Can. J. Math., Vol. XXXIX, No. 2, 1987, pp. 297-308

THE MULTIVALENT CLASS OF GEOMETRICALLY CLOSE-TO-CONVEX FUNCTIONS

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1. Introduction. The class of univalent close-to-convex functions, K, was introduced by Kaplan [**4**] and first studied by him. The first important extension to the class of multivalent close-to-convex functions, $K(p)$ where p is a positive integer, was considered by Livingston [**7**]. Somewhat later, Styer [**15**] introduced the more general class, $K_w(p)$, of weakly close-to-convex functions by simply taking the closure of Livingston's class $K(p)$ in the topology of locally uniform convergence in $\mathbf{B} = \{z: |z| < 1\}$.

In 1936 Biernacki [**2**] introduced his class of linearly accessible functions. A function f is *linearly accessible* if f is univalent in $\mathbf{B}$, $f(0) = 0$, and $\mathbf{C} - f(\mathbf{B})$ where $\mathbf{C}$ is the complex plane, is a union of closed (Euclidean) rays with disjoint interiors. In an interesting result, Lewandowski [**6**] showed that the classes of univalent close-to-convex functions and linearly accessible functions are equal.

Let P be a nonconstant polynomial. A curve l is called a *P-line* if P maps l one-to-one onto a straight line. We define (closed) P-rays similarly. Recently, Lyzzaik [**8**] extended the concept of a linear accessible function to the multivalent case. Accordingly, a function F is *linearly accessible of order* p if $F(0) = 0$, F' has exactly $p - 1$ zeros (counting multiplicity) in $\mathbf{B}$, and $F = P \circ \phi$ where P is a polynomial of degree p and ϕ is a univalent function in $\mathbf{B}$ such that $\phi(0) = 0$ and $\mathbf{C} - \phi(\mathbf{B})$ is a union of P-rays of disjoint interiors. He also showed that every $F \in K(p)$ is linearly accessible of order p. However, the truth of the converse of this result still poses an open problem.

The object of this paper is to consider several ways to define a multivalent class of close-to-convex functions that is equal to the class of linearly accessible functions of order p. This will extend Lewandowski's criteria of close-to-convexity completely to the multivalent case.

2. Basic definitions. This section is devoted to the known classes of functions that we shall use.

Definition 2.1. Let S be the familiar class of functions f univalent in $\mathbf{B}$ that satisfy $f(0) = 0$ and $f'(0) = 1$.

Received December 6, 1984 and in revised form November 6, 1985.

Definition 2.2. Let S^* be the class of functions f that satisfy one of the following conditions:

(a) f is univalent in $\mathbf{B}$, $f(0) = 0$, and $f(\mathbf{B})$ is starshaped with respect to the origin.

(b) f is analytic in $\mathbf{B}$, admits one zero there (counting multiplicity), and

$$\mathrm{Re}(zf'/f) > 0 \quad \text{for all } z \in \mathbf{B}.$$

Note that the normalization $f'(0) = 1$ is not required in this definition.

Throughout, the zeros of functions are counted according to multiplicity.

We will adopt the following definition of *annular p-valent starlike functions* (see [**3**]).

Definition 2.3. A function f is said to belong to $S_a(p)$ if f is analytic in $\mathbf{B}$, has p zeros there, and there is an annulus

$$A_\rho = \{z:\rho < |z| < 1\}$$

such that $\mathrm{Re}(zf'/f) > 0$ for all $z \in A_\rho$.

Let

$$\Psi(z, \zeta) = (z - \zeta)(1 - \bar{\zeta}z)/z,$$

and let $\Psi(z, 0) = 1$.

Hummel [**3**] has extended the class $S_a(p)$ to *weakly starlike functions of order p* as follows:

Definition 2.4. A function f is said to belong to $S_w(p)$ if f is analytic in $\mathbf{B}$, has p zeros there, and satisfies one of the conditions:

(a) $\liminf\limits_{r \to 1^-} [\min\limits_{|z|=r} \mathrm{Re}(zf'/f)] \geqq 0.$

(b) There exists a sequence $(f_n)_{n=1}^{\infty}$ where $f_n \in S_a(p)$ for all n, such that $f_n \to f$ locally uniformly in $\mathbf{B}$.

(c) There is $h \in S^*$ such that

$$f(z) = [h(z)]^p \prod_{i=1}^{p} \Psi(z, z_i), \quad |z_i| < 1, \ 1 \leqq i \leqq p.$$

Observe that this class is not closed in the topology of locally uniform convergence in $\mathbf{B}$. A useful extension of the class $S_w(p)$ has been developed by Styer [**15**].

Definition 2.5. A function f is said to belong to $S_{wc}(p)$ if f is analytic in $\mathbf{B}$ and satisfies one of the conditions:

(a) There exists a sequence $(f_n)_{n=1}^{\infty}$, where $f_n \in S_a(p)$ for all n, such that $f_n \to f$ locally uniformly in $\mathbf{B}$.

(b) There is $h \in S^*$ such that

$$f(z) = [h(z)]^p \prod_{i=1}^{p} \Psi(z, z_i), \quad |z_i| \leqq 1, \ 1 \leqq i \leqq p.$$

Note that functions $f \in S_{wc}(p)$ may have any number of zeros less than or equal to p; and that $f \in S_w(p)$ if and only if $f \in S_{wc}(p)$ and f has exactly p zeros in $\mathbf{B}$.

Next we define Livingston's class of *multivalent close-to-convex functions.*

Definition 2.6. A function F belongs to $K(p)$ if F is analytic in $\mathbf{B}$, $F(0) = 0$, and F satisfies one of the conditions:

(a) There is $f \in S_a(p)$, with $f(0) = 0$, and an annulus A_ρ such that $\mathrm{Re}(zF'/f) > 0$ for all $z \in A_\rho$.

(b) F' has $p - 1$ zeros in $\mathbf{B}$, and for any $\theta_1 < \theta_2$ and $\rho < r < 1$

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}(1 + re^{i\theta}F''(re^{i\theta})/F'(re^{i\theta}))d\theta > -\pi.$$

Note that $K(1) = K$, and that $K(p)$ is not closed in the topology of locally uniform convergence in $\mathbf{B}$. This has led to the following extension, the class of *weakly close-to-convex functions of order p* (see [**15**]).

Definition 2.7. Let F be a nonconstant function analytic in $\mathbf{B}$ with $F(0) = 0$. F is said to belong to $K_w(p)$ if one of the following conditions is satisfied:

(a) There is $f \in S_w(p)$, with $f(0) = 0$, such that

$$\lim_{r \to 1^-} \inf[\min_{|z|=r} \mathrm{Re}(zF'/f)] \geqq 0.$$

(b) There are functions $F_n \in K(p)$ and $f_n \in S_a(p)$, with each $f_n(0) = 0$, such that $F_n \to F$ and $f_n \to f$, locally uniformly in $\mathbf{B}$, $f \in S_w(p)$, and

$$\mathrm{Re}(zF_n'/f_n) > 0 \quad \text{for all } z, \ 0 < \rho_n < |z| < 1.$$

(c) F is the limit of a sequence of functions in $K(p)$ in the topology of locally uniform convergence in $\mathbf{B}$.

(d) There is $H \in K$ and $g \in S_{wc}(p - 1)$ such that

$$F(z) = \int_0^z g(z)H'(z)dz.$$

(e) There is $f \in S_{wc}(p)$, with $f(0) = 0$, such that

$$\mathrm{Re}(zF'/f) > 0 \quad \text{in } \mathbf{B}.$$

(f) There is $h \in S^*$ such that

$$\lim_{r \to 1^-} \inf[\min_{|z|=r} \mathrm{Re}(zF'(z)/[h(z)]^p)] \geqq 0.$$

3. The class $K_g(p)$. In this section we consider several ways to define the multivalent class of geometrically close-to-convex functions, $K_g(p)$. To do so we prove:

THEOREM 3.1. *Let F be a nonconstant function analytic in* **B** *with $F(0) = 0$. The following conditions are equivalent*:

(A) *F' has $p - 1$ zeros in* **B**, *and $F \in K_w(p)$.*

(B) *F' has $p - 1$ zeros in* **B**, *and there is $f \in S_w(p)$, with $f(0) = 0$, such that*

$$\lim_{r \to 1} \inf[\min_{|z|=r} \operatorname{Re}(zF'/f)] \geqq 0.$$

(C) *F' has $p - 1$ zeros in* **B**, *and there are functions $F_n \in K(p)$ and $f_n \in S_a(p)$, with $f_n(0) = 0$ for all n, such that $F_n \to F$ and $f_n \to f$ locally uniformly in* **B**, *$f \in S_w(p)$, and*

$$\operatorname{Re}(zF_n'/f_n') > 0 \quad \text{for all } z,\ 0 < \rho_n < |z| < 1.$$

(D) *F' has $p - 1$ zeros in* **B**, *and F is the limit of a sequence of functions in $K(p)$ in the topology of locally uniform convergence in* **B**.

(E) *There exists $H \in K$ and $g \in S_w(p - 1)$ such that*

$$F(z) = \int_0^z g(z)H'(z)dz.$$

(F) *There exists $f \in S_w(p)$, with $f(0) = 0$, such that*

$$\operatorname{Re}(zF'/f) > 0 \quad \text{in } \mathbf{B}.$$

(G) *There are functions $F_n \in K(p)$ and ρ, $0 < \rho < 1$, such that the modulus of every zero of F_n' is at most ρ for all n, and $F_n \to F$ locally uniformly in* **B**.

(H) *F is the limit in the topology of locally uniform convergence in* **B** *of a sequence of functions*

$$F_n(z) = A_n \int_0^z \prod_{j=1}^{m(n)} (1 - \zeta_{n,j}t)(1 - z_{n,j}t)^{-\gamma_{n,j}}$$

$$\times \prod_{j=1}^{p-1} (t - \beta_{n,j})(1 - \bar{\beta}_{n,j}t)dt,$$

where $A_n \neq 0$, the numbers $\zeta_{n,1}, \zeta_{n,2}, \dots, \zeta_{n,m(n)}$ and $z_{n,1}, z_{n,2}, \dots, z_{n,m(n)}$ are located alternately on **B**, *and for all n and $1 \leqq j \leqq p$ we have*

$$|\beta_{n,j}| < \rho < 1 \quad \text{and} \quad 1 \leqq \gamma_{n,j} \leqq 2p + 1.$$

Furthermore, for each n

$$\sum_{j=1}^{m(n)} \gamma_{n,j} = 2p + m(n).$$

(I) *F' has $p - 1$ zeros in* $\mathbf{B}$, *and*

$$\liminf_{r\to 1^-}\left[\min_{\theta_1 \leqq \theta_2} \int_{\theta_1}^{\theta_2} \mathrm{Re}(1 + re^{i\theta}F''(re^{i\theta})/F'(re^{i\theta}))d\theta\right] \geqq -\pi.$$

(J) *F' has $p - 1$ zeros in* $\mathbf{B}$, *and there is a pair P, ϕ where P is a polynomial of degree p and $\phi \in S$ such that $F = P \circ \phi$. In this case,* $\mathbf{C} - \phi(\mathbf{B})$ *is a union of P-rays of disjoint interiors each of which starts from* $\partial\phi(\mathbf{B})$.

(K) *F is linearly accessible of order p.*

(L) *F' has $p - 1$ zeros in* $\mathbf{B}$, *and there is a pair P, ϕ where P is a polynomial degree p and $\phi \in S$ such that $F = P \circ \phi$. In this case,* $\mathbf{C} - \phi(\mathbf{B})$ *is a union of P-rays such that for any two rays either they have disjoint interiors or one is a subset of the other.*

(M) *F' has $p - 1$ zeros in* $\mathbf{B}$, *and there is a pair P, ϕ where P is a polynomial of degree p and $\phi \in S$ such that $F = P \circ \phi$. In this case, for any two points in* $\partial\phi(\mathbf{B})$ *we can find a P-ray or two P-rays containing the points and not meeting in* $\mathbf{C} - \phi(\mathbf{B})$.

Finally, if F satisfies any of the above conditions, then there is a unique pair P, ϕ where P is a polynomial of degree p and $\phi \in S$ such that $F = P \circ \phi$.

Before proving the theorem we make some remarks and give three lemmas. Conditions (A), (B), (C), (D), (E), (F), and (G) are motivated by Definition 2.7. Conditions (H), (J), (K), and (L) are suggested by [**8**] and [**9**]. Condition (I) is an extention of Kaplan's criteria for the class K, and later Livingston's criteria for the class $K(p)$ (see [**4**] and [**7**]). Condition (M) is motivated by the work of Sheil-Small on linear accessibility (see [**14**]).

LEMMA 3.1. *Let $\epsilon \geqq 0$, and let T be a real-valued function on $(-\infty, \infty)$ that satisfies*

(a) $T(\theta + 2\pi) - T(\theta) = 2p\pi$ *for all θ, and*

(b) $T(\theta_1) - T(\theta_2) < \pi + \epsilon$ *for all $\theta_1 < \theta_2$.*

Then there exists a real-valued function on $(-\infty, \infty)$ which is nondecreasing and satisfies

(c) $S(\theta + 2\pi) - S(\theta) = 2p\pi$ *for all θ, and*

(d) $|S(\theta) - T(\theta)| \leqq (\pi + \epsilon)/2$ *for all θ.*

Furthermore, if T is continuous, then S is also continuous.

This lemma slightly generalizes a result of Kaplan [**4**], and its proof is essentially the same as the latter.

Via this lemma, and a procedure established by Kaplan [**4**, pp. 172-177] and later used by Livingston [**7**, pp. 165-169], we conclude

LEMMA 3.2. *Suppose F is a function analytic in the closure of* $\mathbf{B}$, $\mathrm{Cl}(\mathbf{B})$, *with $p - 1$ critical values in* $\mathbf{B}$. *Also, suppose T is a real-valued function on* $(-\infty, \infty)$ *defined by*

$$T(\theta) = \theta + \arg F'(e^{i\theta})$$

and T satisfies

$$T(\theta_1) - T(\theta_2) < \pi + \epsilon, \quad \text{for all } \theta_1 < \theta_2.$$

Then there exists a function $f \in S_w(p)$ such that f is analytic in $\mathrm{Cl}(\mathbf{B})$ *and*

$$|\arg(zF'/f)| < (\pi + \epsilon)/2 \quad \text{in } \mathrm{Cl}(\mathbf{B}).$$

The following is an extension of a result due to Sheil-Small [**14**, pp. 270-272], and its proof is essentially the same. So, we give it without proof.

LEMMA 3.3. *Let P be a polynomial and D be a domain. Suppose that for any two points in ∂D we can find a P-ray or two disjoint P-rays containing the points and not meeting D. Then for any n points $z_1, z_2, \dots, z_n$ in ∂D there exist a finite number of mutually disjoint P-rays containing the points and not meeting D.*

Proof of theorem. The theorem would follow if we prove the following sequence of equivalences and implications: (A) $\Leftrightarrow$ (B), (A) $\Leftrightarrow$ (C), (A) $\Leftrightarrow$ (D), (A) $\Leftrightarrow$ (E), (A) $\Leftrightarrow$ (F), (A) $\Leftrightarrow$ (G), (F) $\Leftrightarrow$ (I), (G) $\Rightarrow$ (H) $\Rightarrow$ (A), (A) $\Rightarrow$ (J) $\Rightarrow$ (K) $\Rightarrow$ (L) $\Rightarrow$ (M) $\Rightarrow$ (A).

It is straightforward from Definition 2.7 that condition (A) is equivalent to each of the conditions (B), (C), and (D).

(A) $\Leftrightarrow$ (E). Let $F \in K_w(p)$, and suppose F' has exactly $p - 1$ zeros in $\mathbf{B}$. From Definition 2.7 (d) there exist $H \in K$ and $g \in S_{wc}(p - 1)$ such that $F' = gH'$ in $\mathbf{B}$. Since $H' \neq 0$ in $\mathbf{B}$, g has exactly $p - 1$ zeros in $\mathbf{B}$. Hence $f \in S_w(p)$ and F satisfies condition (E).

Conversely, (E) $\Rightarrow$ (A) follows by reversing the previous argument.

(A) $\Leftrightarrow$ (F). Let $F \in K_w(p)$, and suppose F' has $p - 1$ zeros in $\mathbf{B}$. From Definition 2.7 (e) there exist $f \in S_{wc}(p)$, with $f(0) = 0$, and a function h of positive real part such that $zF' = fh$ in $\mathbf{B}$. Since F' has exactly $p - 1$ zeros in $\mathbf{B}$, f has exactly p zeros in $\mathbf{B}$. Hence $f \in S_w(p)$ and F satisfies condition (F).

Conversely, (F) $\Rightarrow$ (A) follows by reversing the previous argument.

(A) $\Leftrightarrow$ (G). This is straightforward from Definition 2.7 (c) and the argument principle.

(G) $\Rightarrow$ (H). Suppose F satisfies condition (G). Then there are functions $F_n \in K(p)$ and ρ, $0 < \rho < 1$, such that the modulus of every zero of F_n' is less than ρ for all n, and $F_n \to F$ locally uniformly in $\mathbf{B}$. As a result of Lemma 3.3 and the proof of Theorem 4.1 in [**8**], for every F_n there is a sequence of those functions appearing in the statement of condition (H) which converges to F_n locally uniformly in $\mathbf{B}$. This implies that there is a similar sequence that converges to F locally uniformly in $\mathbf{B}$. Hence F satisfies condition (H).

(H) $\Rightarrow$ (A). As an implication of the proof of Theorem 4.1 of [**8**] functions F_n appearing in the statement of condition (H) belong to $K_{w'}(p)$. If $F_n \to F$ locally uniformly in $\mathbf{B}$, then $F \in K_{w'}(p)$ since $K_{w'}(p)$ is closed. Also, by the argument principle F' has exactly $p - 1$ zeros in $\mathbf{B}$, and F satisfies condition (A).

(F) $\Leftrightarrow$ (I). First we show (F) $\Rightarrow$ (I). Suppose F and f are as in the statement of condition (F). Let $\epsilon > 0$. From Definition 2.4 (a) it follows that there is r_0 such that

$$\mathrm{Re}(re^{i\theta}f'(re^{i\theta})/f(re^{i\theta})) \geqq -\epsilon/2\pi$$

for all r, $0 < r_0 < r < 1$, and all θ. This gives

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}(re^{i\theta}f'(re^{i\theta})/f(re^{i\theta}))d\theta \geqq -\epsilon$$

whenever $\theta_1 < \theta_2$ because

$$\int_0^{2\pi} \mathrm{Re}(re^{i\theta}f'(re^{i\theta})/f(re^{i\theta}))d\theta = 2p\pi.$$

Since ϵ is arbitrary

$$\lim_{r\to 1^-} \inf\left[\min_{\theta_1<\theta_2} \int_{\theta_1}^{\theta_2} \mathrm{Re}(re^{i\theta}f'(re^{i\theta})/f(re^{i\theta}))d\theta\right] \geqq 0.$$

Let ρ be the maximum of the moduli of all the zeros of f, and let $\rho < r < 1$. From condition (F) we have

$$|\arg(zF'/f)| < \pi/2 \quad \text{in } \mathbf{B}.$$

This implies

$$\begin{aligned}&\arg re^{i\theta_2}F'(re^{i\theta_2}) - \arg re^{i\theta_1}F'(re^{i\theta_1})\\ &\geqq -\pi + \arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1})\end{aligned}$$

for all $\theta_1 < \theta_2$. That is

$$\begin{aligned}&\int_{\theta_1}^{\theta_2} \mathrm{Re}(1 + re^{i\theta}F''(re^{i\theta})/F'(re^{i\theta}))d\theta\\ &\geqq -\pi + \int_{\theta_1}^{\theta_2} \mathrm{Re}(re^{i\theta}f'(re^{i\theta})/f(re^{i\theta}))d\theta.\end{aligned}$$

By taking the minimum of both sides of inequality over all intervals $[\theta_1, \theta_2]$ followed by limit infimum over r we conclude that F satisfies condition (I).

Conversely, suppose F is a function that satisfies condition (I), and let

z_i, $1 \leqq i \leqq p - 1$, be the zeros of F' in $\mathbf{B}$. There is a sequence $(r_n)_{n=1}^{\infty}$, with

$$\max\{|z_i|:1 \leqq i \leqq p - 1\} < r_n$$

for all n, and $r_n \to 1^-$, such that for all θ_1, θ_2, $\theta_1 < \theta_2$ and all n

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}(1 + r_n e^{i\theta} F''(r_n e^{i\theta})/F'(r_n e^{i\theta}))d\theta \geqq -\pi - 1/n.$$

For every n let $F_n(z) = F(r_n z)$. Then F_n is analytic in $\mathrm{Cl}(\mathbf{B})$ and F_n' has $p - 1$ zeros z_i/r_n, $1 \leqq i \leqq p - 1$. Moreover, if for every n we choose

$$T_n(\theta) = \theta + \arg F_n'(e^{i\theta})$$

as a differentiable function, then T_n resembles T in Lemma 3.1 with $\epsilon = 1/n$. From Lemma 3.2 it follows that there are functions $f_n \in S_{w'}(p)$ such that

$$|\arg(zF_n'/f_n)| < \pi/2 + 1/2n, \quad z \in \mathbf{B}.$$

It is evident that each f_n can be normalized so that the first nonzero coefficient of its Maclaurian's expansion is of modulus 1. Let $Q_n = zF_n'/f_n$. Observe that $\{Q_n(0):n \in \mathbf{N}\}$ is bounded. Hence, because of the latter inequality there is an increasing sequence $(n(k))_{k=1}^{\infty}$ of positive integers and a function Q of positive real part such that $Q_{n(k)} \to Q$ locally uniformly in $\mathbf{B}$. If we write $f_n = zF_n'/Q_n$, then since $F_n \to F$ locally uniformly in $\mathbf{B}$ and Q, Q_n are never zero for all n,

$$f_{n(k)} \to f = zF'/Q$$

locally uniformly in $\mathbf{B}$. Note that $f(0) = 0$ and f has exactly p zeros in $\mathbf{B}$. By Definition 2.4 $f \in S_{w'}(p)$. Since

$$\operatorname{Re}(zF'/f) = \operatorname{Re} Q > 0,$$

F satisfies condition (F).

(A) $\Rightarrow$ (J). Suppose F satisfies condition (A). It follows directly from Theorem 2.8 in [**9**] that there exists a pair of P, ϕ where P is a polynomial of degree p and $\phi \in S$, such that $\mathbf{C} - \phi(\mathbf{B})$ is a union of a collection, W, of P-rays with the properties: Each ray starts from the boundary of $\phi(\mathbf{B})$, and for any two rays either they have disjoint interiors or one is a subset of the other.

We construct via W a ruling, $\mathscr{L}$, of $\mathbf{C} - \phi(\mathbf{B})$ consisting of P-rays which start from $\partial\phi(\mathbf{B})$ and have mutually disjoint interiors.

Let $\mathscr{L}_0$ be the collection of all P-rays or lines, l, such that l is the limit in the Riemann sphere of a sequence in W. It is easy to see that no two members of $\mathscr{L}_0$ intersect; and that every $l \in \mathscr{L}_0$ starts from $\partial\phi(\mathbf{B})$ if it is a ray, and meets $\partial\phi(\mathbf{B})$ if it is a line. Because $\phi(\mathbf{B})$ is connected, there are at most $2p$ disjoint lines in $\mathscr{L}_0$. Delete from $\mathscr{L}_0$ every ray that is contained in a line in $\mathscr{L}_0$, and let $\mathscr{L}_1$ be the resulting collection. Consider the relation

R $: l \mathscr{R} l'$ for any $l, l' \in \mathscr{L}_1$ if either $l \subset l'$ or $l' \subset l$. It is easy to verify that $\mathscr{R}$ is an equivalence relation on $\mathscr{L}_1$.

Let $\mathscr{L}_2$ be the collection of all P-rays or lines, l, such that l is the set union of all members of an equivalence class of $\mathscr{L}_1$. Note that no ray of $\mathscr{L}_2$ is contained in another, a line belongs to $\mathscr{L}_2$ if and only if it belongs to $\mathscr{L}_1$, and $\mathscr{L}_2 \subset \mathscr{L}_1$. Moreover, $\mathscr{L}_2$ may contain pairs l, l' such that $l \cap l'$ is a proper line segment with end points the initial points of l and l'. Then $d = l \cup l'$ is a P-line. Observe that l, l' is the only pair of lines in $\mathscr{L}_2$ that determines d; and that any two lines d, d' are either disjoint or coinciding. But there are at most $2p$ P-lines in $\mathbf{C}$ meeting $\partial\phi(\mathbf{B})$. Therefore, by replacing every pair l, l' above and every line in $\mathscr{L}_2$ by a pair of rays of disjoint interiors starting from $\partial\phi(\mathbf{B})$ and whose union is d and the line, respectively, we obtain the desired collection $\mathscr{L}$. This completes the proof.

(J) $\Rightarrow$ (K) $\Rightarrow$ (L) $\Rightarrow$ (M). This is trivial.

(M) $\Rightarrow$ (A). Suppose F satisfies condition (M). Since $\partial\phi$ is an infinite and closed subset of $\mathbf{C}$, there is a countable dense subset $\{z_n\}_{n=1}^{\infty}$ of $\partial\phi$ whose closure is $\partial\phi$. By virtue of Lemma 3.3, for every n there is a finite set of mutually disjoint P-rays containing the points $z_1, z_2, \ldots, z_n$ and not meeting $\phi(\mathbf{B})$. Let ϕ be the conformal map from $\mathbf{B}$ onto the plane cut along these rays such that $\phi_n(0) = 0$ and $\phi_n'(0) > 0$; and let

$$F_n = P \circ \phi_n.$$

It follows directly from the Carathéodory Kernel Theorem that $\phi_n \rightarrow \phi$ locally uniformly in $\mathbf{B}$, and consequently $F_n \rightarrow F$ locally uniformly in $\mathbf{B}$. But from Theorem 2.8 in **[9]** every $F_n \in K_w(P)$. Since K_w is closed, F belongs to $K_w(P)$, and F satisfies condition (A).

The proof of the last part of the theorem follows from **[10]**. This completes the proof.

We close this section by the following:

Definition 3.1. Let $K_g(p)$ be the class of functions F analytic in $\mathbf{B}$, with $F(0) = 0$, such that F' has exactly $p - 1$ zeros in $\mathbf{B}$, and F satisfies one of the conditions (A), (B), $\ldots$, (M). We call $K_g(p)$ the class of *geometrically close-to-convex functions of order p*.

4. $K_g(p)$ and Bazilevič functions. We deal here with the special class, $B'(\alpha)$, of *Bazilevič functions of order* α.

Definition. 4.1. Let $B'(\alpha)$, $0 < \alpha < \infty$, be the class of all functions

$$\phi(z) = \left[\alpha \int_0^z g(\zeta) h^{\alpha}(\zeta) \zeta^{-1} d\zeta\right]^{1/\alpha},$$

where $h(\zeta) = \zeta + \ldots \in S^*$, and $g(\zeta) = 1 + a_1\zeta + \ldots$ satisfies

$$\mathrm{Re}(e^{i\beta} g) > 0 \quad \text{for some } \beta \in \mathbf{R}.$$

The class $K_g(p)$ leads to the following subclass of S:

Definition 4.2. Let $B_g(p)$ be the class of all functions $\phi \in S$ such that $P \circ \phi \in K_g(p)$ for some polynomial P of degree p.

We relate the classes $B'(p)$ and $B_g(p)$ as follows:

THEOREM 4.1. (a) $B'(1) = B_g(1) = K$.
(b) *For* $p \geqq 2$, $B'(p)$ *is a proper subset of* $B_g(p)$. *In particular*,

$$B'(p) = \{\phi \in S{:}a\phi^p \in K_g(p), a \in \mathbf{C}\}.$$

Proof. (a) This follows at once from the above two definitions.
(b) We show first the set equality. Let $\phi \in B'(p)$. Because of Definition 4.1 we can write:

$$\operatorname{Re}\{e^{i\beta}z([h(z)]^p)'/[\phi(z)]^p\} > 0, \quad z \in \mathbf{B},$$

where $h(z) = z + \ldots \in S^*$. Since $h^p \in S_a(p) \subset S_{w'}(p)$, Theorem 3.1 implies

$$e^{i\beta}\phi^p \in K_g(p).$$

Conversely, if for some function ϕ, $A\phi^p \in K_g(p)$, then again by Theorem 3.1 (condition (F)) $\phi \in B'(p)$. This proves our set equality.

It remains to show that $B'(p)$ is a proper subset of $B_g(p)$. We do so by constructing a function $\phi \in B_g(p)$. Let Q be a polynomial of degree p, with $Q(0) = 0$, which is not of the form az^p. It is not hard to see that there is a Q-ray, L, starting from the origin which is neither a euclidean ray nor contains any of the critical values of Q except, possibly, zero. Let l be a proper subray of L, and let ψ be a conformal map from $\mathbf{B}$ onto $\mathbf{C} - l$ such that $\psi(0) = 0$. Also, let

$$\phi = \psi/\psi'(0) \quad \text{and} \quad P(z) = Q(\psi'(0)z),$$

so that ϕ belongs to S and it maps $\mathbf{B}$ univalently onto the complex plane minus a P-ray which is a rotation of l. According to Theorem 3 (Condition (J), (K), or (L)) $P \circ \phi \in K_g(p)$. Hence $\phi \in B_g'(p)$. On the other hand, if $\phi^p \in K_g(p)$, then l is a z^p-ray. This implies that L is also a z^p-ray. Since L starts from the origin, L must be a radial slit; a contradiction.

Because of the randomness of P and l in the proof, one concludes that $B_g(p)$ is too large in comparison with $B'(p)$.

COROLLARY 4.1. $B'(p)$ *is a subset of* S.

This is a special case of a more general result due to Bazilevič [**1**] (see also [**11**]).

COROLLARY 4.2. *A univalent function* ϕ *belongs to* $B'(p)$ *if and only if* $\mathbf{C} - \phi(\mathbf{B})$ *is a union of* z^p*-rays of disjoint interiors.*

This result was first proved in the general for all p, $0 < p < \infty$, by Prokhorov [**12**]. Prokhorov's proof uses Sheil-Small's characterization of Bazilevič functions (see [**13**]) and Lewandowski's method of constructing a subordinating homotopy chain.

Here we give an alternative proof. The proof is based on Corollary 4.2 and uses a couple of lemmas. The first is geometric in nature and is not difficult to prove. So, we state it without proof. The second is due to Keogh and Miller [**5**], and it is analogous to the first for the classes $B'(\alpha)$.

LEMMA 4.1. *Let α be a positive real number, and let m be a positive integer. Suppose that $f \in S$. Let g be the m-fold symmetric function of f, that is,*

$$g(z) = [f(z^m)]^{1/m}.$$

The $\mathbf{C} - f(\mathbf{B})$ is a union of $z^{\alpha/m}$-rays of disjoint interiors if and only if $\mathbf{C} - g(\mathbf{B})$ is a union of z^{α}-rays of disjoint interiors.

LEMMA 4.2. *Under the assumptions of the above lemma, the function g belongs to $B'(\alpha)$ if and only if f belongs to $B'(\alpha/m)$.*

Now we have:

Proof of Corollary 4.2 *for any positive real p.* It suffices to consider the case when $p = n/m$, where n and m are positive integers, for the irrational case would then follow by a natural limiting procedure (see [**12**]). Suppose that $f \in B'(n/m)$, and let

$$g = [f(z^m)]^{1/m}.$$

By Lemma 4.2 $g \in B'(n)$. Then from Corollary 4.2 and Lemma 4.1 it follows at once that $\mathbf{C} - f(\mathbf{B})$ is a union of $z^{\alpha/m}$-rays of disjoint interiors. Conversely, suppose that f satisfies the latter property. Then from Lemma 4.1 it follows that f satisfies that $\mathbf{C} - g(\mathbf{B})$ is a union of z^n-rays of disjoint interiors. Again, by Corollary 4.2 $g \in B'(n)$. This by Lemma 4.2 puts f in $B'(n/m)$, and the proof is complete.

5. *Acknowledgement.* The author thanks the referee for his helpful suggestions, King Fahd University of Petroleum and Minerals for its support, and the Open University for its hospitality during the time this paper has been revised.

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