THE SEMIGROUP OF ENDOMORPHISMS OF A BOOLEAN RING

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1. Introduction

The family $\mathscr{E}(R)$ of all endomorphisms of a ring R is a semigroup under composition. It follows easily that if R and T are isomorphic rings, then $\mathscr{E}(R)$ and $\mathscr{E}(T)$ are isomorphic semigroups. We devote ourselves here to the converse question: 'If $\mathscr{E}(R)$ and $\mathscr{E}(T)$ are isomorphic, must R and T be isomorphic?' As one might expect, the answer is, in general, negative. For example, the ring of integers has precisely two endomorphisms – the zero endomorphism and the identity automorphism. Since the same is true of the ring of rational numbers, the two endomorphism semigroups are isomorphic while the rings themselves are certainly not.

One might expect, however, that there exist nontrivial classes of rings such that any two rings from the same class are isomorphic if and only if their endomorphism semigroups are isomorphic. One purpose of this paper is to show that the family of all Boolean rings is such a class. This is the content of Theorem A in section 2. By a Boolean ring, we mean any ring with identity with the property that every element is idempotent.

In section 3, we derive a result for endomorphisms of finite Boolean rings which is analogous to Howie's Theorem for transformations on a finite set. In [3], Howie has shown that any transformation on a finite set which is not bijective, is the product of idempotent transformations (where multiplication is composition). It is shown here in Section 3 that any endomorphism of a finite Boolean ring which is not an automorphism is the product of idempotent endomorphisms. We remark that Howie's Theorem implies immediately that any such endomorphism is the product of idempotent transformations. The task is to show that the transformations can, in fact, be taken to be endomorphisms.

2. The isomorphism theorem

THEOREM A. *Two Boolean rings R and T are isomorphic if and only if the endomorphism semigroups $\mathscr{E}(R)$ and $\mathscr{E}(T)$ are isomorphic.*

It seems to be convenient to record as lemmas several facts which play an important role in the proof of the theorem. First, we settle on some notation. The

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symbol $\mathscr{B}(X)$ is used to denote the Boolean ring of all clopen subsets (sets which are simultaneously both closed and open) of a topological Hausdorff space X where addition and multiplication are defined by

$$A+B = (A \cup B)-(A \cap B),\ AB = A \cap B.$$

The symbol $P(X)$ is used to denote the semigroup, under composition of all continuous functions whose domain is a clopen subset of X and whose range is contained in X. The 'empty function' will belong to this semigroup and will be denoted by the letter e.

LEMMA 1. *Let X be a compact totally disconnected space. Then the semigroup $\mathscr{E}(\mathscr{B}(X))$ of all endomorphisms of $\mathscr{B}(X)$ is anti-isomorphic to $P(X)$.*

PROOF. Suppose φ is any endomorphism of $\mathscr{B}(X)$. Then for any $A \in \mathscr{B}(X)$, we have

$$\varphi(X) = \varphi(A+\mathscr{C}_X A) = \varphi(A)+\varphi(\mathscr{C}_X A) \tag{1}$$

and

$$\phi = \varphi(\phi) = \varphi(A \cdot \mathscr{C}_X A) = \varphi(A) \cdot \varphi(\mathscr{C}_X A) \tag{2}$$

where $\mathscr{C}_X A$ denotes the complement of A with respect to X. Therefore, if we denote $\varphi(X)$ by Y, it follows from (1) and (2) that

$$\varphi(\mathscr{C}_X A) = \mathscr{C}_Y \varphi(A). \tag{3}$$

We assume that φ is a nonzero endomorphism and since X is the identity of $\mathscr{B}(X)$, it follows that $Y = \varphi(X) \neq \phi$.

Next, we define a function h from Y into X. For each point $y \in Y$, let

$$M = \{A \in \mathscr{B}(X) : y \notin \varphi(A)\}. \tag{4}$$

If $A \notin M$, then $y \in \varphi(A)$ and it follows from (3) that $y \notin \varphi(\mathscr{C}_X A)$. Hence $\mathscr{C}_X A \in M$ and since $X = A+\mathscr{C}_X A$, we conclude that the ideal generated by M together with A is all of $\mathscr{B}(X)$. Thus, M is a maximal ideal of $\mathscr{B}(X)$. By theorem 16.17 [2, p. 247], the clopen subsets of X form a basis and it follows that X is 0-dimensional in the sense of [5, p. 755]. Consequently, theorem (2.6) of [5, p. 757] implies that there exists a (necessarily unique) point $z \in X$ such that

$$M = \{A \in \mathscr{B}(X) : z \notin A\}. \tag{5}$$

We define the function h mapping Y into X by $h(y) = z$. From (4) and (5) we see that the following three statements are successively equivalent:

$$y \in \varphi(A),\ A \notin M,\ h(y) \in A.$$

Thus $\varphi(A) = h^{-1}[A]$ for each $A \in \mathscr{B}(X)$. Since the clopen subsets of X form a basis, it follows that h is a continuous mapping from Y into X. Therefore, we have shown that *for each nonzero endomorphism* φ *of* $\mathscr{B}(X)$, *there exists a (necessarily unique) continuous function* h *mapping* $\varphi(X)$ *into* X *such that*

$$\varphi(A) = h^{-1}[A] \text{ for each } A \in \mathscr{B}(X). \tag{6}$$

We are now in a position to define a mapping Φ from $\mathscr{E}(\mathscr{B}(X))$ into $P(X)$. We simply define $\Phi(\varphi) = h$ for φ different from the zero homomorphism φ_0 and put $\Phi(\varphi_0)$ equal to the empty function e. With this convention, condition (6) holds for φ_0 as well as for the nonzero endomorphisms. To show that Φ is an anti-homomorphism, suppose $\Phi(\varphi_1) = h_1$ and $\Phi(\varphi_2) = h_2$. Then for any $A \in \mathscr{B}(X)$,

$$\begin{aligned}(\varphi_1 \circ \varphi_2)(A) &= \varphi_1(\varphi_2(A)) \\ &= \varphi_1(h_2^{-1}[A]) = h_1^{-1}[h_2^{-1}[A]] \\ &= (h_2 \circ h_1)^{-1}[A].\end{aligned}$$

It follows that

$$\Phi(\varphi_1 \circ \varphi_2) = h_2 \circ h_1$$

and hence that

$$\Phi(\varphi_1 \circ \varphi_2) = \Phi(\varphi_2) \circ \Phi(\varphi_1).$$

If $\varphi_1 \neq \varphi_2$, then for some $A \in \mathscr{B}(X)$,

$$h_1^{-1}[A] = \varphi_1(A) \neq \varphi_2(A) = h_2^{-1}[A].$$

Thus, $h_1 \neq h_2$, i.e., $\Phi(\varphi_1) \neq \Phi(\varphi_2)$. Finally, if h is any continuous function from a clopen subset of X into X, one easily checks that the mapping φ defined by

$$\varphi(A) = h^{-1}[A]$$

for each $A \in \mathscr{B}(X)$ is an endomorphism ot $\mathscr{B}(X)$. Since $\Phi(\varphi) = h$, it follows that Φ is an anti-isomorphism from $\mathscr{E}(\mathscr{B}(X))$ onto $P(X)$.

REMARK. Implicit in the proof of Lemma 1 is the fact that if X and Y are compact totally disconnected spaces, then for each nonzero homomorphism φ from $\mathscr{B}(X)$ into $\mathscr{B}(Y)$, there exists a unique continuous function h mapping a nonempty clopen subset of Y into X such that $\varphi(A) = h^{-1}[A]$ for each $A \in \mathscr{B}(X)$.

For any space X, we use the symbol $S(X)$ to denote the semigroup, under composition, of all continuous selfmaps of X. Thus, $S(X)$ is the subsemigroup of $P(X)$ consisting of all those functions whose domains are all of X. The following result characterizes $S(X)$ algebraically within $P(X)$.

LEMMA 2. *A function* f *in* $P(X)$ *belongs to* $S(X)$ *if and only if for each* $g \neq e$ *in* $P(X)$, $f \circ g \neq e$.

PROOF. It is evident that if $f \in S(X)$ and $g \neq e$, then $f \circ g \neq e$. Suppose, on the other hand, $f \in P(X) - S(X)$. Then $H = C_X \mathscr{D}(f) \neq \phi$ ($\mathscr{D}(f)$ denotes the domain of f). Now let g be any continuous function from H into H (e.g., the identity function on H). We see that $g \neq e$ but $f \circ g = e$.

Now we are in a position to prove the theorem. It is immediate that if two Boolean rings R and T are isomorphic, then $\mathscr{E}(R)$ and $\mathscr{E}(T)$ are isomorphic. Suppose, conversely, that $\mathscr{E}(R)$ and $\mathscr{E}(T)$ are isomorphic. By the well-known representation theorem of Stone [6, p. 351], there exists a compact totally disconnected space X_R such that R is isomorphic to $\mathscr{B}(X_R)$ the Boolean ring of all clopen subsets of X_R. Let X_T denote the corresponding space for the Boolean ring T. It follows that $\mathscr{E}(R)$ is isomorphic to $\mathscr{E}(\mathscr{B}(X_R))$ and similarly that $\mathscr{E}(T)$ is isomorphic to $\mathscr{E}(\mathscr{B}(X_T))$. Consequently, $\mathscr{E}(\mathscr{B}(X_R))$ and $\mathscr{E}(\mathscr{B}(X_T))$ are isomorphic and Lemma 1 now implies that $P(X_R)$ and $P(X_T)$ are isomorphic. But we see from Lemma 2 that $S(X_R)$ can be characterized algebraically within $P(X_R)$. A similar remark holds for $S(X_T)$ and it follows that any isomorphism from $P(X_R)$ onto $P(X_T)$ must carry $S(X_R)$ isomorphically onto $S(X_T)$. We now appeal to Theorems 1 and 2 of [4, p. 295, 296] and conclude that X_R and X_T are homeomorphic. It follows immediately from this that $\mathscr{B}(X_R)$ and $\mathscr{B}(X_T)$ are isomorphic. Since R is isomorphic to the former and T to the latter, we conclude that R and T are isomorphic.

3. Endomorphisms of finite Boolean rings

In [3, p. 708, Theorem I] J. M. Howie proved that every transformation on a finite set which is not bijective is the product (multiplication in this case is composition) of a finite number of idempotent transformations. The theorem of this section gives the analogous result for endomorphisms of finite Boolean rings.

THEOREM B. *Every endomorphism of a finite Boolean ring which is not an automorphism is the product (multiplication is composition) of a finite number of idempotent endomorphisms.*

REMARK. Let R be a finite Boolean ring. As we mentioned in the introduction, it follows immediately from Howie's result that any endomorphism of R which is not an automorphism is the product of idempotent transformations. The result we prove states something more – the transformations can actually be taken to be endomorphisms.

PROOF. We appeal once again to Stone's Representation Theorem to conclude that there exists a compact totally disconnected space X_R such that R is isomorphic to $\mathscr{B}(X_R)$ the Boolean ring of all clopen subsets of X_R. Since R is finite, X_R must be finite. This, together with the fact that X_R is Hausdorff implies that it is discrete. It follows from Lemma 1 that $\mathscr{E}(R)$ is anti-isomorphic to $P(X_R)$ which, in this case, is the semigroup, under composition, of all functions whose domains and ranges are subsets of X_R. Now the nonunits of $\mathscr{E}(R)$ are the endomorphisms which are not automorphisms and the nonunits of $P(X_R)$ are the functions mapping subsets of X_R into subsets of X_R which are not bijections from X_R onto X_R. Since an anti-isomorphism between $\mathscr{E}(R)$ and $P(X_R)$ must correspond nonunits, we can

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complete the proof by showing that if f is any function such that $\mathscr{D}(f) \subset X_R$ and $\mathscr{R}(f) \subset X_R$ (where $\mathscr{D}(f)$ and $\mathscr{R}(f)$ denote respectively the domain and range of f) and f is not a bijection of X_R, then f is the product of idempotent functions. We consider three cases

(1) $\mathscr{R}(f)$ is not contained in $\mathscr{D}(f)$

(2) $\mathscr{R}(f)$ is properly contained in $\mathscr{D}(f)$

(3) $\mathscr{R}(f) = \mathscr{D}(f)$.

Case (2) follows immediately from Howie's Theorem I [3, p. 708]. To handle case (1), let $\mathscr{D}(g) = \mathscr{R}(f) \cup \mathscr{D}(f)$ and define

$$g(x) = f(x) \text{ for } x \in \mathscr{D}(f)$$
$$g(x) = x \text{ for } x \in \mathscr{R}(f) - \mathscr{D}(f).$$

Since $\mathscr{R}(f) - \mathscr{D}(f) \neq \phi$, it follows that g is a mapping from $\mathscr{D}(g)$ into $\mathscr{D}(g)$ which is not bijective. It then follows from Howie's theorem that g is the product of idempotent functions with domains equal to $\mathscr{D}(g)$ and ranges contained in $\mathscr{D}(g)$. Let $\mathscr{D}(i) = \mathscr{D}(f)$ and define $i(x) = x$ for each $x \in \mathscr{D}(f)$. Then i is also idempotent and since $f = g \circ i$, we conclude that f is the product of idempotents.

As for case (3), $\mathscr{R}(f) = \mathscr{D}(f)$ cannot be all of X_R since then f would be a bijection. Choose $p \in X_R - \mathscr{D}(f)$ and let g be any function such that

$$\mathscr{D}(g) = \mathscr{D}(f) \cup \{p\}$$
$$g(x) = f(x) \text{ for } x \in \mathscr{D}(f)$$
$$g(p) \in \mathscr{R}(f).$$

Then g is a mapping from $\mathscr{D}(g)$ into $\mathscr{D}(g)$ which is not a bijection and is therefore the product of idempotents. As above, let $\mathscr{D}(i) = \mathscr{D}(f)$ and $i(x) = x$ for all $x \in \mathscr{D}(f)$. Since $f = g \circ i$, the proof is complete.

Without the restriction that the rings under consideration are finite, the statement of Theorem B becomes false. To see this, let X be any infinite set and let $\mathscr{B}_X$ denote the Boolean ring of all subsets of X. Let h be any function which maps X injectively onto a proper subset of itself and define an endomorphism φ of $\mathscr{B}_X$ by $\varphi(A) = h[A]$ for each $A \in \mathscr{B}_X$. Then φ is injective but is not an automorphism. It follows that φ is not the product of idempotent endomorphisms. In fact, one can show if an endomorphism ψ is injective and is the product of idempotent endomorphisms, then ψ must be the identity automorphism. For suppose $\psi = \alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_n$ where each α_i is idempotent. Since ψ is injective, each α_i is injective and since each α_i is idempotent, each one is the identity on its range. These two facts imply that the range of each α_i is all of $\mathscr{B}_X$ and hence that each α_i is the identity automorphism.

We close with some remarks about Lemma 1. This result indicates that for a Boolean ring R, $\mathscr{E}(R)$ can have many varied subsemigroups. In fact, *given any semigroup S, one can produce a Boolean ring R such that S is isomorphic to a subsemigroup of $\mathscr{E}(R)$. Moreover, if S is finite, then R can be taken to be finite.* As above, let $\mathscr{B}_X$ denote the Boolean ring of all subsets of a finite set X. By Lemma 1, $\mathscr{E}(\mathscr{B}_X)$ is anti-isomorphic to $P(X)$ which in this case consists of all functions with domains and ranges contained in X. $P(X)$ contains as a subsemigroup the family $S(X)$ of all selfmaps of X. But $S(X)$ is anti-isomorphic to $\mathscr{T}_X$, the full transformation semigroup on X [1, p. 2]. Since every finite semigroup can be embedded in any $\mathscr{T}_X$ when the cardinality of X exceeds that of the given semigroup, it follows that any finite semigroup can be embedded in $\mathscr{E}(\mathscr{B}_X)$ for finite X with suitably many elements.

In case the semigroup under consideration is infinite, we modify the previous argument somewhat. First of all, let X be any discrete space and let βX denote its Stone-Čech compactification. We show that $S(X)$ can be embedded in $S(\beta X)$. Each f in $S(X)$ can be regarded as a continuous function from X into βX and, by a well-known property of βX, has an extension f^E to a continuous selfmap of βX. Define a mapping φ from $S(X)$ into $S(\beta X)$ by $\varphi(f) = f^E$. We observe that for all f, g in $S(X)$, $(f \circ g)^E$ and $f^E \circ g^E$ agree on the dense subset X and hence must be identical. Thus, φ is a homomorphism and since it is injective, we conclude that $S(\beta X)$ contains an isomorphic copy of $S(X)$. We recall once again that $S(X)$ is anti-isomorphic to $\mathscr{T}_X$, the full transformation semigroup on X. Since any semigroup can be embedded in a full transformation semigroup on a suitably large set, it follows that any semigroup is anti-isomorphic to a subsemigroup of $S(\beta X)$ for large enough X. Since this is a subsemigroup of $P(\beta X)$ which, by Lemma 1, is anti-isomorphic to $\mathscr{E}(\mathscr{B}(\beta X))$, the desired conclusion follows.

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