Iterative square roots of functions

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Abstract. An iterative square root of a self-map f is a self-map g such that $g(g(\cdot)) = f(\cdot)$.
We obtain new characterizations for detecting the non-existence of such square roots for
self-maps on arbitrary sets. They are used to prove that continuous self-maps with no
square roots are dense in the space of all continuous self-maps for various topological
spaces. The spaces studied include those that are homeomorphic to the unit cube in $\mathbb{R}^m$
and to the whole of $\mathbb{R}^m$ for every positive integer m. However, we also prove that every
continuous self-map on a space homeomorphic to the unit cube in $\mathbb{R}^m$ with a fixed point
on the boundary can be approximated by iterative squares of continuous self-maps.

1. Introduction
Given a map $f : X \to X$ on a non-empty set X and a positive integer n, an *iterative root
of order n* (or simply *nth root*) of f is a map $g : X \to X$ such that

$$g^n(x) = f(x) \quad \text{for all } x \in X, \tag{1.1}$$

where for each non-negative integer k, g^k denote the kth order iterate of g defined
recursively by $g^0 = \text{id}$, the identity map on X, and $g^k = g \circ g^{k-1}$. For $n = 2$, a solution of
equation (1.1) is called an *iterative square root* (or simply a *square root*) of f. The *iterative
root problem* (1.1), which is closely related to the embedding flow problem [13, 22, 53],
the invariant curve problem [24], and linearization [9], is of particular interest [4, 8, 22,
47, 48, 54] both in the theory of the functional equations and in that of dynamical systems,
and has applications in statistics, signal processing, computer graphics, and information
techniques [16, 19, 38]. Since the initial works of Babbage [3], Abel [1], and Königs [20],
various researchers have paid increasing attention to the iterative root problem, and many
advances have been made on its solutions for various classes of maps, e.g. continuous
maps on intervals and in particular those which are piecewise monotone [6, 21, 28, 30–34,
36, 37, 57, 59], continuous complex maps [40, 45, 46, 52], series and transseries [12],

continuous maps on planes and $\mathbb{R}^m$ [25, 26, 39, 51], and set-valued maps [18, 27, 29, 35]. In particular, various properties of the iterative roots have also been studied, see for instance, approximation [60], stability [50, 58], differentiability [23, 55, 56], and category and measure [7, 41].

Given a topological space X, let $C(X)$ denote the set of all continuous self-maps of X. For each positive integer n, let $\mathcal{W}(n; X) := \{f^n : f \in C(X)\}$, the set of nth iterates of all the maps in $C(X)$, and $\mathcal{W}(X) := \bigcup_{n=2}^{\infty} \mathcal{W}(n; X)$. Then it is interesting to ask how big these spaces are inside $C(X)$. We need a metric or topology on $C(X)$ to make this question more precise.

Let X be a compact metric space with metric d. Then it is natural to have the *supremum metric* (or *uniform metric*)

$$\rho(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$$

on $C(X)$, and this is our convention unless mentioned otherwise. Humke and Laczkovich considered the above question when $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$. This paper is heavily motivated by their results. They proved in [14, 15] that $\mathcal{W}(2; [0, 1])$ is not dense in $C([0, 1])$, $\mathcal{W}(2; [0, 1])$ contains no balls of $C([0, 1])$ (that is, its complement $(\mathcal{W}(2; [0, 1]))^c$ is dense in $C([0, 1])$), and $\mathcal{W}(n; [0, 1])$ is an analytic non-Borel subset of $C([0, 1])$ for $n \geq 2$. Simon proved in [42–44] that $\mathcal{W}(2; [0, 1])$ is nowhere dense in $C([0, 1])$ and $\mathcal{W}([0, 1])$ is not dense in $C([0, 1])$. Subsequently, nowhere denseness of $\mathcal{W}([0, 1])$ in $C([0, 1])$ was proved by Blokh [7] using a method different from that of Humke-Laczkovich and Simon.

It is seen that the majority of papers on iterative roots of functions are restricted to continuous self-maps on intervals of the real line. This is not surprising because the setup is simpler to deal with. It has several advantages, including the intermediate value theorem, monotonicity of bijections, and so on. Extending these results to higher dimensions or more generic topological spaces is generally complex. We develop a new approach (see Theorem 2.4) to determine the non-existence of square roots of self-maps on arbitrary sets in §2. This approach can be used effectively to construct continuous self-maps on a wide range of topological spaces, which do not even have discontinuous square roots. We demonstrate the procedure in §3 by showing that the continuous self-maps without square roots are dense in the space of all continuous self-maps of the unit cube in $\mathbb{R}^m$ for the supremum metric ρ (see Theorem 3.8). This is a significant generalization of a similar result in [15] (Corollary 5, p. 362) for intervals of the real line. We believe that the most significant contribution of this article is the method itself. The scheme can be used in various other settings. We demonstrate this claim in §4 by proving that the continuous self-maps without square roots are dense in the space of all continuous self-maps of $\mathbb{R}^m$ for the compact-open topology (see Theorem 4.3).

The outline of the paper is as follows. In §2, we study iterates and orbits of self-maps on arbitrary sets (without any topology). Isaacs conducted a detailed investigation on the fractional iterates of such functions in [17]. Here we discuss some of the basic aspects of such an approach. The main new result is Theorem 2.4, which presents a variety of cases in which we can be certain that the given function does not admit any iterative square root. This will be our principal tool in §§3 and 4. Section 2 also contains some

background material and new results on fractional iterates of continuous self-maps on general topological spaces. In particular, Theorem 2.6 shows that in some cases, the problem of the non-existence of iterative roots of continuous maps can be reduced to elementary combinatorics.

In §3, we study iterative square roots of continuous self-maps on the unit cube I^m in $\mathbb{R}^m$. We find it convenient to triangulate I^m and sew together affine linear maps piece by piece. For this purpose, we use some fundamental notions and notation from the theory of simplicial complexes. Given any continuous self-map on I^m, we approximate it with a piecewise affine linear map, and in turn close to any such map, we find another piecewise affine linear map with no square roots (see Theorem 3.8). This is the main result of this section. In contrast, we also show that any continuous self-map on I^m with a fixed point on the boundary can be approximated by iterative squares of continuous self-maps (see Theorem 3.9). A simple example shows that fixed points on the boundary are not required for such approximations.

In §4, we analyze continuous self-maps on $\mathbb{R}^m$. As in §3, we prove that the continuous self-maps with no square roots are dense in the space of all continuous self-maps in a suitable topology, namely the compact-open topology (see Theorem 4.3). Finally, in §5, we prove that the squares of continuous self-maps are L^p dense in the space of all continuous self-maps on I^m, generalizing a result of [**15**] to higher dimensions.

2. *Functions on arbitrary sets and general topological spaces*

In this section, we study iterative roots of self-maps on arbitrary sets and continuous self-maps on general topological spaces. First, we consider X to be a non-empty set with no predefined topology, and prove some new results on the existence and non-existence of iterative square roots of maps in $\mathcal{F}(X)$, the set of all self-maps on X. Surprisingly, some of these results are useful even when X has a topology and we seek for roots inside $C(X)$, the space of all continuous self-maps on X.

Henceforth, let $\mathbb{Z}_+$ denote the set $\{0, 1, 2, \ldots\}$ of non-negative integers, and for each $f \in \mathcal{F}(X)$ and $A \subseteq X$, let $R(f)$ denote the range of f and $f|_A$ the restriction of f to A. Given an $f \in \mathcal{F}(X)$, we define its *graph* as the directed graph $G = (X, E)$, whose vertex set is X and the edge set is $E = \{(x, f(x)) : x \in X\}$. In other words, we are looking at the same mathematical structure from a different viewpoint. This allows us to borrow some notions from graph theory. To begin, consider the connected components of G in the context of graph theory. We call C a connected component of X if it is the vertex set of some connected component of the graph of f. Observe that a pair of vertices x and y are in the same connected component $C \subseteq X$ if and only if there exist $m, n \in \mathbb{Z}_+$ such that $f^m(x) = f^n(y)$. The connected component of X containing x is called the *orbit* of x under f, and is denoted by $O_f(x)$. An extensive analysis of iterative square roots of functions in $\mathcal{F}(X)$ can be found in [**17**]. In what follows, we analyze them in some special cases and present all the results needed for our further discussions.

Let $f \in \mathcal{F}(X)$, and G_1 and G_2 be two connected components of the graph G of f. An *isomorphism* of G_1 and G_2 is a bijective function $\phi : C_1 \to C_2$ between the vertex sets of G_1 and G_2 such that any two vertices x and y are adjacent in G_1 if and only if $\phi(x)$ and $\phi(y)$ are adjacent in G_2. By the definition of graphs, this means that $y = f(x)$ if and only

if $\phi(y) = f(\phi(x))$. In other words, G_2 is isomorphic to G_1 if and only if there exists a bijective function $\phi : C_1 \to C_2$ such that $\phi \circ f = f \circ \phi$.

PROPOSITION 2.1. *Let $f \in \mathcal{F}(X)$ be such that the graph of f has exactly two isomorphic connected components. Then f has an iterative square root in $\mathcal{F}(X)$.*

Proof. Let C_1 and C_2 be the two isomorphic connected components of X corresponding to those of the graph of f. Then $X = C_1 \cup C_2$, $C_1 \cap C_2 = \emptyset$, and as indicated above, there exists a bijective function $\phi : C_1 \to C_2$ such that $\phi \circ f = f \circ \phi$. Define a map $g : X \to X$ by

$$g(x) = \begin{cases} \phi(x) & \text{if } x \in C_1, \\ f \circ \phi^{-1}(x) & \text{if } x \in C_2. \end{cases}$$

Now, we have $g(x) = \phi(x) \in C_2$ for each $x \in C_1$, implying that $g^2(x) = f \circ \phi^{-1} \circ \phi(x) = f(x)$. Similarly, we have $g(x) = f \circ \phi^{-1}(x) \in C_1$ for each $x \in C_2$, and so $g^2(x) = \phi \circ f \circ \phi^{-1}(x) = f(x)$. Therefore, g is an iterative square root of f on X. □

The above proposition leads to the following result, where a fixed point x of f is said to be *isolated* if there is no $y \neq x$ in X such that $f(y) = x$. By convention, the empty set and all infinite sets are assumed to have an even number of elements here and elsewhere.

THEOREM 2.2. *Let $f \in \mathcal{F}(X)$ be such that excluding isolated fixed points of f, the number of isomorphic copies is even for each connected component of the graph of f. Then f has an iterative square root in $\mathcal{F}(X)$.*

Proof. Let $g : X \to X$ be a map defined as $g(x) = x$ for isolated fixed points $x \in X$ of f and g as in the previous proposition on the union of pairs of connected components of X corresponding to pairs of isomorphic copies of those of the graph of f. Since there are an even number of connected components, we can pair them, and the result follows from the previous proposition. □

The above results are given for general $f \in \mathcal{F}(X)$. If we restrict ourselves to injective maps, we can describe the functions with square roots transparently, as seen below. To make our thoughts more concrete, we borrow some ideas and terminologies from [5]. Note that given an injective map $f \in \mathcal{F}(X)$, any two points $x, y \in X$ are in the same orbit under f if and only if there exists an $n \in \mathbb{Z}_+$ such that either $y = f^n(x)$ or $x = f^n(y)$. Indeed, an orbit under f has one of the following forms for some $x \in X$:

(i) $\{x, f(x), f^2(x), \ldots, f^{d-1}(x)\}$ for some $d \in \mathbb{N}$ with $f^d(x) = x$;
(ii) $\{x, f(x), f^2(x), \ldots\}$ with $x \notin R(f)$;
(iii) $\{\ldots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \ldots\}$.

Therefore, it follows that an injective map $f \in \mathcal{F}(X)$ with exactly one orbit $O_f(x)$ for some $x \in X$ is in bijective correspondence with precisely one of the following maps (see [5]).

(a)   *Cyclic permutations*: For $d \in \mathbb{N}$, consider $\mathbb{Z}_d = \{0, 1, 2, \ldots, d-1\}$ with addition modulo d. Define $s_d : \mathbb{Z}_d \to \mathbb{Z}_d$ by $s_d(k) = k + 1 \pmod{d}$. Note that $\mathbb{Z}_1 = \{0\}$ and $s_1(0) = 0$. Then s_d is bijective and has exactly one orbit.

(b) *Unilateral translation/shift*: Define $s_+ : \mathbb{Z}_+ \to \mathbb{Z}_+$ by $s_+(k) = k + 1$. Then s_+ is a shift with a single orbit.

(c) *Bilateral translation*: Define $s : \mathbb{Z} \to \mathbb{Z}$ by $s(k) = k + 1$. Then s is bijective and has exactly one orbit.

More precisely, if $f \in \mathcal{F}(X)$ has exactly one orbit $O_f(x)$, then there exists a bijective function $\phi : O_f(x) \to Z$ defined by $\phi(f^k(x)) = k$, where Z is $\mathbb{Z}_d$ (for some $d \in \mathbb{N}$), $\mathbb{Z}_+$ or $\mathbb{Z}$ according as $O_f(x)$ has the form (i), (ii) or (iii), respectively. Consider any arbitrary non-empty sets M_d, M_+ and M. We define $1_{M_d} \times s_d : M_d \times \mathbb{Z}_d \to M_d \times \mathbb{Z}_d$ by

$$(1_{M_d} \times s_d)(l, k) = (l, k + 1(\mathrm{mod}\, d)),$$

called the *d-cyclic permutation with multiplicity M_d* for each $d \in \mathbb{N}$. Similarly, we define $1_{M_+} \times s_+ : M_+ \times \mathbb{Z}_+ \to M_+ \times \mathbb{Z}_+$ and $1_M \times s : M \times \mathbb{Z} \to M \times \mathbb{Z}$ by

$$(1_{M_+} \times s_+)(l, k) = (l, k + 1) \quad \text{and} \quad (1_M \times s)(l, k) = (l, k + 1),$$

called the *unilateral shift with multiplicity M_+* and the *bilateral translation with multiplicity M*, respectively. Then, given any injective map $f \in \mathcal{F}(X)$, decomposing $X = \bigsqcup_{x \in X} O_f(x)$ into orbits under f, we see that (X, f) is in bijective correspondence with $(Y, \tilde{f})$, where $Y = \bigsqcup_{d \in \mathbb{N}}(M_d \times \mathbb{Z}_d) \bigsqcup (M_+ \times \mathbb{Z}_+) \bigsqcup (M \times \mathbb{Z})$ for some suitable multiplicity spaces M_d, M_+ and M (some of these sets may be absent in the union), and $\tilde{f}$ is $1_{M_d} \times s_d$, $1_{M_+} \times s_+$ and $1_M \times s$ on $M_d \times \mathbb{Z}_d$, $M_+ \times \mathbb{Z}_+$ and $M \times \mathbb{Z}$, respectively. Furthermore, the cardinalities of these multiplicity spaces are uniquely determined. We call $(m_1, m_2, \ldots, m_+, m)$ as the multiplicity sequence of f, where m_d, m_+ and m are the cardinalities of M_d (for $d \in \mathbb{N}$), M_+ and M, respectively.

THEOREM 2.3. *Let $f \in \mathcal{F}(X)$ be an injective map. Then f has an iterative square root in $\mathcal{F}(X)$ if and only if the multiplicity sequence $(m_1, m_2, \ldots, m_+, m)$ of f satisfies that m_d with d even, m_+ and m are even (0 and infinity allowed).*

Proof. Let $f \in \mathcal{F}(X)$ be an injective map and $(m_1, m_2, \ldots, m_+, m)$ be the multiplicity sequence of f. Suppose that $f = g^2$ for some $g \in \mathcal{F}(X)$. Then, clearly g is an injective map on X, and therefore we can associate a multiplicity sequence $(m_1', m_2', \ldots, m_+', m')$ for g. Further, it is easily seen that an orbit under g corresponding to a cyclic permutation on $\mathbb{Z}_d$ gives rise to two orbits (respectively a unique orbit) under g^2 corresponding to cyclic permutation on $\mathbb{Z}_{d/2}$ (respectively on $\mathbb{Z}_d$) whenever d is even (respectively odd). Similarly, an orbit under g corresponding to the unilateral shift on $\mathbb{Z}_+$ (respectively the bilateral translation on $\mathbb{Z}$) gives rise to two orbits under g^2 corresponding to the unilateral shift on $\mathbb{Z}_+$ (respectively the bilateral translation on $\mathbb{Z}$). Therefore, the multiplicity sequence of g^2 is $(m_1' + 2m_2', 2m_4', m_3' + 2m_6', 2m_8', m_5' + 2m_{10}', 2m_{12}', \ldots, 2m_+', 2m')$. Consequently, we must have $m_+ = 2m_+'$, $m = 2m'$, and

$$m_d = \begin{cases} m_d' + 2m_{2d}' & \text{if } d \text{ is odd,} \\ 2m_{2d}' & \text{if } d \text{ is even,} \end{cases}$$

and hence m_d for d even, m_+ and m are even.

Conversely, suppose that the multiplicity sequence $(m_1, m_2, \ldots, m_+, m)$ of f satisfies that m_d for d even, m_+ and m are even. Then X can be decomposed into orbits under f as $X = \bigsqcup_{x \in X} O_f(x)$ and (X, f) is in bijective correspondence with $(Y, \tilde{f})$, where $Y = \bigsqcup_{d \in \mathbb{N}}(M_d \times \mathbb{Z}_d) \bigsqcup (M_+ \times \mathbb{Z}_+) \bigsqcup (M \times \mathbb{Z})$ such that:

- each orbit $O_f(x)$ in X under f has one of the above-mentioned forms (i), (ii) or (iii);
- the cardinalities of multiplicity spaces M_d (for all $d \in \mathbb{N}$), M_+ and M are m_d, m_+ and m, respectively; and
- $\tilde{f}$ is $1_{M_d} \times s_d$, $1_{M_+} \times s_+$ and $1_M \times s$ on $M_d \times \mathbb{Z}_d$, $M_+ \times \mathbb{Z}_+$ and $M \times \mathbb{Z}$, respectively.

Now, define a map $g : X \to X$ as follows. If $d \in \mathbb{N}$ is odd, then for each of the orbits $O_f(x) = \{x, f(x), f^2(x), \ldots, f^{d-1}(x)\}$ corresponding to the cyclic permutation on $\mathbb{Z}_d$, define

$$g(f^l(x)) = f^{(l+(d+1)/2)(\text{mod } d)}(x)$$

for all $0 \le l \le d - 1$. Then,

$$g^2(f^l(x)) = g(f^{(l+(d+1)/2)(\text{mod } d)}(x)) = f^{l+d+1(\text{mod } d)}(x) = f^{l+1} = f(f^l(x))$$

for all $0 \le l \le d - 1$, implying that $g^2 = f$ on $O_f(x)$. If $d \in \mathbb{N}$ is even, then as m_d is even, by pairing any two distinct orbits $O_f(x) = \{x, f(x), f^2(x), \ldots, f^{d-1}(x)\}$ and $O_f(y) = \{y, f(y), f^2(y), \ldots, f^{d-1}(y)\}$ corresponding to the cyclic permutation on $\mathbb{Z}_d$, define g on $O_f(x) \cup O_f(y)$ by

$$g(z) = \begin{cases} \phi(z) & \text{if } z \in O_f(x), \\ f \circ \phi^{-1}(z) & \text{if } z \in O_f(y), \end{cases}$$

where $\phi : O_f(x) \to O_f(y)$ is a bijective function such that $\phi \circ f = f \circ \phi$. Then, by a similar argument as in Proposition 2.1, it follows that $g^2 = f$ on $O_f(x) \cup O_f(y)$. Since m_+ (respectively m) is even, g can be defined similarly on the union $O_f(x) \cup O_f(y)$ of each pair of distinct orbits $O_f(x)$ and $O_f(y)$ corresponding to the unilateral shift on $\mathbb{Z}_+$ (respectively the bilateral translation on $\mathbb{Z}$) such that $g^2 = f$ on $O_f(x) \cup O_f(y)$. Therefore, f has a square root in $\mathcal{F}(X)$. $\qquad \square$

For each $f \in \mathcal{F}(X)$ and $x \in X$, let $f^{-1}(x)$ and $f^{-2}(x)$ denote the usual inverse images defined by $f^{-1}(x) = \{y \in X : f(y) = x\}$ and $f^{-2}(x) = \{y \in X : f^2(y) = x\}$. Further, for each set A, let $\#A$ denote the number of elements or the cardinality of A. So far, we have studied various conditions under which maps in $\mathcal{F}(X)$ have square roots. We now have some instances in which they have no square roots. The following result is very useful for constructing functions without square roots in the next two sections.

THEOREM 2.4. *Let $f \in \mathcal{F}(X)$ be such that $f(x_0) \ne x_0$ for some $x_0 \in X$. Then f has no iterative square roots in $\mathcal{F}(X)$ in the following cases:*

Case (i): $\#f^{-2}(x_0) > 1$, and $\#f^{-1}(x) \le 1$ for all $x \ne x_0$;

Case (ii): $f^{-2}(x_0)$ is infinite, and $f^{-1}(x)$ is finite for all $x \ne x_0$;

Case (iii): $f^{-2}(x_0)$ is uncountable, and $f^{-1}(x)$ is countable for all $x \ne x_0$.

Proof. Suppose that $f = g^2$ for some $g \in \mathcal{F}(X)$. Consider the action of g on various subsets of X around x_0:

$$A_{-2} \xrightarrow{g} B_{-2} \xrightarrow{g} A_{-1} \xrightarrow{g} B_{-1} \xrightarrow{g} \{x_0\} \xrightarrow{g} \{y_0\},$$

where $y_0 = g(x_0)$, $A_{-1} = f^{-1}(x_0)$, $A_{-2} = f^{-2}(x_0)$, $B_{-1} = g(A_{-1})$ and $B_{-2} = g(A_{-2})$. Let $\tilde{A}_{-1} = f(A_{-2})$ and $\tilde{B}_{-1} = g(\tilde{A}_{-1})$. Then $\tilde{A}_{-1} \subseteq A_{-1}$, $\tilde{B}_{-1} \subseteq B_{-1} \subseteq g^{-1}(x_0)$ and $B_{-2} \subseteq g^{-1}(A_{-1})$. Also, since $f(x_0) \neq x_0$, we have $y_0 \neq x_0$, $x_0 \notin A_{-1}$ and $x_0 \notin B_{-1}$.

Case (i): Since $x_0 \notin A_{-1}$ and $\tilde{A}_{-1} \subseteq A_{-1}$, we have $\#f^{-1}(x) \leq 1$ for all $x \in \tilde{A}_{-1}$. Therefore, as $A_{-2} \subseteq \bigcup_{x \in \tilde{A}_{-1}} f^{-1}(x)$ and $\#A_{-2} > 1$, it follows that $\#\tilde{A}_{-1} > 1$.

However, since $y_0 \neq x_0$, we have $\#f^{-1}(y_0) \leq 1$. This implies that $\#\tilde{B}_{-1} = 1$, because $\tilde{B}_{-1} \neq \emptyset$ and $\tilde{B}_{-1} \subseteq B_{-1} \subseteq f^{-1}(y_0)$. Therefore, as $x_0 \notin \tilde{B}_{-1}$, we get that $\#f^{-1}(\tilde{B}_{-1}) = 1$. Consequently, $\#B_{-2} = 1$, implying that $\#\tilde{A}_{-1} = \#f(A_{-2}) = \#g(B_{-2}) = 1$. This contradicts an earlier conclusion. Hence, f has no square roots in $\mathcal{F}(X)$.

Case (ii): Since $x_0 \notin A_{-1}$ and $\tilde{A}_{-1} \subseteq A_{-1}$, we see that $f^{-1}(x)$ is finite for all $x \in \tilde{A}_{-1}$. Therefore, as $A_{-2} \subseteq \bigcup_{x \in \tilde{A}_{-1}} f^{-1}(x)$ and A_{-2} is infinite, it follows that $\tilde{A}_{-1}$ is infinite.

However, since $y_0 \neq x_0$, we have that $f^{-1}(y_0)$ is finite. This implies that $\tilde{B}_{-1}$ is finite, because $\tilde{B}_{-1} \subseteq B_{-1} \subseteq f^{-1}(y_0)$. Also, as $x_0 \notin B_{-1}$ and $\tilde{B}_{-1} \subseteq B_{-1}$, we see that $f^{-1}(x)$ is finite for all $x \in \tilde{B}_{-1}$. Then it follows that $f^{-1}(\tilde{B}_{-1})$ is finite. Consequently, B_{-2} is finite, implying that $\tilde{A}_{-1} = f(A_{-2}) = g(B_{-2})$ is finite. This contradicts the conclusion of the previous paragraph. Hence, f has no square roots in $\mathcal{F}(X)$.

Case (iii): The proof of Case (ii) repeatedly uses the fact that a finite union of finite sets is finite. The proof for this case is similar, using the result that a countable union of countable sets is countable. □

It is worth noting that f^{-2} in the previous theorem cannot be replaced by f^{-1}, as seen from the following.

Example 2.5. Consider the continuous map $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} \dfrac{3}{4} & \text{if } 0 \leq x \leq \dfrac{1}{2}, \\ \dfrac{5}{8} + \dfrac{x}{4} & \text{if } \dfrac{1}{2} < x \leq 1. \end{cases}$$

Then $f(\frac{3}{4}) \neq \frac{3}{4}$, $f^{-1}(\frac{3}{4})$ is uncountable and $f^{-1}(x)$ is finite for all $x \neq \frac{3}{4}$. However, f has a square root

$$g(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \dfrac{1}{2}, \\ \dfrac{5}{4} - \dfrac{x}{2} & \text{if } \dfrac{1}{2} < x \leq 1, \end{cases}$$

on $[0, 1]$ that is even continuous.

The above results are given for an arbitrary set X. In the rest of the section and those that follow, we consider X to be a topological space or a specific metric space and study continuous roots of continuous self-maps on X. It is useful to have some notation in this context. Recall that for a topological space X, the space of all continuous self-maps of X is denoted by $C(X)$. For each $A \subseteq X$, let A^0 denote the interior of A, $\overline{A}$ the closure of A and ∂A the boundary of A. In particular, if X is a metric space equipped with metric d, then for each $x \in X$ and $\epsilon > 0$, let $B_\epsilon(x) := \{y \in X : d(x, y) < \epsilon\}$, the open ball in X around x of radius ϵ. For any index set Λ, let S_Λ denote the group of all permutations (bijections) on Λ, and for convenience, we denote S_Λ by S_k when $\Lambda = \{1, 2, \ldots, k\}$.

We motivate the next theorem through two examples. The map $f_1(x) = 1 - x$ on the unit interval $[0, 1]$ as an element of $C([0, 1])$, with $[0, 1]$ in the usual metric induced by $|\cdot|$, has no continuous square roots on $[0, 1]$ (see [24, pp. 425–426]). However, the map $f_2(x, y) = (1 - x, 1 - y)$ on $[0, 1] \times [0, 1]$ as an element of $C([0, 1] \times [0, 1])$, where $[0, 1] \times [0, 1]$ has the metric induced by the norm $\|(x, y)\|_\infty := \max\{|x|, |y|\}$, has a continuous square root $g(x, y) = (y, 1 - x)$ on $[0, 1] \times [0, 1]$. The reason for these contrasting conclusions can be detected by observing the fixed points of these maps.

Let f be a continuous self-map on a topological space X and Y a non-empty subset of X invariant under f, that is, $f(Y) \subseteq Y$. Then it is clear that if x and x' are path-connected in Y, then so are $f(x)$ and $f(x')$. Therefore, if $Y = \bigcup_{\alpha \in \Lambda} Y_\alpha$ is the decomposition of Y into its path components for some index set Λ, then there exists a unique map $\sigma_{f,Y} : \Lambda \to \Lambda$ such that $f(x) \in Y_\beta$ whenever $x \in Y_\alpha$ and $\sigma_{f,Y}(\alpha) = \beta$ (note that y and y' are path-connected in Y if and only if there exists an $\alpha \in \Lambda$ such that $y, y' \in Y_\alpha$). We call $\sigma_{f,Y}$ as the map *induced* by the map f and the invariant set Y. Let $F(f) := \{x \in X : f(x) = x\}$, the set of all fixed points of f in X, and $E(f) := R(f) \setminus F(f)$, the complement of $F(f)$ in $R(f)$.

THEOREM 2.6. *Let $n \in \mathbb{N}$ and $f \in C(X)$ be such that $E(f)$ is invariant under f. If $f = g^n$ for some $g \in C(X)$, then $E(f)$ is invariant under g and the induced maps satisfy that $\sigma_{f,E(f)} = \sigma_{g,E(f)}^n$. In other words, if $\sigma_{f,E(f)}$ has no nth roots, then f also has no continuous nth roots on X.*

Proof. Let $E(f) = \bigcup_{\alpha \in \Lambda} Y_\alpha$ be the decomposition of $E(f)$ into its path components for some index set Λ. The case $n = 1$ is trivial. So, let $n > 1$, and consider an arbitrary $x \in E(f)$. Then $x = f(x')$ for some $x' \in X$. Clearly, $g(x) \in R(f)$, because $g(x) = g(f(x')) = g(g^n(x')) = g^n(g(x')) = f(g(x'))$. Also, if $g(x) \in F(f)$, then $g(x) = f(g(x)) = g^n(g(x)) = g(f(x))$, implying that $f(x) = g^n(x) = g^n(f(x)) = f^2(x)$, that is, $f(x)$ is a fixed point of f. This contradicts the hypothesis that $E(f)$ is invariant under f. Therefore, $g(x) \in E(f)$.

The equality $\sigma_{f,E(f)} = \sigma_{g,E(f)}^n$ follows from the definitions of $\sigma_{f,E(f)}$ and $\sigma_{g,E(f)}$, and the relation $f = g^n$. □

In the remainder of this section, we illustrate Theorem 2.6 through some simple examples. Remark that $\sigma \in S_k$, the group of permutation of k elements, has a square root if and only if the number of even cycles of σ is even. More generally, $\sigma \in S_k$ has an nth root if and only if for every $m = 1, 2, \ldots$, it is true that the number of m-cycles that σ has

is a multiple of $((m, n))$, where $((m, n)) := \prod_{p \mid m} p^{e(p, n)}$, $e(p, n)$ being the highest power of p dividing n (see [49, pp. 155–158]).

COROLLARY 2.7. *Let $a < b$ be real numbers and $f \in C([a, b])$ be such that $f|_{R(f)}$ is a non-constant strictly decreasing map. Then f has no iterative roots of even orders in $C([a, b])$.*

Proof. Since $f : [a, b] \to [a, b]$ is a non-constant strictly decreasing continuous map, we see that $R(f)$ is a compact interval $[c, d]$ in $\mathbb{R}$ for some $c < d$ in $[a, b]$ and f has a unique fixed point x_0 in (c, d). Also, $E(f) = [c, x_0) \cup (x_0, d]$ is the decomposition of $E(f)$ into path components and $\sigma := \sigma_{f, E(f)}$ is the transposition of the two-point set S_2. Since σ has no iterative roots of even orders in S_2, it follows from Theorem 2.6 that f has no iterative roots of even orders in $C([a, b])$. □

Example 2.8. Consider $[0, 1] \times [0, 1]$ in the topology induced by the norm $\|(x, y)\|_\infty := \max\{|x|, |y|\}$, and let f_1, f_2 be the continuous maps on $[0, 1] \times [0, 1]$ defined by $f_1(x, y) = (1 - x, 0)$ and $f_2(x, y) = (y, x)$. Then Theorem 2.6 is applicable to f_1 and f_2 with

$$E(f_1) = \left\{(x, 0) : 0 \le x < \tfrac{1}{2}\right\} \cup \left\{(x, 0) : \tfrac{1}{2} < x \le 1\right\};$$
$$E(f_2) = \{(x, y) : 0 \le x < y \le 1\} \cup \{(x, y) : 0 \le y < x \le 1\}.$$

It follows that both f_1 and f_2 have no iterative roots of even orders in $C([0, 1] \times [0, 1])$.

3. *Continuous functions on the unit cube in $\mathbb{R}^m$*

As seen in §1, much literature is available on iterative square roots of functions on compact real intervals. In this section, we extend some of the results in [15] to the *unit cube*

$$I^m := \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : x_i \in I \text{ for } 1 \le i \le m\}$$

considered in the metric induced by the norm

$$\|(x_1, x_2, \ldots, x_m)\|_\infty = \max\{|x_i| : 1 \le i \le m\}, \tag{3.1}$$

where $I := [0, 1]$. In studying iterative roots of continuous maps on intervals, one observes that it becomes essential to look at piecewise linear maps. Here we study their analogs in higher dimensions. It is convenient to have the following elementary notions and notation from the theory of simplicial complexes to present our proofs.

As defined in [10], a set $S = \{x_0, x_1, \ldots, x_k\}$ in $\mathbb{R}^m$, where $k \ge 1$, is said to be *geometrically independent* if the set $\{x_1 - x_0, x_2 - x_0, \ldots, x_k - x_0\}$ is linearly independent in $\mathbb{R}^m$. Equivalently, S is geometrically independent if and only if for arbitrary reals α_i, the equalities $\sum_{i=0}^{k} \alpha_i x_i = 0$ and $\sum_{i=0}^{k} \alpha_i = 0$ imply that $\alpha_i = 0$ for all $0 \le i \le k$. A set having only one point is assumed to be geometrically independent. If S is geometrically independent, then there exists a unique k-dimensional hyperplane which passes through all the points of S and each point x on this hyperplane can be expressed uniquely as $x = \sum_{i=0}^{k} \alpha_i x_i$ such that $\sum_{i=0}^{k} \alpha_i = 1$. The real numbers $\alpha_0, \alpha_1, \ldots, \alpha_k$, which are uniquely determined by S, are called the *barycentric coordinates* of the point x with respect to the set S.

Let $S = \{x_0, x_1, \ldots, x_k\}$ be a geometrically independent set in $\mathbb{R}^m$, where $0 \leq k \leq m$. Then the *k-dimensional geometric simplex* or *k-simplex* spanned by S, denoted by σ, is defined as the set of all points $x \in \mathbb{R}^m$ such that $x = \sum_{i=0}^{k} \alpha_i x_i$, where $\sum_{i=0}^{k} \alpha_i = 1$ and $\alpha_i \geq 0$ for all $0 \leq i \leq k$. Equivalently, σ is the convex hull of S. The points $x_0, x_1, \ldots, x_k$ are called the *vertices* of σ, and we usually write $\sigma = \langle x_0, x_1, \ldots, x_k \rangle$ to indicate that σ is the k-simplex with vertices $x_0, x_1, \ldots, x_k$. If $\sigma = \langle x_0, x_1, \ldots, x_k \rangle$ is a k-simplex, then the set of those points of σ for which all barycentric coordinates are strictly positive, is called the *open k-simplex* σ (or the *interior* of σ), and is denoted by σ^0. If σ is a p-simplex and τ is a q-simplex in $\mathbb{R}^m$ such that $0 \leq p \leq q \leq m$, then we say that σ is a *p-dimensional face* (or simply a *p-simplex*) *of* τ if each vertex of σ is also a vertex of τ. If σ is a face of τ with $p < q$, then σ is called a *proper face of* τ. Any zero-dimensional face of a simplex is simply a vertex of the simplex and a one-dimensional face of a simplex is usually called an *edge* of that simplex.

A finite (respectively countable) *simplicial complex* $\mathcal{K}$ is a finite (respectively countable) collection of simplices of $\mathbb{R}^m$ satisfying the following conditions: (i) If $\sigma \in \mathcal{K}$, then all the faces of σ are in $\mathcal{K}$; (ii) If $\sigma, \tau \in \mathcal{K}$, then either $\sigma \cap \tau = \emptyset$ or $\sigma \cap \tau$ is a common face of both σ and τ. Let $\mathcal{K}$ be a finite simplicial complex and $|\mathcal{K}| = \bigcup_{\sigma \in \mathcal{K}} \sigma$ be the union of all simplices of $\mathcal{K}$. Then $|\mathcal{K}|$ is a topological space with the topology inherited from $\mathbb{R}^m$. The space $|\mathcal{K}|$ is called the *geometric carrier* of $\mathcal{K}$. A topological subspace of $\mathbb{R}^m$, which is the geometric carrier of some finite simplicial complex, is called a *rectilinear polyhedron*. A topological space X is said to be a *polyhedron* if there exists a finite simplicial complex $\mathcal{K}$ such that $|\mathcal{K}|$ is homeomorphic to X. In this case, the space X is said to be *triangulable* and $\mathcal{K}$ is called a *triangulation* of X. A space X is said to be *countably triangulable* if there exists a countable triangulation of X, that is, there exists a countable simplicial complex $\mathcal{K}$ of simplices of $\mathbb{R}^m$ such that its geometric carrier $|\mathcal{K}|$ is homeomorphic to X. Further, we do not distinguish between X and its geometric carrier $|\mathcal{K}|$ whenever $X = |\mathcal{K}|$ for some (finite or countable) triangulation $\mathcal{K}$ of X, because in that case, we always consider the identity map id to be the homeomorphism between $|\mathcal{K}|$ and X.

If $\sigma = \langle x_0, x_1, \ldots, x_k \rangle$ is a k-simplex in $\mathbb{R}^m$, then the point $\sum_{i=0}^{k} (1/(k+1)) x_i$ is called the *barycenter* of σ and is denoted by $\dot{\sigma}$. In other words, barycenter of σ is that point of σ whose barycentric coordinates, with respect to each of the vertices of σ, are equal. Given a simplicial complex $\mathcal{K}$, let $\mathcal{K}^{(1)}$ be a simplicial complex whose vertices are barycenters of all simplices of $\mathcal{K}$, and for any distinct simplices $\sigma_1, \sigma_2, \ldots, \sigma_k$ of $\mathcal{K}$, $\langle \dot{\sigma}_1, \dot{\sigma}_2, \ldots, \dot{\sigma}_k \rangle$ is a simplex of $\mathcal{K}^{(1)}$ if and only if σ_i is a face of σ_{i+1} for $i = 0, 1, \ldots, k-1$. Then $\mathcal{K}^{(1)}$ is a simplicial complex and is called the *first barycentric subdivision* of $\mathcal{K}$. By induction, we define the *l*th barycentric subdivision $\mathcal{K}^{(l)}$ of $\mathcal{K}$ to be the first barycentric subdivision of $\mathcal{K}^{(l-1)}$ for each $l > 1$. We also put $\mathcal{K}^{(0)} = \mathcal{K}$ for convenience. For a simplicial complex $\mathcal{K}$, we define the *mesh* of $\mathcal{K}$, denoted by mesh($\mathcal{K}$), as

$$\text{mesh}(\mathcal{K}) = \max\{\text{diam}(\sigma) : \sigma \text{ is a simplex of } \mathcal{K}\},$$

where diam(σ) denote the diameter of σ. Then $\lim_{l \to \infty} \text{mesh}(\mathcal{K}^{(l)}) = 0$ for every non-empty simplicial complex $\mathcal{K}$.

An $f \in \mathcal{F}(\mathbb{R}^m)$ is said to be *affine linear* if $f(\cdot) = T(\cdot) + z$ for some linear map $T \in \mathcal{F}(\mathbb{R}^m)$ and $z \in \mathbb{R}^m$. Clearly, each affine linear map on $\mathbb{R}^m$ is continuous. An affine linear map f is said to be of *full rank* if the associated linear map T is invertible. Note that full rank affine linear maps are open maps, that is, they map open sets to open sets. They also map geometrically independent sets to geometrically independent sets. We sew affine linear maps on simplices piece by piece to get many continuous maps on $\mathbb{R}^m$.

An $\mathbb{R}^m$-valued function f on a subset X of $\mathbb{R}^m$ is said to be *piecewise affine linear* (respectively *countably piecewise affine linear*) if $X = |\mathcal{K}|$ for some finite (respectively countable) simplicial complex $\mathcal{K}$ of simplices of $\mathbb{R}^m$ and $f|_\sigma$ is the restriction of some affine linear map ϕ_σ on $\mathbb{R}^m$ to σ for every $\sigma \in \mathcal{K}$. In this case, we say that f is supported by the simplicial complex $\mathcal{K}$.

Here is a basic construction of piecewise affine linear maps that we use repetitively. Let $\mathcal{K}$ be a simplicial complex in $\mathbb{R}^m$ and $U = \{x_0, x_1, \ldots\}$ be its set of vertices ordered in some arbitrary way. Let $V = \{y_0, y_1, \ldots\}$ be an ordered set of points in $\mathbb{R}^m$. Then we can define a $\mathbb{R}^m$-valued function $f_{U,V}$ on $|\mathcal{K}|$ satisfying $f_{U,V}(x_j) = y_j$ for all $j \in \mathbb{Z}_+$ as follows: If $x \in |\mathcal{K}|$, then x is in the interior of a unique k-simplex $\langle x_{i_0}, x_{i_1}, \ldots, x_{i_k} \rangle$ in $\mathcal{K}$ for some $0 \leq k \leq m$. We write $x = \sum_{j=0}^{k} \alpha_{i_j} x_{i_j}$ with $\sum_{j=0}^{k} \alpha_{i_j} = 1$ and define

$$f_{U,V}(x) = \sum_{j=0}^{k} \alpha_{i_j} y_{i_j}. \tag{3.2}$$

Clearly, $f_{U,V}$ is a continuous piecewise affine linear map on $|\mathcal{K}|$ satisfying that $f_{U,V}(x_j) = y_j$ for every $j \in \mathbb{Z}_+$. Furthermore, if $|\mathcal{K}|$ is convex and $y_j \in |\mathcal{K}|$ for every j, then $f_{U,V}$ is a self-map of $|\mathcal{K}|$.

LEMMA 3.1. *Let $\mathcal{K}$ and $f_{U,V}$ be as described above.*

(i) *If $\sigma = \langle x_{i_0}, x_{i_1}, \ldots, x_{i_k} \rangle$ is a k-simplex in $\mathcal{K}$ and $\{y_{i_0}, y_{i_1}, \ldots, y_{i_k}\}$ is geometrically independent in $\mathbb{R}^m$, then $f_{U,V}|_\sigma$ is injective and $f_{U,V}(\sigma)^0 = f_{U,V}(\sigma^0)$.*

(ii) *If $V = \{y_0, y_1, \ldots\}$ and $W = \{z_0, z_1, \ldots\}$ are ordered sets in $|\mathcal{K}|$ such that $\|y_j - z_j\|_\infty \leq \eta$ for all $j \in \mathbb{Z}_+$, where $\eta > 0$, then $\|f_{U,V}(x) - f_{U,W}(x)\|_\infty \leq \eta$ for all $x \in |\mathcal{K}|$.*

Proof. The result (i) is trivial. For each $x \in |\mathcal{K}|$, by equation (3.2) we have

$$\|f_{U,V}(x) - f_{U,W}(x)\|_\infty \leq \sum_{j=0}^{k} \alpha_{i_j} \|y_{i_j} - z_{i_j}\|_\infty \leq \eta,$$

proving result (ii). $\square$

LEMMA 3.2. *If $f \in \mathcal{F}(\mathbb{R}^m)$ is an affine linear map such that $f|_S = 0$ for a geometrically independent set $S = \{x_0, x_1, \ldots, x_m\}$ of $\mathbb{R}^m$, then $f = 0$ on $\mathbb{R}^m$.*

Proof. Let $f(\cdot) = T(\cdot) + z$ for some $z \in \mathbb{R}^m$ and a linear map $T \in \mathcal{F}(\mathbb{R}^m)$. Since S is geometrically independent, we have $B = \{x_1 - x_0, x_2 - x_0, \ldots, x_m - x_0\}$ is linearly independent, and therefore it is a Hamel basis for $\mathbb{R}^m$. To prove $f = 0$, consider an arbitrary $x \in \mathbb{R}^m$. Then $x = \sum_{j=1}^{m} \alpha_j (x_j - x_0)$ for some $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$, implying

that

$$f(x) = T\left(\sum_{j=1}^{m} \alpha_j (x_j - x_0) \right) + z$$

$$= \sum_{j=1}^{m} \alpha_j T(x_j) - \sum_{j=1}^{m} \alpha_j T(x_0) + z$$

$$= \sum_{j=1}^{m} \alpha_j (-z) - \sum_{j=1}^{m} \alpha_j (-z) + z = z.$$

In particular, $f(x_0) = z$, and therefore we get that $z = 0$. Hence, $f = 0$ on $\mathbb{R}^m$. □

Using the above lemma, we can deduce the following result on the uniqueness of affine linear maps.

COROLLARY 3.3. *If $f_1, f_2 \in \mathcal{F}(\mathbb{R}^m)$ are affine linear maps such that $f_1|_\sigma = f_2|_\sigma$ for some m-simplex $\sigma \in \mathcal{K}$, then $f_1 = f_2$ on $\mathbb{R}^m$. Consequently, if $f_1|_U = f_2|_U$ for some open subset U of $\mathbb{R}^m$, then $f_1 = f_2$ on $\mathbb{R}^m$.*

Proof. Follows from the above lemma, because $f_1 - f_2$ is an affine linear map on $\mathbb{R}^m$ that vanishes on the geometrically independent set of all vertices of σ. □

COROLLARY 3.4. *Let $f \in \mathcal{F}(\mathbb{R}^m)$ be an affine linear map and σ be an m-simplex in $\mathbb{R}^m$ with vertices $x_0, x_1, \ldots, x_m$ such that $y_0, y_1, \ldots, y_m$ are geometrically independent in $\mathbb{R}^m$, where $y_j = f(x_j)$ for all $0 \le j \le m$. Then the following statements are true.*

(i) $f(U)$ *is open for every open set U in the interior of σ.*
(ii) *f maps geometrically independent subsets of $\mathbb{R}^m$ to geometrically independent subsets.*
(iii) *If $y_j \neq x_j$ for some $0 \le j \le m$ and $\{v_0, v_1, \ldots, v_m\}$ is a geometrically independent subset of σ, then there exists at least one j such that $f(v_j) \neq v_j$.*

Proof. Results (i) and (ii) are trivially true, because geometric independence of $\{y_0, y_1, \ldots, y_m\}$ implies that f is of full rank. If $f(v_j) = v_j$ for all $0 \le j \le m$, then $f = \mathrm{id}$ on $\mathbb{R}^m$, implying that $y_j = f(x_j) = x_j$ for all $0 \le j \le m$. Therefore, (iii) follows. □

For each subset S of $\mathbb{R}^m$, let

$$\mathrm{Aff}(S) := \left\{ \sum_{j=1}^{l} \alpha_j x_j : x_j \in S \text{ for } 1 \le j \le l \text{ and } \sum_{j=1}^{l} \alpha_j = 1 \right\}$$

and

$$\mathrm{Aff}_0(S) := \left\{ \sum_{j=1}^{l} \alpha_j x_j : x_j \in S \text{ for } 1 \le j \le l \text{ and } \sum_{j=1}^{l} \alpha_j = 0 \right\}.$$

Then $\mathrm{Aff}_0(S)$ is a vector subspace of $\mathbb{R}^m$ and $\mathrm{Aff}(S) = \mathrm{Aff}_0(S) + x$ for some $x \in S$, where $\mathrm{Aff}_0(S) + x := \{y + x : y \in \mathrm{Aff}_0(S)\}$. We say that $\mathrm{Aff}(S)$ has dimension k, written as $dim(\mathrm{Aff}(S)) = k$, if $\mathrm{Aff}_0(S)$ is a k-dimensional vector subspace of $\mathbb{R}^m$. A point $z \in \mathbb{R}^m$ is said to be *geometrically independent of S* if $z \notin \mathrm{Aff}(S)$. Then it is clear that z is geometrically independent of S if and only if 0 is geometrically independent of $S - z$.

LEMMA 3.5. *Let S be a countable subset of $\mathbb{R}^m$. Then for each $z \in \mathbb{R}^m$ and $\zeta > 0$, there exists a $y \in B_\zeta(z)$ such that y is geometrically independent of Z for all subsets Z of S such that $\#Z \leq m$.*

Proof. Since $dim(\mathrm{Aff}(Z)) \leq m - 1$, clearly $\mathrm{Aff}(Z) \bigcap \overline{B_{\zeta/2}(z)}$ is a nowhere dense subset of the complete metric space $\overline{B_{\zeta/2}(z)}$ for each subset Z of S such that $\#Z \leq m$. Therefore, by the Baire category theorem, we have

$$\overline{B_{\zeta/2}(z)} \neq \bigcup \left(\mathrm{Aff}(Z) \bigcap \overline{B_{\zeta/2}(z)} \right),$$

where the union is taken over all subsets Z of S such that $\#Z \leq m$. Hence, there exists a $y \in \overline{B_{\zeta/2}(z)}$ such that y is geometrically independent of Z for all subsets Z of S such that $\#Z \leq m$. Since $\overline{B_{\zeta/2}(z)} \subseteq B_\zeta(z)$, the result follows. $\qquad \square$

LEMMA 3.6. *Let $z_1, z_2, \ldots$ be a sequence of points in $\mathbb{R}^m$ and $\zeta_1, \zeta_2, \ldots$ be a sequence of positive reals. Then there exists a sequence $y_1, y_2, \ldots$ in $\mathbb{R}^m$ such that $y_j \in B_{\zeta_j}(z_j)$ for each $j \in \mathbb{N}$ and every subset Z of $S = \{y_1, y_2, \ldots\}$ where $\#Z \leq m + 1$ is geometrically independent.*

Proof. Given $z_1 \in \mathbb{R}^m$ and $\zeta_1 > 0$, consider any $y_1 \in B_{\zeta_1}(z_1)$ such that $y_1 \neq z_1$. Then clearly any subset Z of $S_1 = \{y_1\}$ with $\#Z \leq m + 1$ is geometrically independent. Next, by induction, suppose that $k > 1$, and there exist $y_1, y_2, \ldots, y_k \in \mathbb{R}^m$ with $y_j \in B_{\zeta_j}(z_j)$ for $1 \leq j \leq k$ such that Z is geometrically independent for all subsets Z of $S_k = \{y_1, y_2, \ldots, y_k\}$ with $\#Z \leq m + 1$. Then by Lemma 3.5, there exists $y_{k+1} \in B_{\zeta_{k+1}}(z_{k+1})$ such that y_{k+1} is geometrically independent of Z for all subsets Z of S_k such that $\#Z \leq m$. This implies that Z is geometrically independent for all subsets Z of $S_{k+1} = \{y_1, y_2, \ldots, y_{k+1}\}$ such that $\#Z \leq m + 1$, proving the result for $k + 1$. Thus, continuing the above process, we get a sequence $y_1, y_2, \ldots$ in $\mathbb{R}^m$ such that $y_j \in B_{\zeta_j}(z_j)$ for each $j \in \mathbb{N}$ satisfying the desired property. $\qquad \square$

In addition to the above notions and results on simplicial complexes, we need the following lemma to prove our results.

LEMMA 3.7. *Let X be a metric space with metric d and $f \in C(X)$. If $x \in X$ is not a fixed point of f, then there exists an open set V in X containing x such that $f^{-1}(V) \cap V = \emptyset$ and $f(V) \cap V = \emptyset$.*

Proof. Let $\epsilon = d(x, f(x))$. Since $f(x) \neq x$, clearly $\epsilon > 0$. Since f is continuous at x, there exists a $\delta > 0$ with $\delta < \epsilon/4$ such that

$$d(f(x), f(y)) < \frac{\epsilon}{4} \quad \text{whenever } y \in X \text{ with } d(x, y) < \delta. \tag{3.3}$$

Let $V := B_{\delta/2}(x)$.

Claim. $f^{-1}(V) \cap V = \emptyset$ and $f(V) \cap V = \emptyset$.

Suppose that $y \in f^{-1}(V) \cap V$. Then $f(y) \in V$, implying that $d(y, f(y)) \leq d(y, x) + d(x, f(y)) < \delta$. Also, by equation (3.3), we have $d(f(x), f(y)) < \epsilon/4$. Therefore, $\epsilon = d(x, f(x)) \leq d(x, y) + d(y, f(y)) + d(f(y), f(x)) < 3\epsilon/4$ is a contradiction. Next, suppose that $y \in f(V) \cap V$. Then $y = f(z)$ for some $z \in V$. Therefore, by equation (3.3), we have $d(f(x), y) < \epsilon/4$, implying that $\epsilon = d(x, f(x)) \leq d(x, y) + d(y, f(x)) < \epsilon/2$, which is a contradiction. Thus the claim holds, and the result follows. $\square$

Having the above machinery, we are ready to prove our desired results.

THEOREM 3.8. *If $h \in C(I^m)$, then for each $\epsilon > 0$ the ball $B_\epsilon(h)$ of radius ϵ around h contains a map f having no (even discontinuous) iterative square roots on I^m. In particular, $\mathcal{W}(2; I^m)$ does not contain any ball of $C(I^m)$.*

Proof. Let $h \in C(I^m)$ and $\epsilon > 0$ be arbitrary. Our strategy to prove the existence of an $f \in B_\epsilon(h)$ with no square roots is to make use of Theorem 2.4, and we construct this f in a few steps.

Step 1: Construct a piecewise affine linear map $f_0 \in B_\epsilon(h)$ supported by some suitable triangulation $\mathcal{K}$ of I^m. Since h is uniformly continuous on I^m, there exists a $\delta > 0$ such that

$$\|h(x) - h(y)\|_\infty < \frac{\epsilon}{10} \quad \text{whenever } x, y \in I^m \text{with } \|x - y\|_\infty < \delta. \tag{3.4}$$

Being a polyhedron, I^m is triangulable. Consider a triangulation $\mathcal{K}$ of I^m with vertices $x_0, x_1, x_2, \ldots, x_r$ such that $\|x_{i_0} - x_{i_1}\|_\infty < \delta/4$ for all 2-simplex $\langle x_{i_0}, x_{i_1} \rangle$ of σ and for all m-simplex $\sigma \in \mathcal{K}$. Choose $y_0, y_1, y_2, \ldots, y_r \in I^m$ inductively such that $y_j \neq x_j$ and

$$\|h(x_j) - y_j\|_\infty < \frac{\epsilon}{20} \quad \text{for all } 0 \leq j \leq r, \tag{3.5}$$

and $\{y_{i_0}, y_{i_1}, \ldots, y_{i_m}\}$ is geometrically independent whenever $\langle x_{i_0}, x_{i_1}, \ldots, x_{i_m} \rangle \in \mathcal{K}$ for $0 \leq i_0, i_1, \ldots, i_m \leq r$. This is possible by Lemma 3.6. Here, while choosing points y_j with $y_j \in B_{\epsilon/20}(h(x_j))$ for $0 \leq j \leq r$ satisfying the geometric independence condition, we have some additional restrictions to fulfill, viz. $y_j \neq x_j$ and $y_j \in I^m$ for all $0 \leq j \leq r$. However, the Baire category theorem provides this flexibility as there are plenty of vectors to choose from. Then,

$$\|y_{i_0} - y_{i_1}\|_\infty \leq \|y_{i_0} - h(x_{i_0})\|_\infty + \|h(x_{i_0}) - h(x_{i_1})\|_\infty + \|h(x_{i_1}) - y_{i_1}\|_\infty$$
$$< \frac{\epsilon}{20} + \frac{\epsilon}{10} + \frac{\epsilon}{20} = \frac{\epsilon}{5} \tag{3.6}$$

whenever $\langle x_{i_0}, x_{i_1} \rangle$ is a 2-simplex of σ for all m-simplex $\sigma \in \mathcal{K}$. Let $f_0 : |\mathcal{K}| = I^m \to I^m$ be the map $f_{U,V}$ defined as in equation (3.2), where U and V are the ordered sets $\{x_0, x_1, \ldots, x_r\}$ and $\{y_0, y_1, \ldots, y_r\}$, respectively. Then f_0 is a continuous piecewise affine linear self-map of I^m satisfying that $f_0(x_j) = y_j$ for all $0 \leq j \leq r$. Also, by using

result (i) of Lemma 3.1, we have $R(f_0)^0 \neq \emptyset$ and $f_0|_\sigma$ is injective for all $\sigma \in \mathcal{K}$. Further, from equation (3.6) we have

$$\text{diam}(f_0(\sigma)) < \frac{\epsilon}{5} \quad \text{for all } \sigma \in \mathcal{K}. \tag{3.7}$$

Claim. $\rho(f_0, h) < \epsilon/6$.

Consider an arbitrary $x \in I^m$. Then x is in the interior of some unique k-simplex $\langle x_{i_0}, x_{i_1}, \ldots, x_{i_k} \rangle$ in $\mathcal{K}$ for some $0 \leq k \leq m$. Let $x = \sum_{j=0}^k \alpha_{i_j} x_{i_j}$ with $\sum_{j=0}^k \alpha_{i_j} = 1$. Then by equations (3.4) and (3.5), we have

$$\|f_0(x) - h(x)\|_\infty = \left\| \sum_{j=0}^k \alpha_{i_j} y_{i_j} - \sum_{j=0}^k \alpha_{i_j} h(x) \right\|_\infty$$
$$\leq \sum_{j=0}^k \alpha_{i_j} \{ \|y_{i_j} - h(x_{i_j})\|_\infty + \|h(x_{i_j}) - h(x)\|_\infty \}$$
$$< \frac{\epsilon}{20} + \frac{\epsilon}{10}$$
$$< \frac{\epsilon}{6}, \tag{3.8}$$

since $\|x - x_{i_j}\|_\infty \leq \text{diam}(\langle x_{i_0}, x_{i_1}, \ldots, x_{i_k} \rangle) < \delta/4$ for $0 \leq j \leq k$. This proves the claim and Step 1 follows.

Step 2: Obtain an m-simplex $\Delta \in \mathcal{K}$ containing a non-empty open set $Y \subseteq \Delta^0 \cap R(f_0)^0$ such that $f_0(Y) \subseteq \Delta_1^0$ for some m-simplex Δ_1 in $\mathcal{K}$. Consider any m-simplex $\sigma = \langle x_{i_0}, x_{i_1}, \ldots, x_{i_m} \rangle$ in $\mathcal{K}$. Since $\{y_{i_0}, y_{i_1}, \ldots, y_{i_m}\}$ is geometrically independent by construction, we have $f_0(Z)$ is open in I^m for each open set Z in σ^0. Choose a non-empty open subset Y_0 of $f_0(\sigma)^0$ so small that $Y_0 \subseteq \Delta^0$ for some m-simplex $\Delta \in \mathcal{K}$. Then $f_0(Y_0)$ has non-empty interior. Again, choose a sufficiently small non-empty open subset Y of Y_0 such that $f_0(Y) \subseteq \Delta_1$ for some m-simplex $\Delta_1 \in \mathcal{K}$. This completes the proof of Step 2.

Step 3: Show that there exists a barycentric subdivision $\mathcal{K}^{(s)}$ of $\mathcal{K}$ for some $s \in \mathbb{N}$ with an m-simplex $\sigma_0 \in \mathcal{K}^{(s)}$ such that $\sigma_0 \subseteq Y$, and $f_0^{-1}(\sigma_0) \cap \sigma = \emptyset$ and $f_0(\sigma_0) \cap \sigma = \emptyset$ for all $\sigma \in \mathcal{K}^{(s)}$ with $\sigma \cap \sigma_0 \neq \emptyset$. Since $y_j \neq x_j$ for all $0 \leq j \leq r$, by Corollary 3.4, we have $f_0(z_0) \neq z_0$ for some $z_0 \in Y$. Choose an open set W_0 in I^m such that $z_0 \in W_0 \subseteq Y$ and $f_0(x) \neq x$ for all $x \in W_0$. Then using Lemma 3.7, we get an open set W in I^m with $z_0 \in W \subseteq W_0$ such that $f_0^{-1}(W) \cap W = \emptyset$ and $f_0(W) \cap W = \emptyset$. Perform a barycentric subdivision of $\mathcal{K}$ of order s so large that W contains an m-simplex $\sigma_0 \in \mathcal{K}^{(s)}$ and all m-simplices $\sigma \in \mathcal{K}^{(s)}$ adjacent to it, that is, $\sigma \subseteq W$ for all m-simplices $\sigma \in \mathcal{K}^{(s)}$ such that $\sigma \cap \sigma_0 \neq \emptyset$. Since $\lim_{l \to \infty} \text{mesh}(\mathcal{K}^{(l)}) = 0$, it is possible to obtain such a subdivision of $\mathcal{K}$. This completes the proof of Step 3.

Step 4: Modify f_0 slightly to get a new piecewise affine linear map $f \in B_\epsilon(h)$ that is constant on σ_0. To retain continuity, when we modify f_0 on σ_0, we must also modify it on all m-simplices in $\mathcal{K}^{(s)}$ adjacent to it. Let $A := \bigcup_{j=0}^p \sigma_j$, where σ_j is precisely an m-simplex in $\mathcal{K}^{(s)}$ such that $\sigma_j \cap \sigma_0 \neq \emptyset$ for all $0 \leq j \leq p$. Then, it is clear that $A \subseteq W \subseteq \Delta^0$.

Let the vertices of σ_0 be $u_0, \ldots, u_{m-1}$ and u_m, and let $v_j := f_0(u_j)$ for all $0 \leq j \leq m$. Then $v_j \neq u_j$ for all $0 \leq j \leq m$, because $u_j \in W$ for all $0 \leq j \leq m$ and $f_0(W) \cap W = \emptyset$. Also, since $u_0, u_1, \ldots, u_m$ are geometrically independent vectors in Δ, by result (ii) of Corollary 3.4, we see that $v_0, v_1, \ldots, v_m$ are geometrically independent. Further, they are all contained in the interior of a single simplex Δ_1 of $\mathcal{K}$ by Step 2. Hence, by result (iii) of Corollary 3.4, it follows that $f_0(v_j) \neq v_j$ for some $0 \leq j \leq m$. Without loss of generality, we assume that $f_0(v_0) \neq v_0$.

Extend $\{u_0, \ldots, u_m\}$ to $\{u_0, \ldots, u_{k_0}\}$ with $k_0 > m$ to include the vertices of all the simplices of $\mathcal{K}^{(s)}$, and let $v_j := f_0(u_j)$ for all $0 \leq j \leq k_0$. Now, define f by

$$f(u_j) = \begin{cases} v_0 & \text{if } 0 \leq j \leq m, \\ v_j & \text{if } m < j \leq k_0, \end{cases}$$

and extend it piecewise affine linearly to the whole of $|\mathcal{K}^{(s)}| = I^m$. Then f_0 is modified only on A and $f(x) = v_0$ for all $x \in \sigma_0$.

Consider an arbitrary $x \in A$. Then $x \in \sigma_j$ for some $0 \leq j \leq p$. Let $\sigma_j = \langle u_{j_0}, u_{j_1}, \ldots, u_{j_m} \rangle$ and $x = \sum_{i=0}^m \alpha_{j_i} u_{j_i}$ with $\sum_{i=0}^m \alpha_{j_i} = 1$. Then by equation (3.7), we have

$$\|f(x) - f_0(x)\|_\infty \leq \sum_{i=0}^m \alpha_{j_i} \|f(u_{j_i}) - f_0(u_{j_i})\|_\infty \leq \sum_{i=0}^m \alpha_{j_i} \operatorname{diam}(f_0(\Delta)) < \frac{\epsilon}{5},$$

implying by equation (3.8) that

$$\|f(x) - h(x)\|_\infty \leq \|f(x) - f_0(x)\|_\infty + \|f_0(x) - h(x)\|_\infty < \frac{\epsilon}{5} + \frac{\epsilon}{6} < \frac{\epsilon}{2}.$$

Therefore, $f \in B_\epsilon(h)$ and the proof of Step 4 is completed.

Step 5. Prove that f has no square roots in $C(I^m)$. Since $A \subseteq W$, $f = f_0$ on A^c, and $f_0(W) \cap W = \emptyset$, we see that $f(v_0) \neq v_0$. Also, by Step 3, we have $f_0^{-1}(\sigma_0) \cap \sigma = \emptyset$ for all $\sigma \in \mathcal{K}^{(s)}$ with $\sigma \cap \sigma_0 \neq \emptyset$, implying that $f = f_0$ on $f_0^{-1}(\sigma_0)$. Further, $f_0^{-1}(\sigma_0)$ is infinite, since σ_0 is infinite and $\sigma_0 \subseteq R(f_0)^0$. Therefore, as $f^{-2}(v_0) \supseteq f^{-1}(f^{-1}(v_0)) \supseteq f^{-1}(\sigma_0) \supseteq f_0^{-1}(\sigma_0)$, it follows that $f^{-2}(v_0)$ is infinite. Moreover, $f|_\sigma$ is injective for every $\sigma \in \mathcal{K}^{(s)}$ with $\sigma \neq \sigma_0$ by result (i) of Lemma 3.1, and hence $f^{-1}(x)$ is finite for all $x \neq v_0$ in I^m. Thus, f satisfies the conditions of Case (ii) of Theorem 2.4, proving that it has no square roots in $C(I^m)$. $\qquad \square$

THEOREM 3.9. *Let $h \in C(I^m)$ be such that $h(x_0) = x_0$ for some $x_0 \in \partial I^m$. Then for each $\epsilon > 0$, the ball $B_\epsilon(h)$ of radius ϵ around h contains a map f having a continuous iterative square root on I^m. In other words, the closure of $\mathcal{W}(2; I^m)$ contains all maps in $C(I^m)$ that have a fixed point on the boundary of I^m.*

Proof. Let $h \in C(I^m)$ be such that $h(x_0) = x_0$ for some $x_0 \in \partial I^m$. In view of Step 1 of Theorem 3.8, without loss of generality, we assume that h is piecewise affine linear on I^m. Since h is uniformly continuous on I^m, there is a δ with $0 < \delta < \epsilon/4$ such that

$$\|h(x) - h(y)\|_\infty < \frac{\epsilon}{4} \quad \text{whenever } x, y \in I^m \text{ with } \|x - y\|_\infty < \delta. \tag{3.9}$$

Consider a triangulation $\mathcal{K}$ of I^m with vertices $x_0, x_1, x_2, \ldots, x_r$ such that $\|x_{i_0} - x_{i_1}\|_\infty < \delta/4$ for all 2-simplices $\langle x_{i_0}, x_{i_1} \rangle$ of σ and for all m-simplex $\sigma \in \mathcal{K}$. Let $y_j := h(x_j)$ for all $0 \le j \le r$ and $\sigma_0 = \langle x_0, x_1, \ldots, x_m \rangle$ be an m-simplex in $\mathcal{K}$ having x_0 as a vertex. Let $\phi : I^m \to I^m$ be a homeomorphism such that $\phi(\sigma_0) = \overline{I^m \setminus \sigma_0}$, $\phi(\overline{I^m \setminus \sigma_0}) = \sigma_0$, $\phi = \mathrm{id}$ on $\partial \sigma_0 \bigcap \partial (I^m \setminus \sigma_0)$, and $\phi^2 = \mathrm{id}$ on I^m. This is where we use the assumption that x_0 is on the boundary of I^m. Indeed, a little thought shows that there exists a piecewise affine linear map ϕ that satisfies these conditions. Define a function f_1 by

$$f_1(x_j) = \begin{cases} x_j & \text{if } x_j \in \sigma_0, \\ x_1 & \text{if } x_j \notin \sigma_0 \text{ and } y_j \in \sigma_0^0, \\ y_j & \text{if } x_j \notin \sigma_0 \text{ and } y_j \notin \sigma_0^0, \end{cases}$$

and extend it piecewise affine linearly to the whole of $|\mathcal{K}| = I^m$. Then $f_1 \in C(I^m)$ such that $f_1|_{\sigma_0} = \mathrm{id}$ and $f_1(\overline{I^m \setminus \sigma_0}) \subseteq \overline{I^m \setminus \sigma_0}$.

Claim. $\rho(f_1, h) < \epsilon$.

Let $x \in I^m$ be arbitrary. If $x \in \sigma_0$, then there exist non-negative reals $\alpha_0, \alpha_1, \ldots, \alpha_m$ with $\sum_{j=0}^m \alpha_j = 1$ such that $x = \sum_{j=0}^m \alpha_j x_j$, implying by equation (3.9) that

$$\|f_1(x) - h(x)\|_\infty = \left\| \sum_{j=0}^m \alpha_j f_1(x_j) - \sum_{j=0}^m \alpha_j h(x_j) \right\|_\infty$$

$$\le \sum_{j=0}^m \alpha_j \{ \|x_j - x_0\|_\infty + \|h(x_0) - h(x_j)\|_\infty \}$$

$$< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2},$$

since $\|x_j - x_0\|_\infty \le \mathrm{diam}(\sigma_0) < \delta < \epsilon/4$ for all $0 \le j \le m$. If $x \notin \sigma_0$, then there exists a $\sigma = \langle x_{i_0}, x_{i_1}, \ldots, x_{i_m} \rangle \in \mathcal{K}$ with $\sigma \ne \sigma_0$ such that $x \in \sigma$. Let $x = \sum_{j=0}^m \alpha_{i_j} x_{i_j}$, where $\alpha_{i_0}, \alpha_{i_1}, \ldots, \alpha_{i_m}$ are non-negative and $\sum_{j=0}^m \alpha_{i_j} = 1$. Then by using equation (3.9), we have

$$\|f_1(x) - h(x)\|_\infty = \left\| \sum_{j=0}^m \alpha_{i_j} f_1(x_{i_j}) - \sum_{j=0}^m \alpha_{i_j} h(x_{i_j}) \right\|_\infty \le \sum_{j=0}^m \alpha_{i_j} \|f_1(x_{i_j}) - y_{i_j}\|_\infty$$

$$= \sum_{\substack{0 \le j \le m \\ x_{i_j} \in \sigma_0}} \alpha_{i_j} \|f_1(x_{i_j}) - y_{i_j}\|_\infty + \sum_{\substack{0 \le j \le m \\ x_{i_j} \notin \sigma_0 \text{ and } y_{i_j} \in \sigma_0^0}} \alpha_{i_j} \|f_1(x_{i_j}) - y_{i_j}\|_\infty$$

$$+ \sum_{\substack{0 \le j \le m \\ x_{i_j} \notin \sigma_0 \text{ and } y_{i_j} \notin \sigma_0^0}} \alpha_{i_j} \|f_1(x_{i_j}) - y_{i_j}\|_\infty$$

$$\le \sum_{\substack{0 \le j \le m \\ x_{i_j} \in \sigma_0}} \alpha_{i_j} \|x_{i_j} - y_{i_j}\|_\infty + \sum_{\substack{0 \le j \le m \\ x_{i_j} \notin \sigma_0 \text{ and } y_{i_j} \in \sigma_0^0}} \alpha_{i_j} \|x_1 - y_{i_j}\|_\infty$$

$$+ \sum_{\substack{0 \leq j \leq m \\ x_{i_j} \notin \sigma_0 \text{ and } y_{i_j} \notin \sigma_0^0}} \alpha_{i_j} \| y_{i_j} - y_{i_j} \|_\infty$$

$$\leq \sum_{\substack{0 \leq j \leq m \\ x_{i_j} \in \sigma_0}} \alpha_{i_j} \{ \| x_{i_j} - x_0 \|_\infty + \| h(x_0) - h(x_{i_j}) \|_\infty \}$$

$$+ \sum_{\substack{0 \leq j \leq m \\ x_{i_j} \notin \sigma_0 \text{ and } y_{i_j} \in \sigma_0^0}} \alpha_{i_j} \| x_1 - y_{i_j} \|_\infty < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{3\epsilon}{4},$$

since $\| x_{i_j} - x_0 \|_\infty \leq \text{diam}(\sigma_0) < \delta < \epsilon/4$ for all $0 \leq j \leq m$ such that $x_{i_j} \in \sigma_0$, and $\| x_1 - y_{i_j} \|_\infty \leq \text{diam}(\sigma_0) < \delta < \epsilon/4$ for all $0 \leq j \leq m$ such that $x_{i_j} \notin \sigma_0$ and $y_{i_j} \in \sigma_0^0$. This proves the claim.

Now, define a map $f : I^m \to I^m$ by

$$f(x) = \begin{cases} f_1 \circ \phi(x) & \text{if } x \in \sigma_0, \\ \phi(x) & \text{if } x \notin \sigma_0. \end{cases}$$

Since $f_1 \circ \phi(x) = x = \phi(x)$ for all $x \in \partial \sigma_0 \bigcap \partial(I^m \setminus \sigma_0)$, clearly f is continuous on I^m. We prove that $\rho(f^2, h) < \epsilon$. Consider an arbitrary $x \in I^m$. If $x \in \sigma_0$, then $f(x) \in \overline{I^m \setminus \sigma_0}$, because $\phi(x) \in \overline{I^m \setminus \sigma_0}$ and $f_1(\overline{I^m \setminus \sigma_0}) \subseteq \overline{I^m \setminus \sigma_0}$. Therefore, $f^2(x) \in \sigma_0$, implying by equation (3.9) that

$$\| f^2(x) - h(x) \|_\infty \leq \| f^2(x) - x_0 \|_\infty + \| h(x_0) - h(x) \|_\infty < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2},$$

since $\| f^2(x) - x_0 \|_\infty \leq \text{diam}(\sigma_0) < \delta < \epsilon/4$. If $x \notin \sigma_0$, then $f^2(x) = f_1(x)$, and therefore $\| f^2(x) - h(x) \|_\infty = \| f_1(x) - h(x) \|_\infty < 3\epsilon/4$. This completes the proof. $\qquad \square$

Now it is a natural question to ask what happens if all the fixed points of the map are in the interior of I^m. Theorem 10 of [**15**, p. 364] shows that the map $f(x) = 1 - x$ on the unit interval [0, 1], with only $x = \frac{1}{2}$ as the fixed point, cannot be approximated by squares of continuous maps. In contrast, we have the following.

Example 3.10. In this example we show that, despite having no continuous square roots, the map $f : I^2 \to I^2$ defined by

$$f(x, y) = \left(1 - x, \tfrac{1}{2} \right) \quad \text{for all } (x, y) \in I^2$$

having an interior point $(x, y) = (\frac{1}{2}, \frac{1}{2})$ as the unique fixed point can be approximated by squares of continuous maps on I^2. To prove that f has no continuous square roots, on the contrary, assume that $f = g^2$ for some $g \in C(I^2)$. Since $(1 - x, \frac{1}{2}) \in R(g)$, there exists $(f_1(x), f_2(x)) \in I^2$ such that $g(f_1(x), f_2(x)) = (1 - x, \frac{1}{2})$ for each $x \in I$. Then

$$g\left(1 - x, \tfrac{1}{2} \right) = g^2(f_1(x), f_2(x)) = f(f_1(x), f_2(x)) = \left(1 - f_1(x), \tfrac{1}{2} \right) \quad \text{for all } x \in I,$$

implying that

$$g\left(x, \tfrac{1}{2} \right) = \left(f_1(x), \tfrac{1}{2} \right) \quad \text{for all } x \in I. \tag{3.10}$$

Therefore,

$$\left(1 - x, \tfrac{1}{2}\right) = f\left(x, \tfrac{1}{2}\right) = g^2\left(x, \tfrac{1}{2}\right) = g\left(f_1(x), \tfrac{1}{2}\right) = \left(f_1^2(x), \tfrac{1}{2}\right) \quad \text{for all } x \in I,$$

and hence $f_1^2(x) = 1 - x$ for all $x \in I$. Also, from equation (3.10), it follows that f_1 is continuous on I. Thus, the map $x \mapsto 1 - x$ has a continuous square root on I, a contradiction to Corollary 2.7. Hence, f has no continuous square roots on I^2.

Now, to prove that f can be approximated by squares of continuous maps, consider an arbitrary $\epsilon > 0$. Without loss of generality, we assume that $\epsilon < \tfrac{1}{2}$. The idea is to compress I^2 to the small strip $I \times [\tfrac{1}{2} - \epsilon/2, \tfrac{1}{2} + \epsilon/2]$ and then rotate this strip ninety degrees clockwise with a suitable scaling to stay within the strip. More explicitly, let $g : I^2 \to I \times [\tfrac{1}{2} - \epsilon/2, \tfrac{1}{2} + \epsilon/2]$ be defined by $g = g_1 \circ g_2$, where $g_1 : I \times [\tfrac{1}{2} - \epsilon/2, \tfrac{1}{2} + \epsilon/2] \to I \times [\tfrac{1}{2} - \epsilon/2, \tfrac{1}{2} + \epsilon/2]$ and $g_2 : I^2 \to I \times [\tfrac{1}{2} - \epsilon/2, \tfrac{1}{2} + \epsilon/2]$ are given by

$$g_1(x, y) = \left(\frac{\epsilon - 1}{2\epsilon}, \frac{\epsilon + 1}{2}\right) + (x, y) \begin{pmatrix} 0 & -\epsilon \\ \frac{1}{\epsilon} & 0 \end{pmatrix} = \left(\frac{1}{2} + \frac{2y - 1}{2\epsilon}, \frac{1}{2} - \frac{\epsilon(2x - 1)}{2}\right)$$

and

$$g_2(x, y) = \begin{cases} \left(x, \dfrac{1}{2} - \dfrac{\epsilon}{2}\right) & \text{if } (x, y) \in I \times \left[0, \dfrac{1}{2} - \dfrac{\epsilon}{2}\right], \\[2mm] (x, y) & \text{if } (x, y) \in I \times \left[\dfrac{1}{2} - \dfrac{\epsilon}{2}, \dfrac{1}{2} + \dfrac{\epsilon}{2}\right], \\[2mm] \left(x, \dfrac{1}{2} + \dfrac{\epsilon}{2}\right) & \text{if } (x, y) \in I \times \left[\dfrac{1}{2} + \dfrac{\epsilon}{2}, 1\right]. \end{cases}$$

Then g is continuous on I^2 such that

$$g^2(x, y) = \begin{cases} \left(1 - x, \dfrac{1}{2} + \dfrac{\epsilon}{2}\right) & \text{if } (x, y) \in I \times \left[0, \dfrac{1}{2} - \dfrac{\epsilon}{2}\right], \\[2mm] (1 - x, 1 - y) & \text{if } (x, y) \in I \times \left[\dfrac{1}{2} - \dfrac{\epsilon}{2}, \dfrac{1}{2} + \dfrac{\epsilon}{2}\right], \\[2mm] \left(1 - x, \dfrac{1}{2} - \dfrac{\epsilon}{2}\right) & \text{if } (x, y) \in I \times \left[\dfrac{1}{2} + \dfrac{\epsilon}{2}, 1\right], \end{cases}$$

and it can be easily verified that $\rho(f, g^2) < \epsilon$.

4. *Continuous functions on* $\mathbb{R}^m$

Consider $\mathbb{R}^m$ in the metric induced by the norm $\| \cdot \|_\infty$ defined as in equation (3.1). Since it is a locally compact separable metric space, by Corollary 7.1 of [2], the compact-open topology on $C(\mathbb{R}^m)$ is metrizable with the metric D given by

$$D(f, g) = \sum_{j=1}^{\infty} \mu_j(f, g)$$

such that

$$\mu_j(f, g) = \min\left\{\frac{1}{2^j}, \rho_j(f, g)\right\} \quad \text{for all } j \in \mathbb{N},$$

$$\rho_j(f, g) = \sup\{\|f(x) - g(x)\|_\infty : x \in M_j\} \quad \text{for all } j \in \mathbb{N},$$

where $(M_j)_{j \in \mathbb{N}}$ is a sequence of compact sets in $\mathbb{R}^m$ satisfying that $\mathbb{R}^m = \bigcup_{j=1}^\infty M_j$, and if M is any compact subset of $\mathbb{R}^m$, then $M \subseteq \bigcup_{i=1}^k M_{m_i}$ for some finitely many $M_{m_1}, M_{m_2}, \ldots, M_{m_k}$. For convenience, we assume that M_j is an m-simplex of $\mathbb{R}^m$ for each $j \in \mathbb{N}$, and either $M_i \cap M_j = \emptyset$ or $M_i \cap M_j$ is a common face of both M_i and M_j for all $i, j \in \mathbb{N}$. Considering all the faces M_j for all $j \in \mathbb{N}$, we get a countable triangulation of $\mathbb{R}^m$. Throughout this section, we fix one such triangulation of $\mathbb{R}^m$.

LEMMA 4.1. *Let $\sigma = \langle x_0, x_1, \ldots, x_m \rangle$ be an m-simplex of $\mathbb{R}^m$ and $z \in \sigma$ be such that $z \neq x_j$ for all $0 \leq j \leq m$. Then there exists a simplicial complex $\mathcal{L}_\sigma$ with the set of vertices $\{x_0, x_1, \ldots, x_m, z\}$ such that $|\mathcal{L}_\sigma| = \sigma$. In other words, every m-simplex of $\mathbb{R}^m$ can be triangulated to have a desired point as an extra vertex.*

Proof. Without loss of generality, we assume that $z \in \langle x_0, x_1, \ldots, x_k \rangle^0$ for some $1 \leq k \leq m$, say $z = \sum_{j=0}^k \alpha_j x_j$ for some strictly positive real numbers α_j such that $\sum_{j=0}^k \alpha_j = 1$. Then $V_i := \{x_0, x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_m\}$ is geometrically independent for all $0 \leq i \leq k$. In fact, for a fixed $0 \leq i \leq k$, if the real numbers β_j are arbitrary such that $\sum_{j=0}^m \beta_j = 0$ and

$$\sum_{\substack{0 \leq j \leq m \\ j \neq i}} \beta_j x_j + \beta_i z = 0,$$

then

$$\sum_{\substack{0 \leq j \leq k \\ j \neq i}} (\beta_j + \beta_i \alpha_j) x_j + \beta_i \alpha_i x_i + \sum_{k < j \leq m} \beta_j x_j = 0,$$

and therefore by geometric independence of $\{x_0, x_1, \ldots, x_m\}$, we have $\beta_j + \beta_i \alpha_j = \beta_i \alpha_i = 0$ for all $0 \leq j \leq k$ with $j \neq i$, and $\beta_j = 0$ for all $k < j \leq m$. This implies that $\beta_j = 0$ for all $0 \leq j \leq m$, since $\alpha_i > 0$.

Now, let $\sigma_i := \langle x_0, x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots x_m \rangle$ and $\mathcal{L}_i$ be the simplicial complex of all faces of σ_i for all $0 \leq i \leq k$.

Claim. $\mathcal{L}_\sigma = \bigcup_{i=0}^k \mathcal{L}_i$ is a triangulation of σ with the set of vertices $\{x_0, x_1, \ldots, x_m, z\}$.

Clearly, $\mathcal{L}_\sigma$ is a collection of simplices of $\mathbb{R}^m$ with $\{x_0, x_1, \ldots, x_m, z\}$ as its set of vertices. To show that it is a simplicial complex, consider an arbitrary $\lambda \in \mathcal{L}_\sigma$. Then $\lambda \in \mathcal{L}_i$ for some $0 \leq i \leq k$, implying that $\kappa \in \mathcal{L}_i \subseteq \mathcal{L}_\sigma$ whenever κ is a face of λ, because $\mathcal{L}_i$ is a simplicial complex. Next, consider any two simplices $\lambda, \eta \in \mathcal{L}_\sigma$. If $\lambda, \eta \in \mathcal{L}_i$ for the same $i \in \{0, 1, \ldots, k\}$, then clearly $\lambda \cap \eta$ is either empty or a common face of both λ and η,

because $\mathcal{L}_i$ is a simplicial complex. So, let $\lambda \cap \eta \neq \emptyset$, and suppose that $\lambda \in \mathcal{L}_i$ and $\eta \in \mathcal{L}_j$ for some $i \neq j$, where $0 \leq i, j \leq k$. Further, assume that λ and η are s and t dimensional, respectively, for some $0 \leq s, t \leq m$. Then it is easily seen $\lambda \cap \eta$ is the l-simplex with vertex set $V_i \cap V_j$, where $l = \#(V_i \cap V_j) - 1$. This implies that $\lambda \cap \eta \in \mathcal{L}_i \cap \mathcal{L}_j$, and therefore it is a common face of both λ and η. Hence, $\mathcal{L}_\sigma$ is a simplicial complex.

Now, to prove that $\mathcal{L}_\sigma$ is a triangulation of σ, consider an arbitrary $x \in \sigma$. Then $x \in \lambda := \langle x_{i_0}, x_{i_1}, \ldots, x_{i_s} \rangle$ for some s-dimensional face λ of σ, where $0 \leq s \leq m$. If $\lambda \in \mathcal{L}_i$ for some $0 \leq i \leq k$, then clearly $x \in |\mathcal{L}_i| \subseteq |\mathcal{L}_\sigma|$. If $\lambda \notin \mathcal{L}_i$ for all $0 \leq i \leq k$, then there exist at least two $i \in \{0, 1, \ldots, k\}$ such that $(V_\lambda \backslash \{z\}) \cap (V_i \backslash \{z\}) \neq \emptyset$, where $V_\lambda = \{x_{i_0}, x_{i_1}, \ldots, x_{i_s}\}$. This implies that $x \in \lambda \subseteq \bigcup_{i \in \Gamma} |\mathcal{L}_i| \subseteq |\mathcal{L}_\sigma|$, where $\Gamma := \{i : 0 \leq i \leq k$ and $(V_\lambda \backslash \{z\}) \cap (V_i \backslash \{z\}) \neq \emptyset\}$. Thus the claim holds and result follows. $\square$

LEMMA 4.2. *Let $\mathbb{R}^m = \bigcup_{j=1}^\infty M_j$ be the decomposition of $\mathbb{R}^m$ fixed above, and let $\delta_1, \delta_2, \ldots$ be a sequence of positive real numbers. Then there exists a countable triangulation $\mathcal{K} = \bigcup_{j=1}^\infty \mathcal{K}_j$ of $\mathbb{R}^m$ such that $\mathcal{K}_j$ is a triangulation of M_j and $diam(\sigma) < \delta_j$ for all $\sigma \in \mathcal{K}_j$ and $j \in \mathbb{N}$.*

Proof. Perform a barycentric subdivision of the m-simplex M_j of sufficient order to get a simplicial complex $\mathcal{L}_j$ such that $\text{diam}(\sigma) < \delta_j$ for all $\sigma \in \mathcal{L}_j$ and $j \in \mathbb{N}$. Then $\mathcal{L} := \bigcup_{j=1}^\infty \mathcal{L}_j$ is not necessarily a simplicial complex, and therefore need not be a countable triangulation of $\mathbb{R}^m$, as neighboring m-simplices M_j might have undergone barycentric subdivisions of different orders. Let $\{\sigma_1, \sigma_2, \ldots\}$ be the collection of all m-simplices in $\mathcal{L}$ and V be the set of vertices of all simplices in $\mathcal{L}$. Observe that V may contain some points of σ_i that are not its vertices for each $i \in \mathbb{N}$ due to the barycentric subdivision of its neighboring m-simplices. However, there are at most finitely many such points of each σ_i, since every compact subset of $\mathbb{R}^m$ is contained in a finite union of m-simplices M_j. Now, using Lemma 4.1, we refine σ_1 inductively so that we have a triangulation of σ_1, which includes all these points as vertices. Next, we consider σ_2 and repeat the same procedure. Note that these refinements add new simplices but do not increase the number of vertices. So, the refinement of σ_2 does not disturb that of σ_1 done before. Continuing in this way, we obtain a triangulation $\mathcal{K}_j$ of M_j for each $j \in \mathbb{N}$ so that $\mathcal{K} = \bigcup_{j=1}^\infty \mathcal{K}_j$ is a countable triangulation of $\mathbb{R}^m$ with the set of vertices V. This completes the proof. $\square$

Having the above lemmas, we are ready to prove our desired result.

THEOREM 4.3. *If $h \in C(\mathbb{R}^m)$, then for each $\epsilon > 0$, the ball $B_\epsilon(h)$ of radius ϵ around h contains a map f having no (even discontinuous) iterative square roots on $\mathbb{R}^m$. In particular, $\mathcal{W}(2; \mathbb{R}^m)$ does not contain any ball of $C(\mathbb{R}^m)$.*

Proof. Let $h \in C(\mathbb{R}^m)$ and $\epsilon > 0$ be arbitrary. Our strategy to prove the existence of an $f \in B_\epsilon(h)$ with no square roots is to make use of Theorem 2.4, and we construct this f in a few steps as done in Theorem 3.8.

Step 1: Construct a piecewise affine linear map $f_0 \in B_\epsilon(h)$ supported by some suitable triangulation $\mathcal{K}$ of $\mathbb{R}^m$. Consider the decomposition $\mathbb{R}^m = \bigcup_{j=1}^\infty M_j$ of $\mathbb{R}^m$ fixed above.

Since $h|_{M_j}$ is uniformly continuous, there exists a $\delta_j > 0$ such that

$$\|h(x) - h(y)\|_\infty < \frac{1}{2^{j+5}} \quad \text{whenever } x, y \in M_j \text{ with } \|x - y\|_\infty < \delta_j \quad (4.1)$$

for each $j \in \mathbb{N}$. Also, by Lemma 4.2, there exists a triangulation $\mathcal{K}$ of $\mathbb{R}^m$ such that diam$(\sigma) < \delta_j/4$ for all m-simplex $\sigma \in \mathcal{K}$ contained in M_j and for all $j \in \mathbb{N}$.

Let $x_0, x_1, \ldots$ be the vertices of $\mathcal{K}$. Choose $y_0, y_1, \ldots$ in $\mathbb{R}^m$ inductively such that $y_j \neq x_j$ and

$$\|h(x_j) - y_j\|_\infty < \frac{\epsilon}{2^{j+6}} \quad \text{for all } j \in \mathbb{Z}_+, \quad (4.2)$$

and $\{y_{i_0}, y_{i_1}, \ldots, y_{i_m}\}$ is geometrically independent whenever $\langle x_{i_0}, x_{i_1}, \ldots, x_{i_m}\rangle \in \mathcal{K}$ for $i_0, i_1, \ldots, i_m \in \mathbb{Z}_+$. This is possible by Lemma 3.6. Here, while choosing points y_j using Baire Category theorem with $y_j \in B_{\epsilon/2^{j+6}}(h(x_j))$ for all $j \in \mathbb{Z}_+$, satisfying the geometric independence condition, it is also possible to ensure that $y_j \neq x_j$ for all $j \in \mathbb{Z}_+$. Then, for each $j \in \mathbb{N}$, we have

$$\|y_{i_0} - y_{i_1}\|_\infty \leq \|y_{i_0} - h(x_{i_0})\|_\infty + \|h(x_{i_0}) - h(x_{i_1})\|_\infty + \|h(x_{i_1}) - y_{i_1}\|_\infty$$
$$< \frac{\epsilon}{2^{j+5}} + 2\frac{\epsilon}{2^{j+6}} = \frac{\epsilon}{2^{j+4}}, \quad (4.3)$$

whenever $\langle x_{i_0}, x_{i_1}\rangle$ is a 2-simplex of σ for all m-simplex $\sigma \in \mathcal{K}$ contained in M_j. Let $f_0 : |\mathcal{K}| = \mathbb{R}^m \to \mathbb{R}^m$ be the map $f_{U,V}$ defined as in equation (3.2), where U and V are the ordered sets $\{x_0, x_1, \ldots\}$ and $\{y_0, y_1, \ldots\}$, respectively. Then f_0 is a continuous piecewise affine linear self-map of $\mathbb{R}^m$ satisfying that $f_0(x_j) = y_j$ for all $j \in \mathbb{Z}_+$. Also, by using result (i) of Lemma 3.1, we have $R(f_0)^0 \neq \emptyset$, and $f_0|_\sigma$ is injective for all $\sigma \in \mathcal{K}$ contained in M_j for all $j \in \mathbb{N}$. Further, from equation (4.3) we have

$$\text{diam}(f_0(\sigma)) < \frac{\epsilon}{2^{j+4}} \quad \text{for all } \sigma \in \mathcal{K} \text{ contained in } M_j \text{ and for all } j \in \mathbb{N}. \quad (4.4)$$

Claim. $D(f_0, h) < \epsilon/4$.

Consider arbitrary $j \in \mathbb{N}$ and $x \in M_j$. Then x is in the interior of a unique k-simplex $\langle x_{i_0}, x_{i_1}, \ldots, x_{i_k}\rangle$ in $\mathcal{K}$ contained in M_j for some $0 \leq k \leq m$. Let $x = \sum_{j=0}^{k} \alpha_{i_j} x_{i_j}$ with $\sum_{j=0}^{k} \alpha_{i_j} = 1$. Then by equations (4.1) and (4.2), we have

$$\|f_0(x) - h(x)\|_\infty = \left\| \sum_{j=0}^{k} \alpha_{i_j} y_{i_j} - \sum_{j=0}^{k} \alpha_{i_j} h(x) \right\|_\infty$$
$$\leq \sum_{j=0}^{k} \alpha_{i_j} \{\|y_{i_j} - h(x_{i_j})\|_\infty + \|h(x_{i_j}) - h(x)\|_\infty\}$$
$$< \frac{\epsilon}{2^{j+6}} + \frac{\epsilon}{2^{j+5}}$$
$$< \frac{\epsilon}{2^{j+4}}, \quad (4.5)$$

since $\quad \|x - x_{i_j}\|_\infty \leq \operatorname{diam}(\langle x_{i_0}, x_{i_1}, \ldots, x_{i_k} \rangle) < \delta_j \quad$ for all $\quad 0 \leq j \leq k$. Therefore, $\rho_j(f_0, h) \leq \epsilon/2^{j+4} < \epsilon/2^{j+2}$ for all $j \in \mathbb{N}$, implying that $D(f_0, h) < \sum_{j=1}^{\infty}(\epsilon/2^{j+2}) = \epsilon/4$.

Step 2: Obtain an m-simplex $\Delta \in \mathcal{K}$ contained in M_r for some $r \in \mathbb{N}$ containing a non-empty open set $Y \subseteq \Delta^0 \cap R(f_0)^0$ such that $f_0(Y) \subseteq \Delta_1^0$ for some m-simplex Δ_1 in $\mathcal{K}$. Consider any m-simplex $\sigma = \langle x_{i_0}, x_{i_1}, \ldots, x_{i_m} \rangle$ in $\mathcal{K}$. Since $\{y_{i_0}, y_{i_1}, \ldots, y_{i_m}\}$ is geometrically independent by construction, we have $f_0(Z)$ is open in $\mathbb{R}^m$ for each open subset Z of σ^0. Choose a sufficiently small non-empty open subset Y_0 of $f_0(\sigma)^0$ such that $Y_0 \subseteq \Delta^0$ for some m-simplex $\Delta \in \mathcal{K}$ contained in M_r for some $r \in \mathbb{N}$. Then $f_0(Y_0)$ has non-empty interior. Again, choose a non-empty open subset Y of Y_0 so small that $f_0(Y) \subseteq \Delta_1$ for some m-simplex $\Delta_1 \in \mathcal{K}$. This completes the proof of Step 2.

Step 3: Show that there exists a barycentric subdivision $\mathcal{K}^{(s)}$ of $\mathcal{K}$ for some $s \in \mathbb{N}$ with an m-simplex $\sigma_0 \in \mathcal{K}^{(s)}$ such that $\sigma_0 \subseteq Y$, and $f_0^{-1}(\sigma_0) \cap \sigma = \emptyset$ and $f_0(\sigma_0) \cap \sigma = \emptyset$ for all $\sigma \in \mathcal{K}^{(s)}$ with $\sigma \cap \sigma_0 \neq \emptyset$. The proof is similar to that of Step 3 of Theorem 3.8.

Step 4: Modify f_0 slightly to get a new piecewise affine linear map $f \in B_\epsilon(h)$ that is constant on σ_0. The proof is similar to that of Step 4 of Theorem 3.8; however, we give the details here for clarity. To retain continuity, when we modify f_0 on σ_0, we must also modify it on all m-simplices in $\mathcal{K}^{(s)}$ adjacent to it. Let $A := \bigcup_{j=0}^{p} \sigma_j$, where σ_j is precisely an m-simplex in $\mathcal{K}^{(s)}$ such that $\sigma_j \cap \sigma_0 \neq \emptyset$ for all $0 \leq j \leq p$. Then, clearly $A \subseteq W \subseteq \Delta^0$, where W is as chosen in Step 3.

Let the vertices of σ_0 be $u_0, \ldots, u_{m-1}$ and u_m, and let $v_j := f_0(u_j)$ for all $0 \leq j \leq m$. Then $v_j \neq u_j$ for all $0 \leq j \leq m$, because $u_j \in W$ for all $0 \leq j \leq m$ and $f_0(W) \cap W = \emptyset$. Also, as $u_0, u_1, \ldots, u_m$ are geometrically independent vectors contained in Δ, by result (ii) of Corollary 3.4, we see that $v_0, v_1, \ldots, v_m$ are geometrically independent. Further, by Step 2, they are all contained in the interior of a single simplex Δ_1 of $\mathcal{K}$. Hence, by result (iii) of Corollary 3.4, we have $f_0(v_j) \neq v_j$ for some $0 \leq j \leq m$. Without loss of generality, we assume that $f_0(v_0) \neq v_0$.

Extend $\{u_0, \ldots, u_m\}$ to $\{u_0, u_1, \ldots\}$ to include the vertices of all the simplices of $\mathcal{K}^{(s)}$ and let $v_j := f_0(u_j)$ for all $j \in \mathbb{Z}_+$. Now, define f by

$$f(u_j) = \begin{cases} v_0 & \text{if } 0 \leq j \leq m, \\ v_j & \text{if } j > m, \end{cases}$$

and extend it countably piecewise affine linearly to the whole of $|\mathcal{K}^{(s)}| = \mathbb{R}^m$. Then f_0 is modified only on A and $f(x) = v_0$ for all $x \in \sigma_0$.

Consider an arbitrary $x \in A$. Then $x \in \sigma_j$ for some $0 \leq j \leq p$. Let $\sigma_j = \langle u_{j_0}, u_{j_1}, \ldots, u_{j_m} \rangle$ and $x = \sum_{i=0}^{m} \alpha_{j_i} u_{j_i}$ with $\sum_{i=0}^{m} \alpha_{j_i} = 1$. Then by equation (4.4), we have

$$\|f(x) - f_0(x)\|_\infty \leq \sum_{i=0}^{m} \alpha_{j_i} \|f(u_{j_i}) - f_0(u_{j_i})\|_\infty \leq \sum_{i=0}^{m} \alpha_{j_i} \operatorname{diam}(f_0(\Delta)) < \frac{\epsilon}{2^{r+4}},$$

implying by equation (4.5) that

$$\|f(x) - h(x)\|_\infty \leq \|f(x) - f_0(x)\|_\infty + \|f_0(x) - h(x)\|_\infty < \frac{\epsilon}{2^{r+4}} + \frac{\epsilon}{2^{r+4}} < \frac{\epsilon}{2^{r+3}}.$$

Therefore, $\rho_r(f, h) < \epsilon/2^{r+2}$. Also, we have $\rho_j(f, h) = \rho_j(f_0, h) < \epsilon/2^{j+2}$ for $j \neq r$. Hence, $D(f, h) < \sum_{j=1}^{\infty}(\epsilon/2^{j+2}) = \epsilon/4$, proving that $f \in B_\epsilon(h)$.

Step 5. Prove that f has no square roots in $C(\mathbb{R}^m)$. It is clear that $f(v_0) \neq v_0$, because $A \subseteq W$, $f = f_0$ on A^c and $f_0(W) \cap W = \emptyset$. Also, by Step 3, we have $f_0^{-1}(\sigma_0) \cap \sigma = \emptyset$ for all $\sigma \in \mathcal{K}^{(s)}$ with $\sigma \cap \sigma_0 \neq \emptyset$, implying that $f = f_0$ on $f_0^{-1}(\sigma_0)$. Further, $f_0^{-1}(\sigma_0)$ is uncountable, since σ_0 is uncountable and $\sigma_0 \subseteq R(f_0)^0$. Therefore, as $f^{-2}(v_0) \supseteq f^{-1}(f^{-1}(v_0)) \supseteq f^{-1}(\sigma_0) \supseteq f_0^{-1}(\sigma_0)$, it follows that $f^{-2}(v_0)$ is uncountable. Moreover, $f|_\sigma$ is injective for every $\sigma \in \mathcal{K}^{(s)}$ with $\sigma \neq \sigma_0$ by result (i) of Lemma 3.1, and hence $f^{-1}(x)$ is countable for all $x \neq v_0$ in $\mathbb{R}^m$. Thus, f satisfies the hypothesis of Case (iii) of Theorem 2.4, proving that f has no square roots in $C(\mathbb{R}^m)$. $\square$

We have seen in Theorem 3.8 (respectively Theorem 4.3) that each open neighborhood of each map in $C(I^m)$ (respectively in $C(\mathbb{R}^m)$) has a continuous map which does not have even discontinuous square roots. Additionally, if X and Y are locally compact Hausdorff spaces, and $\phi : X \to Y$ is a homeomorphism, then the map $f \mapsto \phi \circ f \circ \phi^{-1}$ is a homeomorphism of $C(X)$ onto $C(Y)$, where both $C(X)$ and $C(Y)$ have the compact-open topology, and moreover $f = g^2$ on X if and only if $\phi \circ f \circ \phi^{-1} = (\phi \circ g \circ \phi^{-1})^2$ on Y. Hence, it follows that Theorem 3.8 (respectively Theorem 4.3) is true if I^m (respectively $\mathbb{R}^m$) is replaced by any topological space homeomorphic to it. For the same reason, Theorem 3.9 is also true if I^m is replaced by any topological space homeomorphic to it.

5. L^p denseness of iterative squares in $C(I^m)$

In this section, we prove that the iterative squares of continuous self-maps on I^m are L^p dense in $C(I^m)$. Given any nontrivial compact subintervals $I_1, I_2, \ldots, I_m$ of I, let $I_1 \times I_2 \times \cdots \times I_m$ denote the closed rectangular region in I^m defined by

$$I_1 \times I_2 \times \cdots \times I_m = \{(x_1, x_2, \ldots, x_m) : x_i \in I_i \text{ for } 1 \leq i \leq m\}.$$

Consider $C(I^m)$ in the L^p norm defined by

$$\|f\|_p = \left(\int_{I^m} \|f(x)\|_\infty^p \, d\mu(x) \right)^{1/p},$$

where μ is the Lebesgue measure on $\mathbb{R}^m$. First, we prove a result which gives a sufficient condition for extending a given function to a square.

THEOREM 5.1. *Let K be a proper closed subset of I^m and $f : K \to I^m$ be continuous. Then there exists a $g \in C(I^m)$ such that $f = g^2|_K$.*

Proof. Let $I_1 \times I_2 \times \cdots \times I_m$ be a closed rectangular region in $I^m \setminus K$ and $\tilde{g}$ be a homeomorphism of K into $I_1 \times I_2 \times \cdots \times I_m$. Extend $\tilde{g}$ to $K \cup R(\tilde{g})$ by setting $g(x) = f(\tilde{g}^{-1}(x))$ for $x \in R(\tilde{g})$. Then g is continuous on the closed set $K \cup R(\tilde{g})$, and therefore, by an extension of Tietze's extension theorem by Dugundji (see Theorem 4.1 and Corollary 4.2 of [11, pp. 357–358]), it has an extension to a continuous self-map of I^m, which also

we denote by g. Further, for each $x \in K$, we have

$$
\begin{aligned}
g^2(x) &= g(g(x)) \\
&= g(\tilde{g}(x)) \\
&= f(\tilde{g}^{-1}(\tilde{g}(x))) \quad \text{(since } \tilde{g}(x) \in R(\tilde{g})) \\
&= f(x),
\end{aligned}
$$

implying that $f = g^2|_K$. □

It is worth noting that the assumption in the above theorem that K is a proper subset of I^m cannot be dropped. Indeed, if $K = I$, then the continuous map $f(x) = 1 - x$ on K has no square roots in $C(I)$ (see [24, pp. 425–426] or Corollary 2.7). Such maps can be constructed on $K = I^m$ for general m using Theorem 2.6. We now prove our desired result.

THEOREM 5.2. $\mathcal{W}(2; I^m)$ *is* L^p *dense in* $C(I^m)$.

Proof. Consider an arbitrary $f \in C(I^m)$. Then, by Theorem 5.1, for each $\epsilon > 0$, there exists a map $g_\epsilon \in C(I^m)$ such that $f|_{I_\epsilon \times I_\epsilon \times \cdots \times I_\epsilon} = g_\epsilon^2|_{I_\epsilon \times I_\epsilon \times \cdots \times I_\epsilon}$, where $I_\epsilon = [\epsilon, 1]$. Now,

$$
\begin{aligned}
\|g_\epsilon^2 - f\|_p^p &= \int_{I^m} \|g_\epsilon^2(x) - f(x)\|_\infty^p \, d\mu(x) \\
&= \int_{(I_\epsilon \times I_\epsilon \times \cdots \times I_\epsilon)^c} \|g_\epsilon^2(x) - f(x)\|_\infty^p \, d\mu(x) \\
&\leq \mu((I_\epsilon \times I_\epsilon \times \cdots \times I_\epsilon)^c) \\
&= 1 - (1 - \epsilon)^m,
\end{aligned}
$$

implying that $\|g_\epsilon^2 - f\|_p \to 0$ as $\epsilon \to 0$. Hence, $\lim_{\epsilon \to 0} g_\epsilon^2 = f$ in L^p, and the result follows. □

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