effected on the same day by persons aged x. Let δ_1, δ_2, δ_3, &c., be the tabular number of deaths in the first, second, &c., years of the existence of the policies, or the number of deaths that would happen if there were no withdrawals, and let λ_{n-1} denote the number of policies in existence at the end of $n-1$ years, that is, after deaths *and* withdrawals have taken place. Suppose w_n, the withdrawals in the nth year, to be distributed equally over the year, and the deaths δ_n to be so likewise; then, if t be any portion of the nth year, reckoned from its commencement, the deaths at the instant between t and $t+dt$ would, if there were no withdrawals, be $\frac{\delta_n}{l_{x+n-1}}\lambda_{n-1}.dt$, and the deaths at the same moment among the w_nt persons who have withdrawn since the beginning of the year would be $\frac{\delta_n}{l_{x+n-1}}w_nt.dt$; hence, subtracting the latter from the former, and integrating, we get

$$\frac{\delta_n}{l_{x+n-1}}\left(\lambda_{n-1}\int_0^1 dt-w_n\int_0^1 tdt\right)=\frac{\delta_n}{l_{x+n-1}}(\lambda_{n-1}-\tfrac{1}{2}w_n)$$

for the number of deaths during the nth year, in respect of each of which the Office would have to return the premium.

We thus get

$$\lambda_n=\lambda_{n-1}-\frac{\delta_n}{l_{x+n-1}}(\lambda_{n-1}-\tfrac{1}{2}w_n)-w_n \quad . \quad . \quad . \quad . \quad (1)$$

and by making n successively equal to 1, 2, 3, &c., the values of λ_1, λ_2, λ_3, &c., may be easily calculated.

Let t, as before, denote any portion of the nth year, and v^t the present value of £1, due at the end of the time t; then, at the commencement of the nth year the value of the risk of death or withdrawal taking place at the instant between t and $t+dt$ will be

$$\frac{\delta_n}{\lambda_{n-1}.l_{x+n-1}}(\lambda_{n-1}-w_nt)v^t.dt+\frac{w_n}{\lambda_{n-1}}v^t.dt.$$

If we integrate for the whole year, and multiply the result by $\frac{\lambda_{n-1}}{l_x}v^{n-1}$, we shall have for the value, at the time of the issue of the policy, of the risk of an individual death or withdrawal during the nth year, the expression

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$$\frac{\delta_n v^{n-1}}{l_x . l_{x+n-1}}\left(\lambda_{n-1}\int_0^1 v^t.dt - w_n\int_0^1 tv^t.dt\right) + \frac{w_n v^{n-1}}{l_x}\int_0^1 v^t.dt$$
$$= \frac{\delta_n v^{n-1}}{l_x . l_{x+n-1}}\left\{\frac{v-1}{\log v}\lambda_{n-1} + \frac{1}{\log v}\left(\frac{v-1}{\log v} - v\right)w_n\right\} + \frac{v-1}{\log v}\cdot\frac{w_n v^{n-1}}{l_x}.*$$

Hence, summing with respect to n (from 1 to n) we get the whole value of the risk of death or withdrawal during the n years,

$$= \frac{v-1}{l_x . \log v}\left(\Sigma\frac{\delta_n\lambda_{n-1}v^{n-1}}{l_{x+n-1}} + \Sigma w_n v^{n-1}\right) + \frac{1}{l_x . \log v}\left(\frac{v-1}{\log v} - v\right)\Sigma\frac{\delta_n w_n v^{n-1}}{l_{x+n-1}} \quad ..(2).$$

From (1) we readily find that

$$\Sigma\frac{\delta_n\lambda_{n-1}v^{n-1}}{l_{x+n-1}} + \Sigma w_n v^{n-1} = \tfrac{1}{2}\Sigma\frac{\delta_n w_n v^{n-1}}{l_{x+n-1}} + \Sigma\lambda_{n-1}v^{n-1} - \frac{1}{v}\Sigma\lambda_n v^n \quad ... (3);$$

hence (2) becomes, by substitution and reduction,

$$\frac{1}{l_x . \log v}\left\{\frac{v-1}{\log v} - \tfrac{1}{2}(v+1)\right\}\Sigma\frac{\delta_n w_n v^{n-1}}{l_{x+n-1}} + \frac{v-1}{l_x . \log v}\left(\Sigma\lambda_{n-1}v^{n-1} - \frac{1}{v}\Sigma\lambda_n v^n\right) \quad ..(4),$$

a formula which involves only two different summations.

This, multiplied by P_x, gives a complete and perfectly general value of the risk of death and withdrawal.

If we suppose the probability of withdrawal in any year to be always the same at the commencement of the year and $=k$, then $\frac{w_n}{\lambda_{n-1}} = k$, or $w_n = k\lambda_{n-1}$; and substituting in (3), we get

$$(1-\tfrac{1}{2}k)\Sigma\frac{\delta_n\lambda_{n-1}v^{n-1}}{l_{x+n-1}} = (1-k)\Sigma\lambda_{n-1}v^{n-1} - \frac{1}{v}\Sigma\lambda_n v^n \quad ... (5);$$

also the quantity $\Sigma\frac{\delta_n w_n v^{n-1}}{l_{x+n-1}}$, in (4), becomes $k\Sigma\frac{\delta_n\lambda_{n-1}v^{n-1}}{l_{x+n-1}}$; which, by (5), is

$$= \frac{k(1-k)}{1-\frac{1}{2}k}\Sigma\lambda_{n-1}v^{n-1} - \frac{k}{v(1-\frac{1}{2}k)}\Sigma\lambda_n v^n;$$

and this substituted in (4), gives, if we put β for $\frac{1}{l_x . \log v}\left(\frac{v-1}{\log v} - \frac{1}{2}(v+1)\right)$,

$$\left(\frac{\beta k(1-k)}{1-\frac{1}{2}k} + \frac{v-1}{l_x . \log v}\right)\Sigma\lambda_{n-1}v^{n-1} - \left(\frac{\beta k}{v(1-\frac{1}{2}k)} + \frac{v-1}{v . l_x . \log v}\right)\Sigma\lambda_n v^n \quad .. (6)$$

for the value of the risk of death or withdrawal on the particular supposition named. Here, only the one summation, $\Sigma\lambda_n v^n$, is required to be made. When this formula is used, the values of λ_1, λ_2, λ_3, &c., are computed from the simple relation

$$\lambda_n = \lambda_{n-1}\{(1-\tfrac{1}{2}k)p_{x+n-1(1)} - \tfrac{1}{2}k\}.$$

If we find Q to be the arithmetical value of either (4) or (6), then we must have

$$Q.P_x + \frac{N_{x+n}}{D_x} = P_x, \text{ or } P_x = \frac{N_{x+n}}{(1-Q)D_x} \quad .... (7).$$

* The general integrals are $\int v^t.dt = \frac{v^t}{\log v}$, and $\int tv^t dt = \frac{tv^t}{\log v} - \frac{1}{\log v}\int v^t dt = \frac{v^t}{\log v}\left\{t - \frac{1}{\log v}\right\}$.

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By way of illustration, let 50 be the age at entry, and 60 the age at which the annuity (£1) is to be entered upon, and suppose it has been found that the premiums on one-twentieth of the policies in force at the beginning of any year are withdrawn during that year; then, using the Carlisle table of mortality, and 4 per cent. interest, we shall find the value of the expression (6) to be ·42556$=$Q; hence, from (7) we get $P_x=9{\cdot}4157$.

In the same case, if the premium were simply returnable at the end of the year of death, and no option of withdrawal were allowed, the single premium, by the ordinary formula, would be

$$P_x=\frac{N_{x+n}}{D_x-M_x+M_{x+n}}=6{\cdot}2631.$$

These numerical results show that a very moderate supposition as to the probability of withdrawal increases the premium more than 50 per cent.

It will be observed, in all that precedes, that $\log v$ denotes the *hyperbolic* logarithm of v.

I am, Sir,
Your obedient servant,
SAMUEL YOUNGER.

316, *Regent Street, London,*
30*th November*, 1865.

THE LATE MR. FINLAISON'S TABLES.

To the Editor.

SIR,—Reference having been made at the last meeting of the Institute to the discrepancy existing between the probabilities of living as computed by the late Mr. Finlaison, and those that would result from the methods of graduation he professes to adopt, I trouble you with a few details of these differences, with a view of promoting some inquiry into a matter which, having regard to the importance of the tables, and as bearing upon the points recently under discussion, may prove of interest to members.

The well-known tables of Mr. Finlaison are embodied in his Report, printed by order of the House of Commons, 31st March, 1829; and are founded on twenty-one "Observations" exhibiting the rate of mortality experienced under the Government schemes of tontines and other annuities, with an annuity table at 4 per cent. deduced from each. The logarithms of the probability of living one year at each age are tabulated, first as deduced from the data, and secondly as adjusted by Mr. Finlaison.

His formulæ for adjustment, written in Milne's notation, are embodied in the Report, and are as follows:—

$$\text{I.}\quad {}_1\overset{=}{a}=\sqrt[3]{\left\{\begin{array}{l}\sqrt[5]{{}_1a_3\times{}_1a_2\times{}_1a_1\times{}_1a\times{}^1_1a\}}\times\\ \sqrt[5]{{}_1a_2\times{}_1a_1\times{}_1a\times{}^1_1a\times{}^2_1a\}}\times\\ \sqrt[5]{{}_1a_1\times{}_1a\times{}^1_1a\times{}^2_1a\times{}^3_1a\}}\end{array}\right\}}.$$

$$\text{II.}\quad {}_1\overset{=}{a}=\frac{(5\,{}_1a+4\,{}_1a_1+4\,{}^1_1a+3\,{}_1a_1+3\,{}^2_1a+2\,{}_1a_3+2\,{}^3_1a+{}_1a_4+{}^4_1a)}{25}.$$

Mr. Finlaison informs us that nineteen of the observations were "all and each" of them adjusted by the first method above; and that two observations only, the earliest for each sex, which were completed in January, 1823, were adjusted by the second method, which he designates as "perhaps quite as good, but more laborious" than the first; and that by