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A MULTITYPE CONTACT PROCESS WITH FROZEN SITES: A SPATIAL MODEL OF ALLELOPATHY

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Abstract

In this paper, we introduce a generalization of the two-color multitype contact process intended to mimic a biological process called allelopathy. To be precise, we have two types of particle. Particles of each type give birth to particles of the same type, and die at rate 1. When a particle of type 1 dies, it gives way to a frozen site that blocks particles of type 2 for an exponentially distributed amount of time. Specifically, we investigate in detail the phase transitions and the duality properties of the interacting particle system.

Keywords: Interacting particle system; multitype contact process; competition model; allelopathy; phase transition; duality

2000 Mathematics Subject Classification: Primary 60K35

1. Introduction

The model we introduce in this paper is a continuous-time Markov process in which the state at time t is a function $\xi_t : \mathbb{Z}^d \to \{0, 1, 2, 3\}$. At time t, a site $x \in \mathbb{Z}^d$ is said to be occupied by a particle of type 1 (a 'type-1') or type 2 (a 'type-2') if $\xi_t(x) = 1$ or, respectively, $\xi_t(x) = 2$, and is said to be empty otherwise. We distinguish two types of empty site. Namely, at time t, a site $x \in \mathbb{Z}^d$ will be called a *free site* if $\xi_t(x) = 0$ and a *frozen site* if $\xi_t(x) = 3$. The evolution rules are defined as follows.

1. Each type-1 or type-2 tries to give birth onto each of its neighboring sites at rate λ_1 or λ_2, respectively. Here, the neighbors of a site $x \in \mathbb{Z}^d$ constitute the set of $y \in \mathbb{Z}^d$ such that $\|x - y\| \leq R$, where $\|\cdot\|$ is a norm and R a positive constant.
2. If the offspring of a type-1 is sent to a site in state 0 or 3, or the offspring of a type-2 is sent to a site in state 0, the birth occurs. Otherwise, it is suppressed.
3. Both types of particle die at rate 1. Type-1s give way to frozen sites and type-2s give way to free sites.
4. Frozen sites (state 3) become free (state 0) at rate $\gamma > 0$.

This process is a generalization of the multitype contact process (Neuhauser (1992)) in which type-1s inhibit the spread of type-2s by freezing the sites they have just occupied. Reciprocally, the multitype contact process is just the extreme case with $\gamma = \infty$, in which the state transition $3 \to 0$ is instantaneous. The interpretation we have in mind is that of a spatial model of *allelopathy*. In biology, allelopathy is defined as a process involving secondary metabolites produced by plants, micro-organisms, viruses, and fungi that influence the growth

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and development of biological systems. In our case, type-1s are the individuals of an *inhibitory* species and type-2s the individuals of a *susceptible* species. The reader especially interested in this biological process may refer to Durrett and Levin (1997). Their stochastic spatial model also is a generalization of the multitype contact process, but has only three states: $0 \equiv$ empty site, $1 \equiv$ inhibitory species, and $2 \equiv$ susceptible species. Particles of type 1 die at rate 1 and those of type 2 die at rate $1 + c \times$ (the number of neighbors in state 1), where $c > 0$ is a constant. That is, the particles of type 1 increase the death rate of the neighboring particles of type 2.

This stochastic spatial process precisely models the competition of the colicin-producing *Escherichia coli* bacterium and colicin-sensitive bacteria. The particle system we introduce in this paper, on the contrary, is more appropriate to the investigation of plant competitions involving inhibitory species such as *Hieracium pilosella*. In this case, the inhibitory species produces toxic substances that prevent susceptible species from developing for a certain amount of time.

To investigate our model, we first observe that if only type-2s are present, the process reduces to the basic contact process with parameter λ_2. In such a case, there exists a critical value $\lambda_c \in (0, \infty)$ such that if $\lambda_2 \leq \lambda_c$ the process converges in distribution to the all-empty state, while if $\lambda_2 > \lambda_c$ there exists a stationary measure μ_2 that concentrates on configurations with infinitely many type-2s (see, e.g. Liggett (1999)). If only type-1s are present, we have almost the same result: if $\lambda_1 \leq \lambda_c$ then the process converges in distribution to the all-empty state, while if $\lambda_1 > \lambda_c$ there exists a nontrivial stationary measure ν_1 that concentrates on configurations with infinitely many type-1s and frozen sites. To construct this measure, we start the process from a configuration with infinitely many type-1s, take the Cesaro average of the distributions from time 0 to time T, and extract a convergent subsequence. Then, by Proposition 1.8 of Liggett (1985), the limit ν_1 is known to be an invariant measure. Moreover, since the type-1s do not see the frozen sites, we obtain $\nu_1(\xi_t(x) = 1) = \mu_2(\xi_t(x) = 2)$, provided that $\lambda_1 = \lambda_2$. To avoid trivialities, we assume from now on that both λ_1 and λ_2 are greater than λ_c and that ξ_0, the configuration at time 0, contains infinitely many type-1s and type-2s.

We first choose rates γ_a and γ_b, $\gamma_a < \gamma_b$, and denote by ξ_t^i the process with parameters λ_1, λ_2, and γ_i, $i = a, b$. Then, if we think of the processes as being generated by Harris's graphical representation, we may run ξ_t^a and ξ_t^b simultaneously, starting from the same initial configuration, in such a way that ξ_t^a has more type-1s and fewer type-2s than ξ_t^b, i.e. for any $x \in \mathbb{Z}^d$, if $\xi_t^a(x) = 2$ then $\xi_t^b(x) = 2$, and if $\xi_t^b(x) = 1$ then $\xi_t^a(x) = 1$. The same coupling argument implies that the process is also monotone with respect to each of the parameters λ_1 and λ_2. These results are summarized in the following theorem.

Theorem 1. *Let $\Theta_t^i = \{x \in \mathbb{Z}^d : \xi_t(x) = i\}$ be the set of sites occupied at time t by a particle of type i. Then the survival probabilities $\mathrm{P}(\Theta_t^i \neq \varnothing$ for all $t \geq 0)$, $i = 1, 2$, are monotone with respect to each of the parameters λ_1, λ_2, and γ.*

In particular, if we set $\gamma_a \in (0, \infty)$ and $\gamma_b = \infty$, then the process ξ_t^a will have more type-1s and fewer type-2s than will ξ_t^b. Now, as explained above, ξ_t^b is the multitype contact process with parameters λ_1 and λ_2. Theorem 1 of Neuhauser (1992) implies that if both $\lambda_1 > \lambda_2$ and we start with infinitely many type-1s, then $\xi_t^b \xrightarrow{\mathrm{W}} \mu_1$, the upper invariant measure of the basic contact process. Here, '$\xrightarrow{\mathrm{W}}$' denotes weak convergence. In particular, we obtain the following result.

Theorem 2. *Assume that ξ_0 contains infinitely many type-1s. If $\lambda_1 > \lambda_2$ and $\gamma \in (0, \infty)$ then $\xi_t \xrightarrow{\mathrm{W}} \nu_1$.*

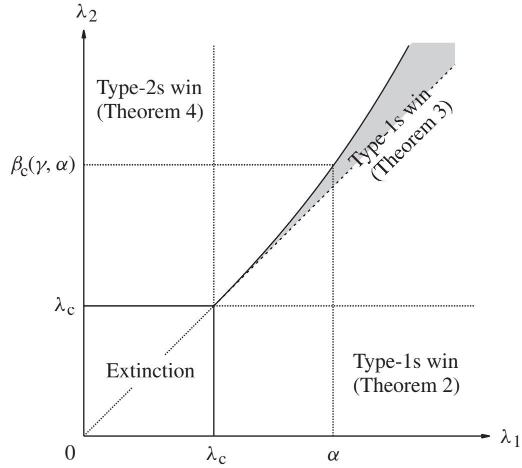

FIGURE 1: Phase diagram.

Consider the case $\lambda_1 = \lambda_2$. Since the evolution rules favor the type-1s, in this case we expect that the processes with and without frozen sites exhibit different behaviors. The following theorem tells us that if $\lambda_1 = \lambda_2$ and $\gamma < \infty$, the type-1s still 'win' in dimensions $d \geq 3$, while type-1s and type-2s coexist if $\gamma = \infty$ (see Theorem 3 of Neuhauser (1992)). We conjecture that the type-1s win in any dimension, but our proof relies heavily on the transience of symmetrical random walks in dimensions $d \geq 3$.

Theorem 3. *Assume that ξ_0 contains infinitely many type-1s and is translation invariant. If $\lambda_1 = \lambda_2$ and $d \geq 3$ then $\xi_t \xrightarrow{\text{w}} \nu_1$.*

The key to the proof of Theorem 3 is duality. The dual process starting at a space–time location (x, t) is defined, from a so-called Harris graphical representation, by going backwards in time, and allows us to keep track of the ancestors of the particle at site x at time t. Similarly to the (basic) multitype contact process, the dual of the process with frozen sites exhibits a tree structure that induces an ancestor hierarchy in which the members are arranged in the order they determine the *color* of the particle at (x, t). To prove that the introduction of frozen sites profoundly alters the limiting behavior of the process when $\lambda_1 = \lambda_2$, the basic idea is to show that the number of frozen sites visited by the first ancestor on its way up to (x, t) tends to infinity with t. This will imply that the probability that site x is occupied by a type-2 vanishes in this limit.

If we now consider the case $\lambda_1 < \lambda_2$, it is not clear that the type-2s win. Theorem 4 tells us that, for $d = 2$, the particles of type 2 win provided that γ is sufficiently large.

Theorem 4. *Assume that ξ_0 contains infinitely many type-2s. If $d = 2$ and $\lambda_1 < \lambda_2$ then there exists a critical value $\gamma_c \in (0, \infty)$ such that $\xi_t \xrightarrow{\text{w}} \mu_2$ for any $\gamma > \gamma_c$.*

To find the implications of our results, we fix $\lambda_1 > \lambda_c$ and $\gamma > 0$ and denote by $\beta_c(\gamma, \lambda_1)$ the infimum of $\{\lambda_2 \geq 0\}$ such that the type-1s die out, with the convention $\inf \varnothing = \infty$. A fairly straightforward application of Theorems 1–4 then implies that the mapping $\lambda_1 \mapsto \beta_c(\gamma, \lambda_1)$ is nondecreasing and $\beta_c(\gamma, \lambda_1) \downarrow \lambda_1$ as $\gamma \uparrow \infty$. In conclusion, the phase diagram we obtain is given by Figure 1, where our results are summarized.

Unfortunately, we do not know the outcome of the competition when the particles evolve in a spatial structure and λ_1 and λ_2 are such that $\lambda_1 < \lambda_2 < \beta_c(\gamma, \lambda_1)$. To deal with this case, we look at the mean field model (Durrett and Levin (1994)), that is, we pretend that all the sites are independent and that the system is spatially homogeneous. The evolution can then be formulated using the following ordinary differential equations, where u_i denotes the density of sites in state i:

$$\begin{aligned}
u_1' &= \lambda_1 u_0 u_1 + \lambda_1 u_3 u_1 - u_1,\\
u_2' &= \lambda_2 u_0 u_2 - u_2,\\
u_3' &= u_1 - \lambda_1 u_1 u_3 - \gamma u_3.
\end{aligned}$$

Let

$$\Omega = \{u = (u_0, u_1, u_2, u_3) \colon u_0 \geq 0,\ u_1 \geq 0,\ u_2 \geq 0,\ u_3 \geq 0,\ u_0 + u_1 + u_2 + u_3 = 1\}$$

be the collection of values we are interested in, and, for a fixed $\gamma > 0$, set

$$\begin{aligned}
D_1 &= \{(\lambda_1, \lambda_2) \colon \lambda_1 > 1,\ (\lambda_2 - \lambda_1)\gamma < (\lambda_1 - 1)\lambda_1\},\\
D_2 &= \{(\lambda_1, \lambda_2) \colon \lambda_2 > 1,\ \lambda_2 > \lambda_1\}.
\end{aligned}$$

A straightforward calculation shows that the system of ordinary differential equations has a nontrivial fixed point u, on the boundary $u_2 = 0$, if and only if $\lambda_1 > 1$ (where 'nontrivial' means that $u \neq (1, 0, 0, 0)$). Moreover, by studying the eigenvalues of the linearization of the equations at point u, we can prove that the equilibrium u is stable if $(\lambda_1, \lambda_2) \in D_1$ and unstable otherwise; that is, the linearization has an unstable direction that points into int Ω, the interior of Ω. Similarly, if $\lambda_2 > 1$ then there is a nontrivial equilibrium u, on the boundary $u_1 = u_3 = 0$, that is stable if $(\lambda_1, \lambda_2) \in D_2$ and unstable otherwise. Finally, the equations have a fixed point belonging to int Ω if and only if

$$\lambda_2 > \lambda_1 > 1 \quad \text{and} \quad \gamma < \frac{\lambda_1 - 1}{\lambda_2 - \lambda_1}\lambda_1,$$

that is, $(\lambda_1, \lambda_2) \in D_1 \cap D_2$. Our mean field model, however, exhibits the same property as the mean field model introduced in Durrett and Levin (1997). That is, the interior fixed point is not locally stable. See Figure 2 for a picture of the solution curves for $\gamma = 1$ and $\gamma = 1.5$ when $\lambda_1 = 2$ and $\lambda_2 = 3$. In words, if $(\lambda_1, \lambda_2) \in D_1 \cap D_2$ then no particle of either species can invade the other one in its equilibrium: if the density of particles of type 1 or type 2 is close to the corresponding equilibrium value, and particles of type 2 or, respectively, type 1 are introduced with a low density, then the density of introduced particles shrinks to 0. In a homogeneously mixing population, the outcome of the competition then depends on the initial densities. Based on the instability of the interior fixed point, the author believes that, for the particle system, given a set of parameters $\lambda_1, \lambda_2 > \lambda_c$, and $\gamma > 0$, there is a stronger species that will win the competition, provided that ξ_0 contains infinitely many type-1s and type-2s. In conclusion, we summarize and complete Theorems 2–4 with the following conjecture.

Conjecture 1. *For any $\lambda_1 > \lambda_c$ and $\lambda_2 > \lambda_c$, there is a critical value γ_c such that if $\gamma < \gamma_c$ then $\xi_t \xrightarrow{\text{w}} \nu_1$, while if $\gamma > \gamma_c$ then $\xi_t \xrightarrow{\text{w}} \mu_2$.*

The rest of the paper is devoted to proving the stated results. In Section 2, we will investigate in greater detail the duality properties of the process. Using the results of Section 2, we will then prove Theorem 3 in Section 3. Finally, the proof of Theorem 4 will be carried out in Section 4.

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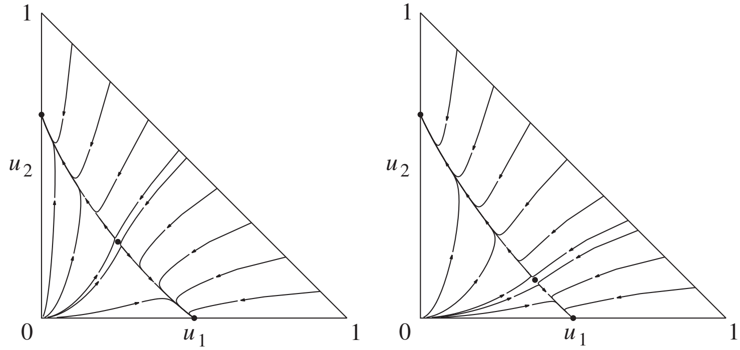

FIGURE 2: Projections of the solution curves for the mean field model, with $\lambda_1 = 2$, $\lambda_2 = 3$, and $\gamma = 1$ (*left*), and $\lambda_1 = 2$, $\lambda_2 = 3$, and $\gamma = 1.5$ (*right*).

2. Graphical representation and duality

The first step in proving Theorem 3 and Theorem 4 is to construct the process from a collection of Poisson processes, in the case $\lambda_1 \le \lambda_2$. For $x, y \in \mathbb{Z}^d$, $\|x - y\| \le R$, we let $\{T_n^{x,y},\ n \ge 1\}$, $\{U_n^x,\ n \ge 1\}$, and $\{V_n^x,\ n \ge 1\}$ be the arrival times of Poisson processes with respective rates λ_2, 1, and γ. At times $T_n^{x,y}$, we draw an arrow from x to y and, with probability $(\lambda_2 - \lambda_1)/\lambda_2$, label the arrow with a '2' (making it a '2-arrow'). If at time $T_n^{x,y}$ the site x is occupied by a type-1, the site y is empty (that is, free or frozen), and the arrow is unlabeled, then y becomes occupied by a type-1; while if x is occupied by a type-2 and y is free, then y becomes occupied by a type-2. At times U_n^x we put a cross at x to indicate that a death occurs, i.e. a type-1 gives way to a frozen site or a type-2 to a free site. Finally, at times V_n^x we put a dot at x to indicate that a frozen site becomes free. A result of Harris (1972) implies that such a graphical representation can be used to construct the process starting from any initial configuration $\xi_0 \colon \mathbb{Z}^d \to \{0, 1, 2, 3\}$ (see Figure 3 for an illustration).

With the graphical representation in hand, we are now ready to define the dual process. We say that two points (y, s) and (x, t) in $\mathbb{Z}^d \times \mathbb{R}^+$ are *connected* or that there is a *path* from (y, s) to (x, t) if there exists a sequence of times $s_0, \dots, s_{n+1}$, $s_0 = s < s_1 < s_2 < \cdots < s_n < s_{n+1} = t$, and spatial locations $x_0, \dots, x_n$, $x_0 = y, x_1, x_2, \cdots, x_n = x$, such that the following conditions hold.

1. For $i = 1, 2, \dots, n$, there is an arrow from x_{i-1} to x_i at time s_i.
2. For $i = 0, 1, \dots, n$, the vertical segments $\{x_i\} \times (s_i, s_{i+1})$ do not contain any crosses.

If there is a path from $(y, t - s)$ to (x, t), we say that there is a *dual path* from (x, t) to $(y, t - s)$, and define the *dual process starting at* (x, t), as for the multitype contact process, by setting

$$\tilde{\xi}_s^{(x,t)} = \{y \in \mathbb{Z}^d : \text{there is a dual path from } (x, t) \text{ to } (y, t - s)\}$$

for any $0 \le s \le t$. Since it will sometimes be easier to work with a forward process rather than a backward process, we also introduce the dual $\hat{\xi}_s^{(x,0)}$, defined by

$$\hat{\xi}_s^{(x,0)} = \{y \in \mathbb{Z}^d : \text{there is a path from } (x, 0) \text{ to } (y, s)\}.$$

The reader will note that the processes $\tilde{\xi}_s^{(x,t)}$ and $\hat{\xi}_s^{(x,0)}$ have the same law. Observe that $\{(\tilde{\xi}_s^{(x,t)}, s),\ 0 \le s \le t\}$ exhibits a tree structure that allows us to equip $\tilde{\xi}_s^{(x,t)}$ with an order

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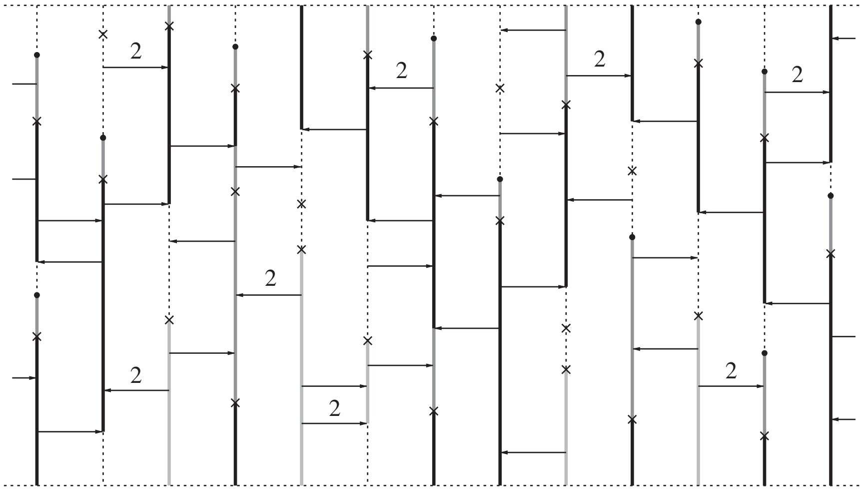

FIGURE 3: Harris's graphical representation. The black lines refer to type-1s, the pale gray lines to type-2s, the dark gray lines to frozen sites, and the dotted lines to free sites.

relation by which the members are arranged in the order they determine the color of site x at time t (see Neuhauser (1992); also, see the left-hand diagram of Figure 4 for an illustration of ancestor hierarchy). From now on, the tree

$$\Gamma = \{(\tilde{\xi}_s^{(x,t)}, s),\ 0 \le s \le t\}$$

will be called the *upper tree starting at* (x, t), and the elements of $\tilde{\xi}_s^{(x,t)}$ the *upper ancestors*. We let $\tilde{\xi}_s^{(x,t)}(n)$ denote the nth member of the ordered ancestor set, and call the first upper ancestor the *distinguished particle*.

The main difference with the multitype contact process is that type-1s now produce sites in state 3 that are forbidden for the type-2s. In particular, the color of (x, t) does not depend solely on the state of the upper ancestors at time 0. To determine from where the particle at site x originates (and prepare for the proof of Theorem 3), the basic idea is to extend the notion of path in the following way. If, instead of condition 2 holding,

3. the set $\bigcup_{i=0}^{n} \{x_i\} \times (s_i, s_{i+1})$ contains exactly one cross,

then we say that (y, s) and (x, t) are *weakly connected*. In such a case, the tree starting at (y, s) will be called a *lower tree* and the elements of $\tilde{\xi}_s^{(y,s)}$ the *lower ancestors*. We observe that, unlike in the multitype contact process, the state of some sites (free or frozen) strongly depends on the lower ancestors. That is, in view of the cross's effect, if (y, s) and (x, t) are weakly connected then a particle of type 1 at site y at time s can *freeze* the path of the distinguished particle at some particular points; this prevents type-2s from determining the color of (x, t).

To conclude this section, we describe an algorithm to determine the color of (x, t) in the case $\lambda_1 \le \lambda_2$. We say that an arrow from x to y is *bad for the type-2s* if its target site y is frozen. First, we determine whether the site that the distinguished particle lands on at time 0 is (i) in state 1, (ii) in state 2, or (iii) in state 0 or 3. In case (i), the distinguished particle will 'paint' (x, t) the color 1 if it does not cross a 2-arrow. Otherwise, we follow the path of the distinguished particle on its way up to (x, t) until a 2-arrow is first encountered, look backwards in time starting from the point where this arrow is attached, and discard all the ancestors of

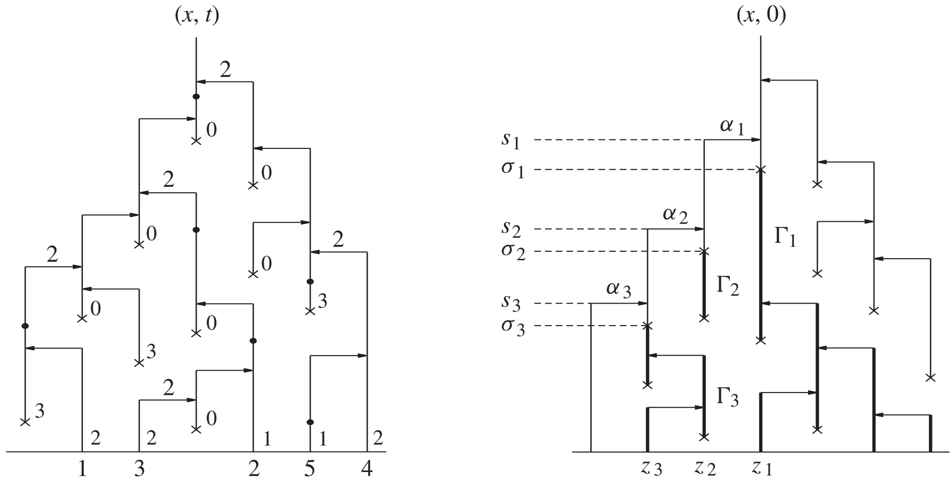

FIGURE 4: The dual process.

this point. We discard these ancestors because they are now blocked on their way up to (x, t) by a particle of type 1. In case (ii), the distinguished particle will paint (x, t) the color 2 if it does not cross any arrow bad for the type-2s. Otherwise, we similarly discard all the ancestors of the point where the first bad arrow is attached, since these ancestors are now blocked by a particle of type 2. Finally, in case (iii), the distinguished particle cannot paint (x, t) any color. If, after the first trial, the distinguished particle has not painted (x, t) any color, we repeat the same procedure with the first upper ancestor that is left after discarding, and so on.

We refer the reader to the left-hand diagram of Figure 4 for an illustration of this algorithm. The distinguished particle lands on a type-2 (case (ii)), but crosses the arrow bad for the type-2s that points from $x - 2$ to $x - 3$ before reaching (x, t), so we discard all the ancestors of this arrow. Since the only such ancestor is the distinguished particle, we now focus on the second ancestor of the hierarchy. The second ancestor lands on a type-1 (case (i)), but crosses the 2-arrow that points from x to $x - 1$. Since the ancestors of this 2-arrow are the second and third ancestors, we look at the fourth ancestor. The fourth ancestor lands on a type-2 (case (ii)) and does not cross any arrow bad for the type-2s, so (x, t) will be of type 2. The reader will note that, due to the dot under its tip, the 2-arrow on the right-hand side of the tree is not bad for the type-2s.

3. Proof of Theorem 3

To establish Theorem 3, our strategy is to prove that if the upper tree, Γ, lives forever, then with probability 1 the distinguished particle will jump infinitely often to a frozen site. To do this, we will focus on the structure of the lower trees and show that the number of lower ancestors that freeze the sites visited by the distinguished particle tends to infinity as $t \to \infty$. We will then conclude by showing that there exists an upper ancestor that will bring a type-1 to (x, t).

From now on, we denote by λ the common value of λ_1 and λ_2, and suppose that Γ lives forever. The reader will observe that such an event occurs with positive probability, since $\lambda > \lambda_c$. For more convenience, in this section we use the dual process $\hat{\xi}_s^{(x,0)}$ starting at $(x, 0)$. The main objective is to prove that the number of frozen sites visited by the distinguished particle tends to infinity as $t \to \infty$. We follow the path of the distinguished particle, starting

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from $(x, 0)$, and denote by α_n, $n \geq 1$, the nth arrow we cross (see the right-hand diagram of Figure 4). We let z_n and s_n respectively be the arrival site and the temporal location of the arrow α_n, and denote by N_t the number of arrows α_n that by time t point to a frozen site, i.e.

$$N_t = \mathrm{card}\{n \geq 1 \colon \xi_{s_n}(z_n) = 3 \text{ and } s_n \leq t\}.$$

By construction, N_t also denotes the number of frozen sites visited by the distinguished particle by time t. The main result we have to prove is the following proposition.

Proposition 1. *If $d \geq 3$ then $\lim_{t\to\infty} N_t = \infty$ almost surely.*

The intuitive idea of the proof is that the lower ancestors provide enough type-1s to freeze the path of the distinguished particle at infinitely many points. We denote by σ_n the arrival time of the first cross located under the tip of α_n, i.e.

$$\sigma_n = \min\{U_k^{z_n} \colon U_k^{z_n} \geq s_n\},$$

and let Γ_n be the lower tree starting at (z_n, σ_n), i.e.

$$\Gamma_n = \{(y, s) \in \mathbb{Z}^d \times [\sigma_n, \infty) \colon \text{there is a path from } (z_n, \sigma_n) \text{ to } (y, s)\}$$

(see Figure 4). We say that Γ_n is *good* if the following two conditions are satisfied.

1. Γ_n lives forever.
2. The vertical segment $\{z_n\} \times (s_n, \sigma_n)$ does not contain any dots.

As we will see, these conditions will allow us to freeze the site z_n at time s_n. Let G_n be the event that the nth lower tree is good.

Lemma 1. *We have* $\mathrm{P}(\limsup_{n\to\infty} G_n) = 1$.

Proof. We denote by A_n the event that Γ_n lives forever, and by B_n the event that $\{z_n\} \times (s_n, \sigma_n)$ does not contain any dots. The first step is to prove that, for any $n \geq 1$, there almost surely exists an integer $m \geq n$ such that A_m occurs. To do this, we set $\Gamma_{n_1} = \Gamma_n$ and, while Γ_{n_k} is bounded, we denote by $\Gamma_{n_{k+1}}$ the first lower tree that is born after Γ_{n_k} dies. Note that if A_{n_k} does not occur, then $\Gamma_{n_{k+1}}$ is well defined and the event $A_{n_{k+1}}$ is determined by parts of the graph that succeed the death of Γ_{n_k}, meaning that A_{n_k} and $A_{n_{k+1}}$ are independent. More generally, since the trees $\Gamma_{n_1}, \Gamma_{n_2}, \dots, \Gamma_{n_{k+1}}$ are disjoint, the events $A_{n_1}, A_{n_2}, \dots, A_{n_{k+1}}$ are independent. Moreover, the probability that A_{n_k} occurs is given by the survival probability $p(\lambda)$ of the basic contact process with parameter λ, starting from one infected site. Hence,

$$\begin{aligned} \mathrm{P}(A_n^{\mathrm{c}} \cap A_{n+1}^{\mathrm{c}} \cap \cdots) &\leq \lim_{k\to\infty} \mathrm{P}(A_{n_1}^{\mathrm{c}} \cap A_{n_2}^{\mathrm{c}} \cap \cdots \cap A_{n_k}^{\mathrm{c}}) \\ &\leq \prod_{k=1}^{\infty} \mathrm{P}(A_{n_k}^{\mathrm{c}}) \\ &= \lim_{k\to\infty} (1 - p(\lambda))^k = 0 \end{aligned}$$

for $\lambda > \lambda_{\mathrm{c}}$. In particular,

$$\mathrm{P}\Big(\limsup_{n\to\infty} A_n\Big) = \lim_{n\to\infty} \mathrm{P}(A_n \cup A_{n+1} \cup \cdots) = 1.$$

This proves that, with probability 1, there exist infinitely many lower trees Γ_n that live forever. Furthermore, since $\sigma_n - s_n$ is exponentially distributed with parameter 1, we have

$$\mathrm{P}(B_n) = \mathrm{P}(\sigma_n - s_n \leq V_1^{z_n}) = \gamma^{-1}(1+\gamma)^{-1} > 0.$$

By independence, we can finally conclude that $\mathrm{P}(\limsup_{n\to\infty} A_n \cap B_n) = 1$.

To complete the proof of Proposition 1, we now consider, for any $n \geq 1$ and $s \geq \sigma_n$, the time translation dual process

$$\hat{\xi}_s^{(z_n,\sigma_n)} = \{y \in \mathbb{Z}^d : \text{there is a path from } (z_n, \sigma_n) \text{ to } (y, s)\},$$

and denote by $\zeta_s(n)$ the associated distinguished particle, namely the first ancestor of (z_n, σ_n). Observe that if the lower tree Γ_n lives forever, then $\zeta_s(n)$ is well defined for any $s \geq \sigma_n$. Moreover, if we suppose that Γ_n is good and that $\zeta_s(n)$ lands on a type-1 then, in view of condition 2 above, the site z_n will be frozen at time s_n. In particular, if Γ_{n_k} is a subsequence of good trees given by Lemma 1, the proof of Proposition 1 can be completed with the aid of the following result.

Lemma 2. *Let* $\Omega_s = \{\zeta_s(n_k) : \sigma_{n_k} \leq s\}$ *and* Θ_s^1 *be the set of sites occupied at time* s *by a type-1. If* ξ_0 *is translation invariant and* $d \geq 3$*, then, starting from infinitely many type-1s, we have*

$$\lim_{t\to\infty} \operatorname{card}(\Omega_t \cap \Theta_0^1) = \infty \quad \textit{almost surely.}$$

Proof. By Proposition 2.1 of Neuhauser (1992), the path of $\zeta_s(n_k)$ can be broken into independent and identically distributed pieces in such a way that the process $\zeta_s(n_k)$ is transient in dimension $d \geq 3$ (see Neuhauser (1992, Sections 4 and 5) for a proof). This, together with Lemmas 7 and 8 of Lanchier (2005), implies that $\operatorname{card}(\Omega_t) \to \infty$. Finally, since ξ_0 is translation invariant, Lemma 9.14 of Harris (1976) tells us that, with probability 1,

$$\operatorname{card}(\Omega_t \cap \Theta_0^1) \to \infty \quad \text{as } t \to \infty.$$

To conclude the proof of Theorem 3, we now use the dual process $\tilde{\xi}_t^{(x,t)}$ and construct a sequence of upper ancestors $\eta_t^{(x,t)}(k)$, $k \geq 0$, that are candidates to paint (x, t) the color 1. The first member of the sequence will be the distinguished particle. Next, we renumber the sequence of frozen points (z_k, s_k), $k \geq 1$, visited by the distinguished particle by going forward in time, and denote by n_t the number of frozen points encountered. For each k, $1 \leq k \leq n_t$, we look backwards in time, starting from the location where the arrow α_k is attached, and discard all the ancestors of this particular point; we then define $\eta_t^{(x,t)}(k)$ to be the first upper ancestor that is left after discarding. Let $\eta_t = \{\eta_t^{(x,t)}(k),\ 0 \leq k \leq n_t\}$. Proposition 1 tells us that $\lim_{t\to\infty} n_t = \infty$ with probability 1. This, together with Lemmas 7 and 8 of Lanchier (2005), implies that the cardinality of η_t can be made arbitrarily large by choosing t to be sufficiently large. In particular, a new application of Lemma 9.14 of Harris (1976) yields

$$\lim_{t\to\infty} \mathrm{P}(\eta_t \cap \Theta_0^1 = \varnothing) = 0.$$

Hence, there exists at least one candidate that lands on a type-1. We denote by $\eta_t^{(x,t)}(k_0)$ the first upper ancestor belonging to η_t. Since the arrow α_{k_0} is bad for the type-2s, the upper ancestor $\eta_t^{(x,t)}(k_0)$ will finally paint (x, t) the color 1. This completes the proof of Theorem 3.

4. Proof of Theorem 4

In this section, we assume that $d = 2$, set $\lambda_1 < \lambda_2$, and prove that there is a $\gamma_c < \infty$ such that, for any $\gamma > \gamma_c$, the type-2s win. In view of the evolution rules, the survival of type-2s is not clear, and tools such as coupling and duality cannot be used to prove Theorem 4. We will instead rely on the rescaling argument described in Durrett and Neuhauser (1997, Section 3), which is valid in the case $\gamma = \infty$, and then prove that taking $\gamma > 0$ to be sufficiently large does not affect the process too much. We start by introducing suitable space and time scales. We let L be a positive integer and, for $z = (z_1, z_2)$ in $\mathbb{Z}^2$, set

$$\Phi(z) = (Lz_1, Lz_2), \qquad B = [-L, L]^2, \qquad B(z) = \Phi(z) + B.$$

Moreover, we tile $B(z)$ with $L^{1/10} \times L^{1/10}$ squares by setting

$$\pi(w) = (L^{1/10} w_1, L^{1/10} w_2),$$
$$D = (-\tfrac{1}{2}L^{1/10}, \tfrac{1}{2}L^{1/10}]^2, \qquad D(w) = \pi(w) + D,$$
$$I_z = \{w \in \mathbb{Z}^2 \colon D(w) \subseteq B(z)\}.$$

We say that $B(z)$ is *good* if $B(z)$ is devoid of type-1s and has at least one particle of type 2 in each of the squares $D(w)$, $w \in I_z$. For $z = (z_1, z_2) \in \mathbb{Z}^2$, with z_1 and z_2 both even for even k, and z_1 and z_2 both odd for odd k, we say that (z, k) is *occupied* if $B(z)$ is good at time kT, where T is an integer to be picked later. Moreover, we require this event to occur for the process restricted to the region $\Phi(z) + [-ML, ML]^2$. We start by assuming that $\gamma = \infty$.

Proposition 2. (Durrett and Neuhauser (1997).) *If $\lambda_2 > \lambda_1$ and $T = L^2$ then, for any $\varepsilon > 0$, the parameters L and M can be chosen so that the set of occupied sites dominates the set of open sites in an M-dependent oriented percolation process with parameter* $1 - \varepsilon$.

See Durrett and Neuhauser (1997, Proposition 3.1 and Lemma 3.7). To generalize the comparison to sufficiently large $\gamma > 0$, we only need to prove that, with probability close to 1, the process behaves like the multitype contact process (i.e. none of the type-2s is blocked by a frozen site) inside the space–time box

$$J(z) \times [0, T] \qquad \text{where } J(z) = \Phi(z) + [-\tfrac{1}{3}ML, \tfrac{1}{3}ML]^2$$

(see Lemma 3.7 of Durrett and Neuhauser (1997)). The event we are interested in occurs if and only if each site $x \in J(z)$ pointed to by an arrow by time T is not in state 3. To ensure that this occurs, we follow the line $\{x\} \times [0, T]$ forwards in time, and, each time we encounter a cross, put a dot at x before meeting the next arrow tip. Let $K(x, t)$ be the number of arrows that point to site x by time t. By decomposing according to whether or not $K(x, T) > 2\lambda_2 T$, we obtain

$$\begin{aligned} \mathrm{P}(\text{any of the type-2s is blocked}) &\le \sum_{x \in J(z)} \mathrm{P}(K(x, T) > 2\lambda_2 T) + 2\lambda_2 T \sum_{x \in J(z)} \mathrm{P}(U_1^x < V_1^x) \\ &\le (\tfrac{2}{3}ML)^2 (C\mathrm{e}^{-\alpha T} + 2\lambda_2 T \gamma^{-1}(\gamma + 1)^{-1}) \\ &\le \tfrac{1}{3}\varepsilon \end{aligned}$$

for T and γ sufficiently large and appropriate constants $C < \infty$ and $\gamma > 0$. By this point, we have proved that if $\lambda_1 < \lambda_2$ and γ is sufficiently large, then there exist an L and an M such that the set of occupied sites dominates the set of open sites in an M-dependent oriented percolation process with parameter $1-\varepsilon$. This almost proves Theorem 4. Our last problem is that oriented site percolation has a positive density of unoccupied sites. To prove that there is a region devoid of type-1s expanding in all directions, we apply a result of Durrett (1992) which shows that unoccupied sites do not percolate when ε is sufficiently close to 0. Since particles of neither type can appear spontaneously, once a region is devoid of one type, this type can only reappear in the region through invasion from the outside. This implies that our process has the desired property, and completes the proof of Theorem 4.

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