SERIES EXPANSIONS OF GENERALIZED TEMPERATURE FUNCTIONS IN *N* DIMENSIONS

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1. Introduction. The generalized heat equation is given by

$$\Delta_x u(x, t) = \partial u(x, t)/\partial t, \tag{1.1}$$

where $\Delta_x f(x) = f''(x) + (2\nu/x)f'(x)$, ν a fixed positive number. In a recent paper **(5)**, the author established criteria for representing solutions of (1.1) in either the form

$$\sum_{n=0}^{\infty} a_n P_{n,\nu}(x, t) \tag{1.2}$$

or

$$\sum_{n=0}^{\infty} b_n W_{n,\nu}(x, t) \tag{1.3}$$

where $P_{n,\nu}(x, t)$ is the polynomial solution of (1.1) given explicitly by

$$P_{n,\nu}(x, t) = \sum_{k=0}^{n} 2^{2k} \binom{n}{k} \frac{\Gamma(\nu + \frac{1}{2} + n)}{\Gamma(\nu + \frac{1}{2} + n - k)} x^{2n-2k} t^k, \tag{1.4}$$

and $W_{n,\nu}(x, t)$ is its Appell transform; cf. **(1)**. Our object is to generalize these results by extending them to higher dimensions. D. V. Widder **(8)** studied the problem for the ordinary heat equation.

Consider the Euclidean space E^{n+1} of points $(\mathbf{x}, t) = (x_1, \ldots, x_n, t)$. Here the generalized heat equation becomes

$$\Delta_{\mathbf{x}} u(\mathbf{x}, t) = \sum_{i=1}^{n} \Delta_{x_i} u(\mathbf{x}, t) = \frac{\partial}{\partial t} u(\mathbf{x}, t). \tag{1.5}$$

A function $u(\mathbf{x}, t)$ of class C^2 in a region of E^{n+1} is said to belong to class H there and is called a generalized temperature function if and only if it is a solution of (1.5). The fundamental solution of (1.5) is the function $G(\mathbf{x}, \mathbf{y}; t)$ given by

$$G(\mathbf{x}, \mathbf{y}; t) = \prod_{i=1}^{n} G(x_i, y_i; t) \tag{1.6}$$

where each of the factors $G(x_i, y_i; t)$, $i = 1, \ldots, n$, is the fundamental solution of (1.1) and is given by

$$G(x_i, y_i; t) = \left(\frac{1}{2t}\right)^{\nu+\frac{1}{2}} \exp\left(-\frac{x_i^2 + y_i^2}{4t}\right) \mathfrak{J}\left(\frac{x_i y_i}{2t}\right)$$

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with

$$\mathfrak{J}(z) = 2^{\nu-\frac{1}{2}}\Gamma(\nu + \tfrac{1}{2})z^{\frac{1}{2}-\nu}I_{\nu-\frac{1}{2}}(z),$$

$I_\alpha(z)$ being the Bessel function of imaginary argument of order α. A subclass H^* of H important to our theory is that of those generalized temperature functions $u(\mathbf{x}, t)$, $a < t < b$, for which

$$u(\mathbf{x}, t) = \int_0^\infty \ldots \int_0^\infty G(\mathbf{x}, \mathbf{y}; t - t')u(\mathbf{y}, t')d\mu(y_1)d\mu(y_2) \ldots d\mu(y_n),$$

with

$$d\mu(y) = \{2^{\nu-\frac{1}{2}}/\Gamma(\nu + \tfrac{1}{2})\}y^{2\nu}dy,$$

for every pair of numbers t, t', $a < t' < t < b$, with the multiple integral converging absolutely. If $u(\mathbf{x}, t) \in H^*$, $a < t < b$, then $u(\mathbf{x}, t)$ is said to have the Huygens property there.

Our principal result is that a necessary and sufficient condition for representing a function $u(\mathbf{x}, t)$ by the absolutely convergent multiple series

$$\sum_{m_n=0}^{\infty} \ldots \sum_{m_1=0}^{\infty} a_{m_1} \ldots a_{m_n} P_{m_1,\nu}(x_1, t) \ldots P_{m_n,\nu}(x_n, t)$$

where $P_{m_i,\nu}(x_i, t)$, $i = 1, \ldots, n$, is defined by (1.4), is that $u(\mathbf{x}, t) \in H^*$ for $|t| < \sigma$. In addition, we derive a corresponding theorem for expanding $u(\mathbf{x}, t)$ in the series

$$\sum_{m_n=0}^{\infty} \ldots \sum_{m_1=0}^{\infty} b_{m_1} \ldots b_{m_n} W_{m_1,\nu}(x_1, t) \ldots W_{m_n,\nu}(x_n, t)$$

where $W_{m_i,\nu}(x_i, t)$, $i = 1, \ldots, n$, is the Appell transform of $P_{m_i,\nu}(x_i, t)$.

2. Notation and definitions. We make use of the following vector notation:

(2.1) $$\mathbf{m}! = \prod_{i=1}^{n} m_i!,$$

(2.2) $$|\mathbf{m}| = \sum_{i=1}^{n} m_i,$$

(2.3) $$||\mathbf{x}||^2 = \sum_{i=1}^{n} x_i^2,$$

(2.4) $$\mathbf{x}^{\mathbf{m}} = \prod_{i=1}^{n} x_i^{m_i},$$

(2.5) $$\mathbf{xy} = (x_1 y_1, x_2 y_2, \ldots, x_n y_n).$$

(2.6) $$f(\mathbf{x})^{\mathbf{1}} = \prod_{i=1}^{n} f(x_i),$$

(2.7) $$a_{\mathbf{m}} = \prod_{i=1}^{n} a_{m_i},$$

(2.8) $$\sum_{\mathbf{m}=\mathbf{0}}^{\infty} a_{\mathbf{m}} = \sum_{m_n=0}^{\infty} \ldots \sum_{m_1=0}^{\infty} a_{m_1} \ldots a_{m_n}.$$

(2.9) $$\int_{\mathbf{0}}^{\infty} f(\mathbf{x})d\mathbf{x} = \int_0^\infty \ldots \int_0^\infty f(x_1, \ldots, x_n)dx_1\, dx_2 \ldots dx_n.$$

We write $\mathbf{m} \to \infty$ to mean that the m_i tend to ∞, $i = 1, \ldots, n$, independently.

3. Regions of convergence. In this section, we consider the regions of convergence of the multiple series

$$\sum_{\mathbf{m}=0}^{\infty} a_{\mathbf{m}} P_{\mathbf{m},\nu}(\mathbf{x}, t) \tag{3.1}$$

and

$$\sum_{\mathbf{m}=0}^{\infty} b_{\mathbf{m}} W_{\mathbf{m},\nu}(\mathbf{x}, t), \tag{3.2}$$

where we say that a multiple series is convergent if and only if every simple series formed from all of its terms converges absolutely. We find that the series (3.1) converges in the "strip" $|t| < \sigma$, whereas the series (3.2) converges in the half-space $t > \sigma$. Since these results are direct generalizations of the two-dimensional case, and are based on extensions of the order properties of the coefficients and of the bounds for the $P_{m,\nu}(x, t)$ and the $W_{m,\nu}(x, t)$, we state the theorems without proof.

THEOREM 3.1. *If*

$$\lim_{\mathbf{m}\to\infty} \left(|a_{\mathbf{m}}|\mathbf{m}^{\mathbf{m}}\right)^{1/|\mathbf{m}|} = \frac{e}{4\sigma}, \tag{3.3}$$

then the series (3.1) *converges for* $|t| < \sigma$. *It diverges for* $t > \sigma$. *For* $t < -\sigma$, *it cannot converge for any set of points of positive measure.*

Similarly, we have the region of convergence of the series (3.2).

THEOREM 3.2. *If*

$$\overline{\lim_{\mathbf{m}\to\infty}} \left(|b_{\mathbf{m}}|\mathbf{m}^{\mathbf{m}}\right)^{1/|\mathbf{m}|} = \frac{e\sigma}{4}, \tag{3.4}$$

then the series (3.2) *converges for* $t > \sigma$. *It cannot converge for* $t = t_0$, $0 < t_0 < \sigma$, *on a set of points* $\mathbf{x}$ *of positive measure.*

4. The Huygens property. It is the class H^* of generalized temperature functions that plays the central role in our theory. As in the two-dimensional case, it follows readily that $P_{\mathbf{m},\nu}(\mathbf{x}, t)$ and $W_{\mathbf{m},\nu}(\mathbf{x}, t)$ are both members of H^* for $0 < t < \infty$. Further, as a simple extension of (**3**, Theorem 6.2), we note that any function $u(\mathbf{x}, t)$ having a Poisson–Hankel–Stieltjes integral representation

$$u(\mathbf{x}, t) = \int_{\mathbf{0}}^{\infty} G(\mathbf{x}, \mathbf{y}; t) d\alpha(\mathbf{y})^{1} \tag{4.1}$$

belongs to H^* in the "strip" of absolute convergence of the integral (4.1). In particular, as an immediate consequence of (**2**, Theorem 9.1), we know that every positive generalized temperature function has a representation (4.1) and hence belongs to H^*. In addition, as an extension of (3, Theorem 8.1), it follows that members $u(\mathbf{x}, t)$ of H, $a < t < b$, for which

$$\int_0^\infty |u(\mathbf{x}, t)| G(\mathbf{x}; b - t)[d\mu(\mathbf{x})]^1 < M, \tag{4.2}$$

have the representation (4.1) and consequently belong to H^* there also.

A readily applicable criterion for the Huygens property is given by the following result.

THEOREM 4.1. *A necessary and sufficient condition that $u(\mathbf{x}, t) \in H^*$ for $a < t < b$ is that $u(\mathbf{x}, t) \in H$ and that the function*

$$F_c(t) = \int_0^\infty G(\mathbf{y}; c - t)|u(\mathbf{y}, t)|[d\mu(\mathbf{y})]^1 \tag{4.3}$$

be non-increasing for $a < t < c$ for every c, $a < c < b$.

Proof. To prove the necessity for the conditions, note that if $u \in H^*$, then, for $a < t < c < b$,

$$u(\mathbf{0}, c) = \int_0^\infty G(\mathbf{y}; c - t) u(\mathbf{y}, t)[d\mu(\mathbf{y})]^1, \tag{4.4}$$

with the integral (4.4) converging absolutely. Hence the integral (4.3) exists. Further, $F_c(t)$ is non-increasing, for we have, using the fact that $u \in H^*$ for $a < t' < t < b$,

$$\begin{aligned} F_c(t) &= \int_0^\infty G(\mathbf{y}; c - t)|u(\mathbf{y}, t)|[d\mu(\mathbf{y})]^1 \\ &\leqslant \int_0^\infty G(\mathbf{y}; c - t)[d\mu(\mathbf{y})]^1 \int_0^\infty G(\mathbf{y}, \mathbf{z}; t - t')|u(\mathbf{z}, t')|[d\mu(\mathbf{z})]^1 \\ &= \int_0^\infty G(\mathbf{z}; c - t')|u(\mathbf{z}, t')|[d\mu(\mathbf{z})]^1 = F_c(t'). \end{aligned}$$

The interchange in order of integration is valid by Fubini's Theorem. Hence $F_c(t)$ is non-increasing and the necessity of the condition is proved.

Conversely, assume that $F_c(t)$ is given by (4.3) and choose a', c such that $a < a' < c < b$. Then, by assumption, we have, for $a' < t < c$,

$$F_c(t) = \int_0^\infty G(\mathbf{y}; c - t)|u(\mathbf{y}, t)|[d\mu(\mathbf{y})]^1 \leqslant F_c(a').$$

By the remarks preceding the theorem, it follows that $u(\mathbf{x}, t)$ has the absolutely convergent integral representation

$$u(\mathbf{x}, t) = \int_0^\infty G(\mathbf{x}, \mathbf{y}; t)[d\alpha(\mathbf{y})]^1, \qquad a' < t < c.$$

Hence $u(\mathbf{x}, t) \in H^*$ for $a' < t < c$ and since a', c are arbitrary, it follows that $u(\mathbf{x}, t) \in H^*$ for $a < t < b$, as was to be proved.

We complete this section with a theorem establishing the independence of t of an integral involving functions of H^*. The proof is direct and follows that of the two-dimensional case given in (**3**, Theorem 7.6).

THEOREM 4.2. *Let* $u(\mathbf{x}, t) \in H^*$ *for* $a < t < b$ *and* $v(\mathbf{x}, t) \in H^*$ *for* $a < -t < b$. *If*

$$\int_0^\infty |u(\mathbf{y}, t)| [d\mu(\mathbf{y})]^{\mathbf{1}} \int_0^\infty G(\mathbf{y}, \mathbf{z}; t' - t) |v(\mathbf{z}, -t')|] d\mu(\mathbf{z})]^{\mathbf{1}} \tag{4.5}$$

is finite for $a < t < t' < b$, *then*

$$\int_0^\infty u(\mathbf{y}, t) v(\mathbf{y}, -t) [d\mu(\mathbf{y})]^{\mathbf{1}}, \qquad a < t < b, \tag{4.6}$$

is a constant.

Special cases of this theorem are of interest to us.

COROLLARY 4.3. *If* $u(\mathbf{x}, t) \in H^*$ *for* $0 < t < \infty$, *then*

$$\int_0^\infty u(\mathbf{x}, t) P_{\mathbf{m},\nu}(\mathbf{x}, -t) [d\mu(\mathbf{x})]^{\mathbf{1}} \tag{4.7}$$

is a constant.

COROLLARY 4.4. *If* $u(\mathbf{x}, -t) \in H^*$ *for* $0 < t < \infty$, *then*

$$\int_0^\infty u(\mathbf{x}, -t) W_{\mathbf{m},\nu}(\mathbf{x}, t) [d\mu(\mathbf{x})]^{\mathbf{1}} \tag{4.8}$$

is a constant.

5. Even functions of growth $(1, \sigma)$. Certain entire functions enter into our theory and we study them in this section.

DEFINITION 5.1. *An even function* $f(\mathbf{x})$ *belongs to class* $(1, \sigma)$ *if it is represented by the n-fold series*

$$f(\mathbf{x}) = \sum_{\mathbf{m}=0}^{\infty} a_{\mathbf{m}} \mathbf{x}^{2\mathbf{m}}, \tag{5.1}$$

with the coefficients satisfying the inequality

$$\overline{\lim_{\mathbf{m}\to\infty}} \, [|a_{\mathbf{m}}| \mathbf{m}^{\mathbf{m}}]^{1/|\mathbf{m}|} \leqslant e\sigma. \tag{5.2}$$

We determine an order property of functions of growth $(1, \sigma)$.

THEOREM 5.2. *If* $f(\mathbf{x})$ *is an even function of growth* $(1, \sigma)$, *then for any* $\sigma' > \sigma$,

$$f(\mathbf{x}) = O[\exp(\sigma' ||\mathbf{x}||^2], \qquad \mathbf{x} \to \infty, \tag{5.3}$$

Proof. By definition 5.1, there exists to any $\epsilon > 0$ a constant M_ϵ such that

$$|a_{\mathbf{m}}| \leqslant M_\epsilon [e(\sigma + \epsilon)]^{|\mathbf{m}|} (1/\mathbf{m}^{\mathbf{m}}) \tag{5.4}$$

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for all $\mathbf{m} > \mathbf{0}$. Hence, using (5.4) and the simple inequality

$$\sum_{m=0}^{\infty} \frac{r^{2m}}{m^m} \leqslant 2[r^2 e^{r^2/e} + 1], \qquad 0 < r < \infty,$$

we have

$$|f(\mathbf{x})| \leqslant M_\epsilon \sum_{\mathbf{m}=\mathbf{0}}^{\infty} [e(\sigma+\epsilon)]^{|\mathbf{m}|} \frac{1}{\mathbf{m}^{\mathbf{m}}} |\mathbf{x}^{2\mathbf{m}}|$$
$$\leqslant M_\epsilon 2[e(\sigma+\epsilon)\mathbf{x}^{2\mathbf{1}} \exp\{\sigma+\epsilon)||\mathbf{x}||^2\} + 1]$$
$$< 2M_\epsilon \exp\{(\sigma+2\epsilon)||\mathbf{x}||^2\}$$

and the theorem is established.

We complete the section by proving that a certain integral transform of an even function of growth $(1, \sigma)$ has the Huygens property.

THEOREM 5.3. *If $f(\mathbf{x})$ is an even function of growth $(1, \sigma)$ and if*

$$u(\mathbf{x}, t) = \int_{\mathbf{0}}^{\infty} f(\mathbf{y})[\mathfrak{B}(\mathbf{xy})]^{\mathbf{1}} \exp(-t||\mathbf{y}||^2)[d\mu(\mathbf{y})]^{\mathbf{1}}, \tag{5.5}$$

then the integral converges absolutely for $t > \sigma$ and $u(\mathbf{x}, t) \in H^*$ there.

Proof. The convergence of the integral (5.5) follows from the preceding theorem. Further, since $(\mathfrak{B}(\mathbf{xy})]^{\mathbf{1}} \exp(-t||\mathbf{y}||^2)$ is readily shown to be a generalized temperature function, and differentiation with respect to t of the integrand (5.5) results in an integral that by virtue of (5.3) converges uniformly, it follows that $u(\mathbf{x}, t) \in H$. To show that indeed $(\mathbf{x}, t) \in H^*$, note that (5.5) implies, for $\sigma < t' < t$,

$$I = \int_{\mathbf{0}}^{\infty} G(\mathbf{x}, \mathbf{y}; t - t')u(\mathbf{y}, t')[d\mu(\mathbf{y})]^{\mathbf{1}}$$
$$= \int_{\mathbf{0}}^{\infty} G(\mathbf{x}, \mathbf{y}; t - t')[d\mu(\mathbf{y})]^{\mathbf{1}} \int_{\mathbf{0}}^{\infty} f(\omega)[\mathfrak{B}(\omega\mathbf{y})]^{\mathbf{1}} \exp(-t'||\omega||^2)[d\mu(\omega)]^{\mathbf{1}}.$$

Now since

$$\int_{\mathbf{0}}^{\infty} |f(\omega)| \exp(-t'||\omega||^2)[d\mu(\omega)]^{\mathbf{1}} < \infty$$

by (5.3) we may apply Fubini's theorem to obtain

$$I = \int_{\mathbf{0}}^{\infty} f(\omega)\exp(-t'||\omega||^2)[d\mu(\omega)]^{\mathbf{1}} \int_{\mathbf{0}}^{\infty} [\mathfrak{B}(\omega\mathbf{y})]^{\mathbf{1}} G(\mathbf{x}, \mathbf{y}; t - t')[d\mu(\mathbf{y})]^{\mathbf{1}}$$
$$= \int_{\mathbf{0}}^{\infty} f(\omega)\exp(-t'||\omega||^2)\exp\{-(t-t')||\omega||^2\}[\mathfrak{B}(\mathbf{x}\omega)]^{\mathbf{1}}[d\mu(\omega)]^{\mathbf{1}} = u(\mathbf{x}, t),$$

which is the identity needed to complete the proof.

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6. Series expansions in terms of the generalized heat polynomials. The criterion for expanding a generalized temperature function in terms of the polynomials considered is found to be membership in class H^* over the region of convergence of the series. We omit the proof, which is based on a simple extension of (**5**, Theorem 5.1).

THEOREM 6.1. *A necessary and sufficient condition that*

$$u(\mathbf{x}, t) = \sum_{\mathbf{m}=\mathbf{0}}^{\infty} a_{\mathbf{m}} P_{\mathbf{m},\nu}(\mathbf{x}, t), \tag{6.1}$$

the series converging for $|t| < \sigma$, *is that* $u(\mathbf{x}, t) \in H^*$ *there. The coefficients* $a_{\mathbf{m}}$ *have either the determination*

$$a_{\mathbf{m}} = \frac{1}{(2\mathbf{m})!} \left(\frac{\partial}{\partial \mathbf{x}}\right)^{2\mathbf{m}} u(\mathbf{x}, t)\bigg|_{\substack{\mathbf{x}=\mathbf{0}\\ t=0}} \tag{6.2}$$

or

$$a_{\mathbf{m}} = k_{\mathbf{m}} \int_0^{\infty} u(\mathbf{y}, -t) W_{\mathbf{m},\nu}(\mathbf{y}, t) [d\mu(\mathbf{y})]^{\mathbf{1}}, \qquad 0 < t < \sigma, \tag{6.3}$$

where $k_{\mathbf{m}}$ *is given by*

$$k_{\mathbf{m}} = \frac{[\Gamma(\nu + \frac{1}{2})]^n}{2^{4|\mathbf{m}|}\mathbf{m}!\Gamma(\nu + \frac{1}{2} + \mathbf{m})^{\mathbf{1}}}. \tag{6.4}$$

An example illustrating the theorem is given by

$$u(x_1, x_2, t) = \left(\frac{1}{1 - 4t^2}\right)^{\nu+\frac{1}{2}} \exp\left(\frac{t(x_1^{\,2} + x_2^{\,2})}{1 - 4t^2}\right) \mathfrak{J}\left(\frac{x_1 x_2}{1 - 4t^2}\right), \qquad |t| < \tfrac{1}{2}.$$

It is easy to ascertain that $u \in H$ for $|t| < \frac{1}{2}$, and since $u > 0$ also, it belongs to H^*. Hence u must have an expansion in terms of the generalized heat polynomials. Note that for $t = 0$, we have

$$u(x_1, x_2, 0) = \mathfrak{J}(x_1, x_2) = \Gamma(\nu + \tfrac{1}{2}) \sum_{k=0}^{\infty} \frac{(x_1 x_2)^{2k}}{2^{2k} k!\Gamma(\nu + \frac{1}{2} + k)}.$$

We thus derive the coefficients a_{m_1,m_2} using (6.2) and find that

$$u(x_1, x_2, t) = \sum_{m=0}^{\infty} \frac{\Gamma(\nu + \frac{1}{2})}{2^{2m} m!\Gamma(\nu + \frac{1}{2} + m)} P_{m,\nu}(x_1, t) P_{m,\nu}(x_2, t).$$

7. Expansions in terms of $W_{m,\nu}(\mathbf{x}, t)$. Membership in H^* is not sufficient for the expansion of generalized temperature functions in terms of $W_{\mathbf{m},\nu}(\mathbf{x}, t)$. We need, in addition, an integrability condition. In order to derive a dual to Theorem 6.1, we first state, without proof, a series representation theorem with conditions of a different nature.

THEOREM 7.1. *A necessary and sufficient condition that*

$$u(\mathbf{x}, t) = \sum_{\mathbf{m}=0}^{\infty} b_{\mathbf{m}} W_{\mathbf{m},\nu}(\mathbf{x}, t), \tag{7.1}$$

the series converging for $0 \leqslant \sigma < t$, *is that*

$$u(\mathbf{x}, t) = \int_0^\infty [\mathfrak{B}(\mathbf{xy})]^{\mathbf{1}} \exp(-t\|\mathbf{y}\|^2)\phi(\mathbf{y})[d\mu(\mathbf{y})]^{\mathbf{1}}, \tag{7.2}$$

where $\phi(\mathbf{y})$ *is an even entire function of growth* $(1, \sigma)$ *and*

$$b_{\mathbf{m}} = \frac{(-1)^{|\mathbf{m}|}}{2^{2|\mathbf{m}|}(2\mathbf{m})!}\left(\frac{\partial}{\partial \mathbf{x}}\right)^{2\mathbf{m}} \phi(\mathbf{x})\bigg|_{\mathbf{x}=\mathbf{0}}. \tag{7.3}$$

Further, for the analogue to Theorem 6.1 for the series in terms of $W_{\mathbf{m},\nu}(\mathbf{x}, t)$, we need the following preliminary lemma, which is a direct extension of (**5**, Lemma 6.3).

LEMMA 7.2. *If*

$$u(\mathbf{x}, t) = \int_0^\infty [\mathfrak{B}(\mathbf{xy})]^{\mathbf{1}} \exp(-t\|\mathbf{y}\|^2)\phi(\mathbf{y})[d\mu(\mathbf{y})]^{\mathbf{1}} \tag{7.4}$$

with $\phi(\mathbf{y})$ *an even function of growth* $(1, \sigma)$, *then, for each* $c > \sigma$, *there exists a constant* $M(c)$ *such that*

$$|u(\mathbf{x}, t)| \leqslant M(c)(\mathbf{x}^{\mathbf{1}})^{-\nu} \frac{\exp[-1/4(t+c)]^{\|\mathbf{x}\|^2}}{\sqrt{(t-c)}}, \qquad t > c. \tag{7.5}$$

We now have the essential tools to establish the principal theorem, which we state without proof.

THEOREM 7.3. *A necessary and sufficient condition that*

$$u(\mathbf{x}, t) = \sum_{\mathbf{m}=\mathbf{0}}^{\infty} b_{\mathbf{m}} W_{\mathbf{m},\nu}(\mathbf{x}, t), \tag{7.6}$$

the series converging for $t > \sigma \geqslant 0$, *is that* $u(\mathbf{x}, t) \in H^*$ *there and that*

$$\int_0^\infty |u(\mathbf{x}, t)| \exp(\|x\|^2/8t)[d\mu(\mathbf{x})]^{\mathbf{1}} < \infty, \qquad \sigma < t < \infty. \tag{7.7}$$

The coefficients $b_{\mathbf{m}}$ *are determined by*

$$b_{\mathbf{m}} = k_{\mathbf{m}} \int_0^\infty u(\mathbf{y}, t) P_{\mathbf{m},\nu}(\mathbf{y}, -t)[d\mu(\mathbf{y})]^{\mathbf{1}}, \qquad \sigma < t < \infty, \tag{7.8}$$

where $k_{\mathbf{m}}$ *is defined by* (6.4).

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