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UNIVERSAL SPACES FOR CLOSED IMAGES OF σ-DISCRETE METRIC SPACES

Dedicated to Dean A. Okuyama on the occasion of his 60th birthday

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ABSTRACT. We present a proof of a theorem announced by van Douwen concerning existences of universal spaces for certain closed images of σ-discrete metric spaces.

0. **Introduction and definitions.** All spaces in this paper are assumed to be Tychonoff, and all maps are assumed to be continuous. See [3, 4] for undefined terminologies.

For every infinite cardinals κ and λ with $\omega_1 \leq \lambda \leq \kappa^+$ let $C_{\kappa,\lambda}$ be the class of all spaces Y for which there are σ-discrete metric space M of $w(M) \leq \kappa$ and a closed onto map $f: M \longrightarrow Y$ such that $\forall y \in Y[|f^{\leftarrow}(y)| < \lambda]$.

It is the late E. K. van Douwen who communicated in [1] the following theorem (see also [11, 13, 16]).

THEOREM 0. *There is a universal space for every class $C_{\kappa,\lambda}$. Moreover, every space in $C_{\kappa,\lambda}$ embeds as a closed subspace in it.*

The special case when $\kappa = \omega$ (and hence, $\lambda = \omega_1$) in the above theorem answers the following question due to H. Junnila cited in [10, Problem] affirmatively.

QUESTION 0. *Does there exist a universal space for all closed images of countable metrizable spaces?*

The purpose of this paper is to give a proof of the theorem, since it is known that he left no proof, but only its statement in [2] (see [13] for the details). We shall use a method (we shall call it *Watson method* as in [13]), which is known to be effective not only for solving Question 0 [16] but also for showing the existence of a universal space for all strongly 0-dimensional closed images of the *Baire's 0-dimensional space $B(\kappa) = {}^{\omega}\kappa$* of a given weight $\kappa \geq \omega$ (*i.e.* the countable Cartesian product of a discrete space of size κ) [12].

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1. **Preliminaries.** Let us fix a point $* = (0, 0, \ldots) \in B(\kappa)$ and let Q_κ be the subspace of $B(\kappa)$ consisting of all points having all but finitely many coordinates equal to those of $*$. Then by its definition Q_κ is σ-discrete, and $|Q_\kappa| = \kappa$, in particular, Q_ω is homeomorphic to the space of rational numbers Q, and it holds that:

(0) for every infinite $\mu \leq \kappa$ there exists a closed nowhere dense subset $F_\mu = Q_\kappa \cap ({}^\omega I_\mu)$ of Q_κ, where $I_\mu \subset \kappa$ satisfying that $|I_\mu| = \mu$ and $|\kappa \setminus I_\mu| = \kappa$ (hence F_μ is homeomorphic to Q_μ and $|F_\mu| = w(F_\mu) = \mu$).

Using the space Q_κ, we have the following result:

FACT 0 [6, COROLLARY 5]. Every σ-discrete metric space of weight at most κ can be embedded in Q_κ as a closed nowhere dense subset.

On the other hand there is the following characterization theorem (a space X is called *homogeneous with respect to weight* if the weight of any non-empty open subset is equal to the weight of X).

FACT 1 [6, THEOREM 1]. Let X be a σ-discrete metric space of weight κ that is homogeneous with respect to weight. Then X is homeomorphic to Q_κ.

We call a space *strongly homogeneous* (s. h.) when all its non-empty *clopen* (*i.e.* simultaneously closed and open) subspaces are homeomorphic. For example by Fact 1 the space Q_κ is s. h. for any $\kappa \geq \omega$.

REMARK 0. It is easy to see that every first countable s. h. space is homogeneous. In particular, every s. h. metric space is homogeneous (see [9] for more generalizations).

For s. h. metric spaces we need the following homeomorphism extension theorem.

FACT 2 [12, THEOREM 2] AND [14, COROLLARY 1]. Let G and H be closed nowhere dense subsets of a s. h. strongly 0-dimensional metric space X. If $h: G \longrightarrow H$ is a homeomorphism then h can be extended to an autohomeomorphism $\tilde{h}$ of X.

Next we need the following result due to D. M. Hyman [5]. A pair (X, A) for a closed subset A of a space X is called *semicanonical* (s. c. for short) if there exists a collection $\mathcal{U}$ (we call it a *semicanonical cover* for the pair) consisting of open subsets of X satisfying that

(1) $\bigcup \mathcal{U} = X \setminus A$, and

(2) for each $a \in A$ and each neighborhood V of a in X there exists a neighborhood W of a in X such that $\mathrm{st}(W, \mathcal{U}) \subset V$, where $\mathrm{st}(W, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap W \neq \emptyset\}$.

Moreover by [5, Lemma 3] it also holds that

(2A) if $\mathcal{U}$ is a s. c. cover of (X, A) and $B \subset A$ then for each neighborhood V of B in X there exists a neighborhood W of B in X such that $\mathrm{st}(W, \mathcal{U}) \subset V$.

FACT 3. Every pair (X, A) is s. c. when X is a closed image of some metric space.

In particular, for every closed subset A of a metric space X with $\dim X = 0$, there exists a disjoint clopen s. c. cover $\mathcal{U}$ for (X, A). Then for the cover $\mathcal{U}$, where A in nowhere dense in $X = Q_\kappa$, we have the following lemma.

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LEMMA 0. *For any non-empty closed subset $B \subset A$, there exists a closed subset X_B of X and a subcollection $\mathcal{U}_B$ of $\mathcal{U}$ such that $\mathcal{U}_B$ is a* s. c. *cover for (T_B, B), where A is nowhere dense in $X \approx Q_\kappa$, $T_B = B \cup (\bigcup \mathcal{U}_B)$ and $\mathcal{U}$ is any given disjoint clopen* s. c. *cover for (X, A). Moreover, B is nowhere dense in T_B.*

PROOF. There exists a non-empty closed subset Z in X such that

(a) $B = A \cap Z$, $Z \setminus B$ is clopen in $X \setminus B$, and B is nowhere dense in Z.

Indeed, in the product space $X \times Q_\kappa$ put

$$W = X \times \{*\} \cup B \times Q_\kappa.$$

Then, note that

(b) $W \approx Q_\kappa$ by Fact 1, and $A \times \{*\}$ is closed nowhere dense in W. Note also that

(c) $B \times \{*\} = (A \times \{*\}) \cap (B \times Q_\kappa) = (X \times \{*\}) \cap (B \times Q_\kappa)$ is closed nowhere dense in $B \times Q_\kappa$ and $B \times Q_\kappa \setminus B \times \{*\}$ is clopen in $W \setminus B \times \{*\}$.

Then by Fact 2 there exists a homeomorphism $\varphi \colon W \longrightarrow X$ such that $\varphi(a, *) = a$ for each $a \in A$. Put

$$Z = \varphi(B \times Q_\kappa).$$

It is easy to see that Z satisfies (a) (compare the construction in [16, Theorem 1.5]). Let

$$T = \operatorname{st}(Z, \mathcal{U}) \quad \text{and} \quad \mathcal{U}_B = \{U \in \mathcal{U} : U \subset Z\}.$$

Then, it is not difficult to see that $\mathcal{U}_B$ satisfies the required properties by (1), (2), and (2A). ∎

On the other hand, we have one more result of another restriction of s. c. covers.

LEMMA 1. *For any given clopen disjoint* s. c. *cover $\mathcal{U}$ for (X, A), where X is a metric space with* $\dim X = 0$, *take a non-empty clopen set $O_U \subset U$ for each $U \in \mathcal{U}$. Then, the collection $O = \{O_U : U \in \mathcal{U}\}$ satisfies*

(2B) for each neighborhood V of A in S_A, there exists a neighborhood W of A such that $\operatorname{st}(W, O) \subset V$, *where $S_A = A \cup (\bigcup O)$.*

We omit its straightforward proof.

REMARK 1. a) Note that by Fact 1 the above space X_B in Lemma 0 is always homeomorphic to X, when A is nowhere dense in $X = Q_\kappa$. On the other hand, the closed subspace S_A in Lemma 1 is not necessarily homeomorphic to X (see an example in the final section). A cover O satisfying the condition (2B) is called an *approaching anti-cover of* A in [8].

b) Let $\tilde{T}_B$ and $\tilde{S}_B$ be the decomposition spaces, which are obtained by identifying all points of a nowhere dense closed subset B in the original spaces T_B and S_B to a single point, respectively (in the second case we apply Lemma 1 for $X = T_B$, $A = B$, and $\mathcal{U} = \mathcal{U}_B$ before collapsing). Then, note that the two spaces are homeomorphic to the decomposition space $\tilde{X}_{B_0}$, which is obtained by collapsing some closed nowhere dense subset $B_0 \approx B$ of a s. h. space X (*cf.* [16, Theorem 1.5]).

Finally we shall show the following lemma which reduces our proof to a special case (an onto map $f \colon X \longrightarrow Y$ is called *irreducible* if $f(A) \neq Y$ for any proper closed subset A of X).

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LEMMA 2. *For every closed onto map $f: A \longrightarrow Y$ where A is a closed subset of the space $X = Q_\kappa$, there exists an irreducible closed onto map $\tilde{f}: T \longrightarrow Z$ such that $f = \tilde{f}|X \times \{*\}$ and that $T = X^2 \setminus (X \setminus A) \times \{*\}$.*

PROOF. Let Z be the decomposition space with respect to the following identification on T: Its non-trivial equivalence classes are $f^{\leftarrow}(y) \times \{*\}$, where $y \in Y$. Let $\tilde{f}: T \longrightarrow Z$ be the natural quotient map. Then, it is not difficult to see that $\tilde{f}$ is a closed map satisfying the required properties (see [16] for a parallel proof). ∎

REMARK 2. We call the above closed map $\tilde{f}$ a *trivial extension* of a map f. Note that Y is closed in Z and that Lemma 2 holds for every (not necessarily 0-dimensional) space X with no isolated points such that T is normal.

2. **Watson method.**

a) Watson construction. Since (0) holds and the topological sum $\oplus_\kappa Q_\kappa$ of κ many copies of Q_κ is homeomorphic to the space Q_κ, there is a discrete collection $\mathcal{F}$ consisting of nowhere dense closed subsets of Q_κ, satisfying the following $(3)_\mu$ for each μ with $\omega \leq \mu \leq \kappa$ (note that we can assume that $* \notin \bigcup \mathcal{F}$).

$(3)_\mu$ It holds that $|\{F \in \mathcal{F} : F \approx Q_\mu\}| = \kappa$.

Now we shall construct a universal space $D_{\kappa,\lambda}$ for a given $C_{\kappa,\lambda}$. At first we shall construct two collections $\mathcal{F}_i$ and $\mathcal{U}_i$ satisfying the following (4)–(6) recursively.

(4) The mesh $\mathcal{U}_i < 1/2^i$, and $\mathcal{U}_i$ refines $\mathcal{U}_{i-1}$, and $\mathcal{U}_i$ is a disjoint clopen s. c. cover for the pair (Q_κ, E_i), where $F_j = \bigcup \mathcal{F}_j$ for each $j \leq i$ and $E_i = \bigcup_{j \leq i} F_j$.

(5) $\mathcal{F}_i = \bigcup_{U \in \mathcal{U}_{i-1}} \mathcal{F}_U$, where $\mathcal{F}_U = \mathcal{F}$ consists of subsets of U, which satisfies the condition $(3)_\mu$ for each μ with $\omega \leq \mu < \lambda$.

(6) The collection $\mathcal{F}_i$ is discrete in the subspace $Q_\kappa \setminus E_{i-1}$, and $* \notin E_i$.

Let $\mathcal{U}_{-1}$ be a singleton cover $\{Q_\kappa\}$, and let $\mathcal{F}_0$ be a subcollection $\{F \in \mathcal{F} : F \approx Q_\mu$ for some $\omega \leq \mu < \lambda\}$ of $\mathcal{F}$ guaranteed at the beginning of this section. It is easy to take a clopen cover $\mathcal{U}_0$ satisfying (4) by $\dim Q_\kappa = 0$ and Fact 3. Since every element $U \in \mathcal{U}_{i-1}$ is homeomorphic to Q_κ, it is ready to see that there exist $\mathcal{F}_i$ and $\mathcal{U}_i$ satisfying (4)–(6) under the assumption of those for $i-1$. Now, let $W_{\kappa,\lambda}$ be the decomposition space defined by the following identification on Q_κ:

we identify $r, s \in Q_\kappa$ if $r, s \in F \in \mathcal{F}_i$ for some i.

From the above construction this definition of identification makes sense, and let $q: Q_\kappa \longrightarrow W_{\kappa,\lambda}$ be the natural quotient map. Here, we need the following fundamental property of our quotient topology (see [16, Lemmas 1.7 and 1.8]).

LEMMA 3. *For a point $q(x) \in W_{\kappa,\lambda}$ we have*

(a) in case $x \in F \in \mathcal{F}_i$ there exists a clopen neighborhood base $\mathcal{W}$ of $q(x)$ such that $\mathcal{W} = \{q(B) : B \in \mathcal{B}\}$, where $\mathcal{B}$ is a neighborhood base of F in Q_κ such that, for each $B \in \mathcal{B}$, there exists a collection $\mathcal{U}_B \subset \mathcal{U}_i$ satisfying that $B = \bigcup \mathcal{U}_B \cup F$;

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(b) in other cases the collection $\{q(U) : x \in U \in \mathcal{U}_i\}$ *is a clopen neighborhood base of* $q(x)$.

By virtue of this lemma it is not difficult to prove the following lemma (by (0) note that for each $y \in W_{\kappa,\lambda}$ it holds that $|q^{\leftarrow}(y)| < \lambda$, and see [16, §1] for the remaining proof).

LEMMA 4. *The quotient map* $q: Q_\kappa \longrightarrow W_{\kappa,\lambda}$ *is closed irreducible. Hence, the decomposition space* $W_{\kappa,\lambda}$ *is* σ*-discrete, and is a member of* $C_{\kappa,\lambda}$.

Finally let $D_{\kappa,\lambda}$ be the subspace of $W_{\kappa,\lambda}$ consisting of non-trivial equivalent classes *i.e.* $D_{\kappa,\lambda} = \big(q(\bigcup_i F_i)\big)$. In the next section we shall show that both $W_{\kappa,\lambda}$ and $D_{\kappa,\lambda}$ satisfy Theorem 0.

REMARK 3. The space $W_{\kappa,\lambda}$ is not homeomorphic to $D_{\kappa,\lambda}$, since the point $q(*)$ has a countable neighborhood base in $W_{\kappa,\lambda}$, while no point is first countable in $D_{\kappa,\lambda}$.

b) Watson control. Let $f: M \longrightarrow Y$ be a closed onto map satisfying that M is σ-discrete, $w(M) \leq \kappa$, and $|f^{\leftarrow}(y)| < \lambda$ for every $y \in Y$. By Fact 0 we can assume that M is a closed subset of Q_κ. Hence by Lemma 2 we have a trivial extension $\tilde{f}: T \longrightarrow Z$. Since T is homeomorphic to Q_κ by Fact 1, we can reduce our proof to an irreducible closed image of Q_κ. Hence we can assume that f is irreducible so that every fiber $f^{\leftarrow}(y)$ is nowhere dense in $M \approx Q_\kappa$.

In other words, we may consider the spaces Y and $W_{\kappa,\lambda}$ as *upper semicontinuous decompositions* of the model spaces M and Q_κ, respectively. Hence, we shall construct a closed embedding $g: Y \longrightarrow W_{\kappa,\lambda}$ by controlling materials in the model spaces. In this sense we shall call it a *Watson control* that is a combination of a *control sequence* $\{\mathcal{V}_i\}$ of s. c. *covers* (we call it *the first control*), and a *control sequence* $\{h_i\}$ of *homeomorphisms* (we call it *the second control*), satisfying the following conditions (11)–(14).

The first control determines global destination of our embedding g, while the second control adjusts the topology about the corresponding points y and $g(y)$ (see also the final section).

Now we shall construct a Watson control. Note that Y is σ-discrete, since every closed image of a discrete closed subspace is closed discrete. Hence, put $Y = \bigcup_{i \geq 0} Y_i$, where each Y_i is closed discrete in Y and $Y_i \cap Y_j = \emptyset$. Since $\dim Y = 0$ and Fact 3 holds for Y, there exists a sequence $\{\mathcal{G}_i\}$ of clopen disjoint collections of Y such that

(7) $\mathcal{G}_i$ refines $\mathcal{G}_{i-1}$, and $\mathcal{G}_i$ *separates* Y_{i+1}(*i.e.* $\forall y \in Y_{i+1}[|\{G \in \mathcal{G}_i : y \in G\}| = 1]$).

(8) $\mathcal{G}_i$ is a s. c. cover for $(Y, \bigcup_{j \leq i} Y_j)$.

Put $\mathcal{Z}_i = \{Z_y = f^{\leftarrow}(y) : y \in Y_i\}$ and $Z_i = \bigcup \mathcal{Z}_i$. Since it holds that $|Y_i| \leq \kappa$ and $w\big(f^{\leftarrow}(y)\big) = |f^{\leftarrow}(y)| < \lambda$ for each $y \in Y$, we can apply Fact 0 and (5) so that the following property holds.

(9) $\forall Z_y \exists$ a closed embedding $h_y: Z_y \longrightarrow F_y \in \mathcal{F}_0$, and $F_y \cap F_{y'} = \emptyset$ if $y \neq y'$.

Let $h_0: Z_0 \longrightarrow Q_\kappa$ be the closed embedding defined by $h_0 \mid Z_y = f_y$. Applying Lemma 0 for $A = F_0$ and $B = h_0(Z_0) = H_0$, we have a closed subspace T_0 of Q_κ and a subcollection $\mathcal{V}_0$ of $\mathcal{U}_0$, which is a s. c. cover for (T_0, H_0). Note that by Fact 1 the space T_0 is homeomorphic to Q_κ and H_0 is its nowhere dense subset. Applying Fact 2 for $G = Z_0, H = H_0$

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so that we have a homeomorphism $\tilde{h}_0\colon M \longrightarrow T_0$ which is an extension of h_0. Note that the following $(10)_i$ is valid for $i = 0$, since (4) holds and E_j is nowhere dense for each j.

$(10)_i$ For each $G \in \mathcal{G}_i$ we can take a $U_G \in \mathcal{U}_i^\infty$, where $\mathcal{U}_i^\infty = \bigcup_{j\geq i} \mathcal{U}_j$, such that

$$U_G \subset \tilde{h}_i \circ f^{\leftarrow}(G).$$

Let $\mathcal{O}_0 = \{U_G : G \in \mathcal{G}_0\}$. Then by Lemma 1 we have a closed set $S_0 = S_A$. By recursion we have, for each $i \geq 0$, two closed subsets T_i and S_i in Q_κ, two subcollections $\mathcal{V}_i, \mathcal{O}_i \subset \mathcal{U}_i^\infty$, and a homeomorphism $\tilde{h}_i\colon (M \setminus \bigcup_{j<i} Z_j) \longrightarrow (S_i \setminus \bigcup_{j<i} H_j)$ satisfying the following (11)–(14), where $\mathcal{O}_i$ is defined by the property $(10)_i$ analogous to $\mathcal{O}_0$.

(11) $\mathcal{V}_i$ refines $\mathcal{V}_{i-1}$, $S_i = \bigcup \mathcal{O}_i \cup (\bigcup_{j<i} H_j) \subset T_i = \bigcup \mathcal{V}_i \cup (\bigcup_{j<i} H_j) \subset S_{i-1}$, and for each $U_G \in \mathcal{O}_{i-1}$ the collection $\mathcal{V}_G = \{U \in \mathcal{V}_i : U \subset U_G\}$ is a s.c. cover for (T_B, B), where $T_B = U_G \cap T_i$ and $B = U_G \cap H_i$.

(12) $\mathcal{O}_i$ refines $\mathcal{O}_{i-1}$, $\tilde{h}_i(Z_i) = H_i \subset \bigcup_{j\geq 0} F_j$, and for each $U_G \in \mathcal{O}_{i-1}$ the collection $\mathcal{O}_G = \{O \in \mathcal{O}_i : O \subset V_G\}$ is an approaching anti-cover of (S_B, B), where $S_B = U_G \cap S_i$ and $B = U_G \cap H_i$.

(13) For each $y \in G \cap Y_i$, where $G \in \mathcal{G}_{i-1}$ the restriction $\tilde{h}_i \mid Z_y$ is a closed embedding into a set $F_y \in F_U$ given by applying (5) and Fact 0.

(14) For $G \in \mathcal{G}_{i-1}$ the restriction $\tilde{h}_i \mid f^{\leftarrow}(G)$ is a homeomorphism onto the set T_B, where we apply Lemma 0 for $B = \tilde{h}_i(Z_y) \subset A = F_{U_G}$, $X = U_G$, $\mathcal{U} = \{U \in \mathcal{U}_k : U \subset U_G\}$, and k is determined by the relation that $F_{U_G} \in \mathcal{F}_{k+1}$.

Define $g\colon Y \longrightarrow W_{\kappa,\lambda}$ as follows. For every $y \in Y$ there exists a unique i such that $y \in Y_i$. Then let

$$g(y) = q \circ h_i(F_y).$$

3. **Proof of Theorem 0.** Note that the following lemma shows that both $W_{\kappa,\lambda}$ and $D_{\kappa,\lambda}$ satisfy Theorem 0.

LEMMA 5. *The map g is an embedding, satisfying that the set $g(Y)$ is a closed subset of $W_{\kappa,\lambda}$.*

PROOF. By the definition of g it holds that g is one to one, and $g(Y) \subset \mathcal{W}_{\kappa,\lambda}$. Hence, at first we show that the set $g(Y)$ is closed in $W_{\kappa,\lambda}$. We shall show that:

ASSERTION. *Each set $q(H_i)$ is closed discrete in $W_{\kappa,\lambda}$.*

PROOF OF ASSERTION. Indeed, the set $q(H_0)$ is closed, since $\mathcal{F}_0$ is closed discrete collection. Note that the set $q(\bigcup_{j\leq i} H_j)$ is closed, since T_i is closed in Q_κ and $\bigcup_{j\leq i} H_j$ is closed in T_i. Since Y_i, where $i \geq 1$, is disjoint from the set $\bigcup_{j<i} Y_j$ and $\mathcal{G}_{i-1}$ is a s.c. cover for $(Y, \bigcup_{j<i} Y_j)$, there exists a subcollection C of $\mathcal{G}_{i-1}$ such that $B_{i-1} \supset \bigcup_{j<i} Y_j$ and $Y_i \cap B_{i-1} = \emptyset$, where B_{i-1} is a clopen set of Y, defined by $B_{i-1} = (\bigcup_{j<i} Y_j) \cup (\bigcup C)$. Then, by (2B) and (12), we see that $V_{i-1} = \tilde{h}_{i-1}\big(f^{\leftarrow}(B_{i-1})\big)$ and $U_{i-1} = S_{i-1} \setminus V_{i-1}$ are disjoint neighborhoods of H_{i-1} and H_i, respectively, in S_{i-1}. In a same manner we can choose clopen sets $B_{i-2}, \ldots, B_0$ in Y such that $B_j \supset \bigcup_{k\leq j} Y_k$, and $V_j = \tilde{h}_j\big(f^{\leftarrow}(B_j)\big)$ and $U_j = S_j \setminus V_j$ are disjoint neighborhoods of H_j and H_i, respectively, in S_j for each $i-2 \geq j \geq 0$. Hence,

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$U = \bigcap_j U_j$ is a neighborhood of H_i disjoint from $V = \bigcup V_j \supset \bigcup_{j<i} H_j$ in $S_{i-1} = \bigcap_{j<i} S_j$. Therefore $q(H_i)$ is closed in $W_{\kappa,\lambda}$, since S_{i-1} is closed and $q^{\leftarrow}q(U) \cap V = \emptyset$. By (14) and construction of $\mathcal{O}_i$ in (12) it is not difficult to see that $q(H_i)$ is discrete.

The set $g(Y)$ is closed, since $g(Y) = q(\bigcap_i T_i) = \bigcup_i q(H_i)$.

Indeed, since $g(Y) \subset q(S_i)$ and $g(S_i)$ is closed for each i, it suffices to show that

$$K = \mathrm{cl}\big(g(Y)\big) \cap \Big(q\Big(\bigcap_i S_i\Big)\Big) \subset g(Y).$$

Let $x \in K$. Then, since $g(Y) \subset D_{\kappa,\lambda}$, there exists i such that $q^{\leftarrow}(x) \in \mathcal{F}_i$. Hence, we see that $x \notin \mathrm{cl}\big(\bigcup_{j<i} q(H_j)\big)$ by the above assertion and by the fact that $x \notin \bigcup_{j<i} q(H_j)$. Then, by Lemmas 3 and (11) we have $x \in \mathrm{cl}\big(g(S_i)\big) = g(S_i)$. Hence, $x \in g(H_i) \subset g(Y)$.

Next we shall show that $g^{\leftarrow}$ is continuous. Let N be any open set containing $y \in Y_i$, and suppose that $y \in G \in \mathcal{G}_{i-1}$. Then by (14) the set $\tilde{h}_i(N)$ is an open neighborhood of $A = h_i(Z)$. Hence by (11) and (12) there exists a subcollection $\mathcal{V}_N$ of $\mathcal{V}_G$ and a set W of $g(y)$ such that

$$W = \bigcup \mathcal{V}_N \cup A \text{ is a neighborhood of } A \text{ in } T_i \text{ and } W \cap S_i \subset \tilde{h}_i(N).$$

Then, $O = q(W) \cap g(Y)$ is a neighborhood of $g(y)$ in $g(Y)$, since $W = g^{\leftarrow}q(W) \cap T_i$. Hence, $q^{\leftarrow}(O) \subset N$. This relation shows that $g^{\leftarrow}$ is continuous at $g(y)$. It can be shown by a parallel way that g is continuous (compare [16, §1]). ∎

4. **Concluding remarks.**

REMARK 4. The following observation shows that we need the concept of a trivial extension given in Lemma 2 even if we consider *metric* spaces.

OBSERVATION. No countable metric space, which is compact, is an irreducible closed image of the space of national numbers Q.

Indeed, let Y be compact, and assume that it is an irreducible closed image of Q. Then, Y is a perfect image of Q by Hanai-Morita-Stone theorem [4, Theorem 4.4.17], and hence Q must be compact, since every perfect preimage of a compact set is compact. This contradiction shows that our observation is true.

REMARK 5. Strictly speaking a Watson control is performed under not only the two control sequences, but also a control sequence $\{\mathcal{O}_i\}$ of approaching anti-covers in (12). The reason why we emphasize the first two controls is that we believe that it helps the reader to understand an outline of rather complicated Watson method. The following example, which is announced in Remark 1a), shows that the third control is inevitable for our proof.

EXAMPLE 1. Consider the space $Z = X_1 \oplus X_2$ of two copies of $X = Q$. Let A_i be closed nowhere dense subsets such that $A_1 \subset X_1$ is homeomorphic to X and $A_2 \subset X_2$ is homeomorphic to some non-trivial compact set. Let Y be the decomposition space of Z, obtained by collapsing the set $B = A_1 \cup A_2$ to a point $* \in Y$. Let $q: Z \longrightarrow Y$ be the natural

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quotient map, and since A_2 is compact, let $\mathcal{B} = \{B_i\}$ be a countable clopen decreasing base of A_2 in X_2 satisfying that

$$O_i \subset N_i(b) \cap U_i,$$

where $N_i(b)$ is the $1/i$ neighborhood of some fixed point $b \in A_2$, $U_i = X_2 \setminus B_i$, and $\mathcal{O}_2 = \{O_i\}$ consists of non-empty disjoint clopen subsets of X_2. Then, the collections $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$, where $\mathcal{U}_1$ is a s. c. cover for (X_1, A_1) and $\mathcal{U}_2 = \{U_i\}$, and $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$, where $\mathcal{O}_1 = \mathcal{U}_1$, satisfies the condition of Lemma 1 so that the decomposition spaces $\tilde{X}_B$ and $\tilde{T}_B$ are homeomorphic each other. In this situation, however, the closed subspace T_B is not homeomorphic to X, since no points of X are locally compact, while T_B is locally compact at any point of $A_2 \setminus \{b\}$.

REMARK 6. The following corollary, which is the special case $\lambda = \kappa^+$ in our theorem, corresponds main theorems in [12, 16].

COROLLARY. *For each cardinal number $\kappa \geq \omega$ there exists a universal space for all closed images of σ-discrete metric space M of $w(M) \leq \kappa$.*

REMARK 7. For every infinite cardinals κ and λ with $\omega_1 \leq \lambda \leq \kappa^+$ let $\mathcal{K}_{\kappa,\lambda}$ be the class of all spaces Y for which there are complete metric space M of $w(M) \leq \kappa$ and a closed onto map $f : M \longrightarrow Y$ such that $\forall y \in Y\Big[\big|w\big(f^{\leftarrow}(y)\big)\big| < \lambda\Big]$.

Then, we have the following theorem by using a parallel method in this paper as well as results developed in [12] (see [14] for details).

THEOREM 1. *There is a universal space for every class $\mathcal{K}_{\kappa,\lambda}$. Moreover, every space in $\mathcal{K}_{\kappa,\lambda}$ embeds as a closed subspace in it.*

REMARK 8. Using special s. c. covers, our universal spaces $W_{\kappa,\lambda}$ and $D_{\kappa,\lambda}$ are made to be *h-homogeneous*. Moreover, we can improve our construction to satisfy that any non-empty open subset of each space is homeomorphic each other (see [15] for the details).

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