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SUMS OF MULTIPLICATIVE FUNCTIONS OVER A BEATTY SEQUENCE

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Abstract

We study sums involving multiplicative functions that take values over a nonhomogenous Beatty sequence. We then apply our result in a few special cases to obtain asymptotic formulas for quantities such as the number of integers in a Beatty sequence that are representable as a sum of two squares up to a given magnitude.

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1. Introduction

Let $A \geq 1$ be an arbitrary constant, and let $\mathcal{F}_A$ be the set of multiplicative functions f such that $|f(p)| \leq A$ for all primes p and

$$\sum_{n \leq N} |f(n)|^2 \leq A^2 N \quad (N \in \mathbb{N}). \tag{1}$$

Exponential sums of the form

$$S_{\alpha,f}(N) = \sum_{n \leq N} f(n) e(n\alpha) \quad (\alpha \in \mathbb{R},\ f \in \mathcal{F}_A), \tag{2}$$

where $e(z) = e^{2\pi i z}$ for $z \in \mathbb{R}$, occur frequently in analytic number theory. Montgomery and Vaughan have shown (see [8, Corollary 1]) that the upper bound

$$S_{\alpha,f}(N) \ll_A \frac{N}{\log N} + \frac{N(\log R)^{3/2}}{R^{1/2}} \tag{3}$$

holds uniformly for all $f \in \mathcal{F}_A$, provided that $|\alpha - a/q| \leq q^{-2}$ with some reduced fraction a/q for which $2 \leq R \leq q \leq N/R$. In this paper, we use the Montgomery–Vaughan result to estimate sums of the form

$$G_{\alpha,\beta,f}(N) = \sum_{\substack{n \le N \\ n \in \mathcal{B}_{\alpha,\beta}}} f(n), \tag{4}$$

where $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$, $f \in \mathcal{F}_A$, and $\mathcal{B}_{\alpha,\beta}$ is the *nonhomogenous Beatty sequence* defined by

$$\mathcal{B}_{\alpha,\beta} = \{n \in \mathbb{N} : n = \lfloor \alpha m + \beta \rfloor \text{ for some } m \in \mathbb{Z}\}.$$

Our results are uniform over the family $\mathcal{F}_A$ and nontrivial whenever

$$\lim_{N\to\infty} \frac{\log N}{N \log\log N} \left| \sum_{n \le N} f(n) \right| = \infty,$$

a condition which guarantees that the error term in Theorem 1 is smaller than the main term. One can remove this condition, at the expense of losing uniformity with respect to f, and still obtain Theorem 1 for any bounded arithmetic function f (not necessarily multiplicative) for which the exponential sums in (2) satisfy

$$S_{\alpha,f}(N) = o\bigg(\sum_{n \le N} f(n)\bigg) \quad (N \to \infty).$$

The general problem of characterizing functions for which this relation holds appears to be rather difficult; see [1] for Bachman's conjecture and his related work on this problem.

We shall also assume that α is irrational and of finite type τ. For an irrational number γ, the type of γ is defined by

$$\tau = \sup\left\{t \in \mathbb{R} : \liminf_{n\to\infty} n^t \, [\![\gamma n]\!] = 0\right\},$$

where $[\![\cdot]\!]$ denotes the distance to the nearest integer. *Dirichlet's approximation theorem* implies that $\tau \ge 1$ for every irrational number γ. According to theorems of Khinchin [6] and Roth [10], $\tau = 1$ for *almost all* real numbers (in the sense of the Lebesgue measure) and *all* irrational algebraic numbers γ, respectively; also see [2, 11].

Our main result is the following theorem.

THEOREM 1. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$, and suppose that α is irrational and of finite type. Then, for all $f \in \mathcal{F}_A$,*

$$G_{\alpha,\beta,f}(N) = \alpha^{-1} \sum_{n \le N} f(n) + O\left(\frac{N \log\log N}{\log N}\right),$$

where the implied constant depends only on α and A.

The following corollaries are immediate applications of Theorem 1.

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COROLLARY 2. *The number of integers not exceeding N that lie in the Beatty sequence $\mathcal{B}_{\alpha,\beta}$ and can be represented as a sum of two squares is*

$$\#\{n \le N : n \in \mathcal{B}_{\alpha,\beta},\, n = \square + \square\} = \frac{CN}{\alpha\sqrt{\log N}} + O\left(\frac{N \log\log N}{\log N}\right),$$

where

$$C = 2^{-1/2} \prod_{p \equiv 3 \mod 4} (1 - p^{-2})^{-1/2} = 0.76422365\ldots \tag{5}$$

is the Landau–Ramanujan constant.

To state the next result, we recall that an integer n is said to be *k-free* if $p^k \nmid n$ for every prime p.

COROLLARY 3. *For every $k \ge 2$, the number of k-free integers not exceeding N that lie in the Beatty sequence $\mathcal{B}_{\alpha,\beta}$ is*

$$\#\{n \le N : n \in \mathcal{B}_{\alpha,\beta},\ n \text{ is } k\text{-free}\} = \alpha^{-1}\zeta^{-1}(k)N + O\left(\frac{N \log\log N}{\log N}\right),$$

where $\zeta(s)$ is the Riemann zeta function.

Finally, we consider the average value of the number of representations of an integer from a Beatty sequence as a sum of four squares.

COROLLARY 4. *Let $r_4(n)$ denote the number of representations of n as a sum of four squares. Then*

$$\sum_{\substack{n \le N \\ n \in \mathcal{B}_{\alpha,\beta}}} r_4(n) = \frac{\pi^2 N^2}{2\alpha} + O\left(\frac{N^2 \log\log N}{\log N}\right),$$

where the implied constant depends only on α.

Any implied constants in the symbols O and $\ll$ may depend on the parameters α and A but are absolute otherwise. We recall that the notation $X \ll Y$ is equivalent to $X = O(Y)$.

2. Preliminaries

2.1. Discrepancy of fractional parts We define the *discrepancy* $D(M)$ of a sequence of real numbers $b_1, b_2, \ldots, b_M \in [0, 1)$ by

$$D(M) = \sup_{\mathcal{I} \subseteq [0,1)} \left| \frac{\mathcal{V}(\mathcal{I}, M)}{M} - |\mathcal{I}| \right|, \tag{6}$$

where the supremum is taken over all possible subintervals $\mathcal{I} = (a, c)$ of the interval $[0, 1)$, $\mathcal{V}(\mathcal{I}, M)$ is the number of positive integers $m \le M$ such that $b_m \in \mathcal{I}$, and $|\mathcal{I}| = c - a$ is the length of $\mathcal{I}$.

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If an irrational number γ is of finite type, we let $D_{\gamma,\delta}(M)$ denote the discrepancy of the sequence of fractional parts $(\{\gamma m + \delta\})_{m=1}^{M}$. By [7, Theorem 3.2, Ch. 2], we have the following result.

LEMMA 5. *For a fixed irrational number γ of finite type τ and for all $\delta \in \mathbb{R}$,*

$$D_{\gamma,\delta}(M) \le M^{-1/\tau+o(1)} \quad (M \to \infty),$$

where the function defined by $o(\cdot)$ depends only on γ.

2.2. Numbers in a Beatty sequence The following result is standard in characterizing the elements of the Beatty sequence $\mathcal{B}_{\alpha,\beta}$.

LEMMA 6. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$, and set $\gamma = \alpha^{-1}$ and $\delta = \alpha^{-1}(1-\beta)$. Then $n = \lfloor \alpha m + \beta \rfloor$ for some $m \in \mathbb{Z}$ if and only if $0 < \{\gamma n + \delta\} \le \gamma$.*

From Lemma 6, an integer n lies in $\mathcal{B}_{\alpha,\beta}$ if and only if $n \ge 1$ and $\psi(\gamma n + \delta) = 1$, where ψ is the periodic function with period one whose values on the interval $(0, 1]$ are given by

$$\psi(x) = \begin{cases} 1 & \text{if } 0 < x \le \gamma, \\ 0 & \text{if } \gamma < x \le 1. \end{cases}$$

We wish to approximate ψ by a function whose Fourier series representation is well behaved. This will give rise to the aforementioned exponential sum $S_{\alpha,f}(N)$. To this end, we use the result of Vinogradov (see [15, Ch. I, Lemma 12]) which states that for any Δ such that

$$0 < \Delta < \tfrac{1}{8} \quad \text{and} \quad \Delta \le \tfrac{1}{2}\min\{\gamma, 1-\gamma\},$$

there exists a real-valued function Ψ with the following properties:

(i) Ψ is periodic with period one;
(ii) $0 \le \Psi(x) \le 1$ for all $x \in \mathbb{R}$;
(iii) $\Psi(x) = \psi(x)$ if $\Delta \le \{x\} \le \gamma - \Delta$ or if $\gamma + \Delta \le \{x\} \le 1 - \Delta$;
(iv) Ψ can be represented by a Fourier series

$$\Psi(x) = \sum_{k \in \mathbb{Z}} g(k)\mathbf{e}(kx),$$

where $g(0) = \gamma$ and the Fourier coefficients satisfy the uniform bound

$$g(k) \ll \min\{|k|^{-1}, |k|^{-2}\Delta^{-1}\} \quad (k \ne 0). \tag{7}$$

3. Proofs

3.1. Proof of Theorem 1 Using Lemma 6, we rewrite the sum (4) in the form

$$G_{\alpha,\beta,f}(N) = \sum_{n \le N} f(n)\psi(\gamma n + \delta).$$

Replacing ψ by Ψ, we obtain

$$G_{\alpha,\beta,f}(N) = \sum_{n\le N} f(n)\Psi(\gamma n+\delta) + O\bigg(\sum_{n\in V(\Delta,N)} f(n)\bigg), \tag{8}$$

where $V(\Delta, N)$ is the set of positive integers $n \le N$ for which

$$\{\gamma n+\delta\} \in [0, \Delta) \cup (\gamma-\Delta, \gamma+\Delta) \cup (1-\Delta, 1).$$

Since the length of each interval above is at most 2Δ, it follows from definition (6) and Lemma 5 that

$$|V(\Delta, N)| \ll \Delta N + N^{1-1/(2\tau)},$$

where we have used the fact that α and γ have the same type τ. Thus, taking (1) into account, Cauchy's inequality gives

$$\begin{aligned}\bigg|\sum_{n\in V(\Delta,N)} f(n)\bigg| &\le \big|V(I, N)\big|^{1/2}\bigg(\sum_{n\le N}|f(n)|^2\bigg)^{1/2}\\ &\ll ((\Delta N)^{1/2} + N^{1/2-1/(4\tau)})N^{1/2}\\ &= \Delta^{1/2}N + N^{1-1/(4\tau)}.\end{aligned} \tag{9}$$

Next, let $K \ge \Delta^{-1}$ be a large real number (to be specified later), and let Ψ_K be the trigonometric polynomial given by

$$\Psi_K(x) = \sum_{|k|\le K} g(k)\mathbf{e}(kx) = \gamma + \sum_{0<|k|\le K} g(k)\mathbf{e}(kx) \quad (x\in\mathbb{R}). \tag{10}$$

Using (7), we see that the estimate

$$\Psi(x) = \Psi_K(x) + O(K^{-1}\Delta^{-1})$$

holds uniformly for all $x \in \mathbb{R}$; therefore,

$$\sum_{n\le N} f(n)\Psi(\gamma n+\delta) = \sum_{n\le N} f(n)\Psi_K(\gamma n+\delta) + O(K^{-1}\Delta^{-1}N), \tag{11}$$

where we have used the bound $\sum_{n\le N}|f(n)| \ll N$ which follows from (1).

Combining (8), (9), (10) and (11), we derive that

$$G_{\alpha,\beta,f}(N) = \gamma\sum_{n\le N} f(n) + H(N) + O(K^{-1}\Delta^{-1}N + \Delta^{1/2}N + N^{1-1/(4\tau)}),$$

where

$$H(N) = \sum_{0<|k|\le K} g(k)\mathbf{e}(k\delta)S_{k\gamma,f}(N).$$

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Put $R = (\log N)^3$. We claim that if N is sufficiently large, then for every k in the above sum there is a reduced fraction a/q such that $|k\gamma - a/q| \le q^{-2}$ and $R \le q \le N/R$. Assuming this is true for the moment, (3) implies that

$$S_{k\gamma,f}(N) \ll \frac{N}{\log N} \quad (0 < |k| \le K);$$

using (7), we then deduce that

$$H(N) \ll \frac{N \log K}{\log N}.$$

Therefore,

$$G_{\alpha,\beta,f}(N) - \gamma \sum_{n \le N} f(n) \ll \frac{N \log K}{\log N} + K^{-1}\Delta^{-1}N + \Delta^{1/2}N + N^{1-1/4\tau}.$$

To balance the error terms, we choose

$$\Delta = (\log N)^{-2} \quad \text{and} \quad K = \Delta^{-3/2} = (\log N)^3,$$

thus obtaining the bound stated in the theorem.

To prove the claim, let k be an integer with $0 < |k| \le K = (\log N)^3$, and let $r_i = a_i/q_i$ be the ith convergent in the continued fraction expansion of $k\gamma$. Since γ is of finite type τ, for every $\varepsilon > 0$ there is a constant $C = C(\gamma, \varepsilon)$ such that

$$C(|k|q_{i-1})^{-(\tau+\varepsilon)} < [\![\gamma|k|q_{i-1}]\!] \le \big|\gamma|k|q_{i-1} - a_{i-1}\big| \le q_i^{-1}.$$

Put $\varepsilon = \tau$, and let j be the least positive integer for which $q_j \ge R$ (note that $j \ge 2$). Then

$$R \le q_j \ll (|k|q_{i-1})^{2\tau} \le (KR)^{2\tau} = (\log N)^{6\tau},$$

and it follows that $R \le q_j \le N/R$ if N is sufficiently large, depending only on α. This concludes the proof. □

3.2. Proof of Corollary 2 Let $f(n)$ be the characteristic function of the set of integers that can be represented as a sum of two squares. It follows from [4, Theorem 366] that $f(n)$ is multiplicative. Hence Corollary 2 is an immediate consequence of Theorem 1 and the asymptotic formula

$$\sum_{n \le N} f(n) = \frac{CN}{(\log N)^{1/2}} + O\left(\frac{N}{(\log N)^{3/2}}\right)$$

(see, for example, [12, 13]), where C is given by (5). □

3.3. Proof of Corollary 3 Fix $k \ge 2$ and let $f(n)$ be the characteristic function of the set of k-free integers. It is easily proved that $f(n)$ is multiplicative. Thus Corollary 3 follows from Theorem 1 and the following estimate of Gegenbauer [3] for the number of k-free integers not exceeding N:

$$\sum_{n \le N} f(N) = \zeta^{-1}(k)N + O(N^{1/k}). \qquad \square$$

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3.4. Proof of Corollary 4 Put $f(n) = r_4(n)/(8n)$. From Jacobi's formula for $r_4(n)$, namely

$$r_4(n) = 8(2 + (-1)^n) \sum_{\substack{d \mid n \\ d \text{ odd}}} d \quad (n \geq 1),$$

it follows that $f(n)$ is multiplicative and that $f(p) \leq 3/2$ for every prime p. Moreover, using the formula of Ramanujan [9] (see also [14]),

$$\sum_{n \leq N} \sigma^2(n) = \frac{5}{6} \zeta(3) N^3 + O(N^2 (\log N)^2)$$

where σ is the sum of divisors function, we obtain

$$\sum_{n \leq N} |f(n)|^2 \leq \sum_{n \leq N} \frac{\sigma^2(n)}{n^2} = \frac{5}{2} \zeta(3) N + O((\log N)^3)$$

by partial summation. Therefore, $f(n) \in \mathcal{F}_A$ for some constant $A \geq 1$. Applying Theorem 1, we deduce that

$$\sum_{\substack{n \leq N \\ n \in \mathcal{B}_{\alpha,\beta}}} \frac{r_4(n)}{n} = \alpha^{-1} \sum_{n \leq N} \frac{r_4(n)}{n} + O\left(\frac{N \log \log N}{\log N}\right),$$

where the implied constant depends only on α.

From the asymptotic formula

$$\sum_{n \leq N} r_4(n) = \frac{\pi^2 N^2}{2} + O(N \log N)$$

(see for example [5, p. 22]), partial summation gives

$$\sum_{n \leq N} \frac{r_4(n)}{n} = \pi^2 N + O((\log N)^2).$$

Consequently,

$$\sum_{\substack{n \leq N \\ n \in \mathcal{B}_{\alpha,\beta}}} \frac{r_4(n)}{n} = \alpha^{-1} \pi^2 N + O\left(\frac{N \log \log N}{\log N}\right).$$

Using partial summation once more, we obtain the statement of Corollary 4. □

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