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K-THEORY AND ASYMPTOTICALLY COMMUTING MATRICES

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1. Introduction. To shed light on the following unsolved problem, several authors have considered related problems. The problem is that of finding commuting approximants to pairs of asymptotically commuting self-adjoint matrices:

Suppose that H_n and K_n are self-adjoint matrices of dimension $m(n)$, with $||H_n||$, $||K_n|| \leqq 1$, which commute asymptotically in the sense that

$$|| [H_n, K_n] || \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

Must there then exist commuting self-adjoint matrices H_n' and K_n' for which

$$||H_n - H_n'|| \rightarrow 0 \quad \text{and} \quad ||K_n - K_n'|| \rightarrow 0?$$

One may alter the conditions imposed on H_n and K_n, for example, by requiring H_n to be normal and K_n to be self-adjoint, and ask whether commuting approximants H_n' and K_n' can be found satisfying the same conditions. Some of these related problems have been solved. This paper will examine their solutions from a K-theoretic point of view, illustrating the difficulty inherent in modifying them to work for the original problem.

Voiculescu has given an example in [**9**] showing that the corresponding question for two unitaries has a negative answer. Davidson has shown in [**4**] that for three asymptotically commuting self-adjoint matrices, commuting approximants cannot always be found. By using two of the matrices as the real and imaginary parts of an (essentially) normal matrix, he also shows that, for a normal and a self-adjoint, the answer to the corresponding question is again false.

The problem of finding commuting approximants to unitary matrices can be translated into a lifting problem for a certain homomorphism from $C(\mathbf{T}^2)$. Voiculescu remarks that his example seems to depend on the nonzero second-cohomology of the space $\mathbf{T}^2$, and so is unlikely to have any direct bearing on the original problem. The main purpose of this

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paper is to make explicit the role of the second-cohomology in Davidson's and Voiculescu's examples. The K-theory of the related homomorphisms is computed, and it is shown that, even if homotopy is allowed, commuting approximants (of the same type) cannot be found for these examples.

The matrices in Voiculescu's example are S_n and Ω_n where

$$S_n = \begin{bmatrix} 0 & & & & & 1 \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & 1 & 0 \end{bmatrix},$$

$$\Omega_n = \begin{bmatrix} \omega & & & & & & \\ & \omega^2 & & & & & \\ & & \omega^3 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \omega^n \end{bmatrix}, \quad \omega = \exp(2\pi i/n).$$

Voiculescu gives a proof of the following result that is based on the non-quasidiagonality of the unilateral shift.

1.1 THEOREM. *Let S_n and Ω_n be the matrices above. Then*

$$\lim \| [S_n, \Omega_n] \| = 0,$$

but there do not exist unitaries U_n and V_n such that $U_nV_n = V_nU_n$ and

$$\lim \|S_n - U_n\| = \lim \|\Omega_n - V_n\| = 0.$$

This will follow from Theorem 4.3 which states that, for any pair of paths of unitaries from S_n and Ω_n to the identity, the commutator must at some point grow very large. While the second-cohomology of the torus is not explicitly mentioned in the proof of Theorem 4.3, it is always in the background, as explained below.

Let $m(n)$ be any sequence of integers. Voiculescu considers the C^*-algebra

$$\mathscr{A} = \{ (T_n)_1^\infty \mid T_n \in M_{m(n)}, \sup_n \|T_n\| < \infty \}$$

and the ideal $\mathscr{I}$ which consists of sequences (T_n) such that

$$\lim \|T_n\| = 0.$$

Asymptotically commuting unitaries define commuting unitaries in the quotient $\mathscr{A}/\mathscr{I}$, and so define a $*$-homomorphism of $C(\mathbf{T}^2)$ into $\mathscr{A}/\mathscr{I}$.

The approximation question is equivalent to asking whether every *-homomorphsim

$$\varphi:C(\mathbf{T}^2) \to \mathscr{A}/\mathscr{I}$$

can be lifted to $\mathscr{A}$. Lemma 4.1 below will show that any

$$\psi:C(\mathbf{T}^2) \to \mathscr{A}$$

induces a map

$$\psi_*:K_0(C(\mathbf{T}^2)) \to K_0(\mathscr{A})$$

whose kernel contains the second-cohomology of $\mathbf{T}^2$, where $K_0(C(\mathbf{T}^2))$ is identified with the even cohomology of the torus via the Chern character. Thus one requirement for lifting φ is that it must also contain the second-cohomology in its kernel. Stated more concretely, φ cannot be lifted unless $\varphi_*(1)$ and $\varphi_*(e)$ are equivalent projections, where e is the projection to be defined in Section 2.

Now let $\varphi:C(\mathbf{T}^2) \to \mathscr{A}/\mathscr{I}$ denote the *-homomorphism corresponding to S_n and Ω_n. Using the six-term exact sequence for K-theory, it quickly follows that $K_0(\mathscr{A}/\mathscr{I})$ is isomorphic to a subgroup of sequences of integers, where two sequences are identified if they agree except on a finite portion. The content of Theorem 4.2 is that $\varphi_*(e)$ corresponds to the equivalence class of the sequence $(n - 1)$ while clearly $\varphi_*(1)$ corresponds to the equivalence class of the sequence (n). These are not equivalent, and so φ cannot be lifted.

The commutation question for a pair of self-adjoint matrices is equivalent to the lifting problem for maps of $C([0, 1]^2)$ to $\mathscr{A}/\mathscr{I}$. Since the second-cohomology of $[0, 1]^2$ is zero, it will be considerably more difficult to find a nonliftable map in this case.

This K-theoretic proof of Voiculescu's example is presented in as concrete a manner as possible in Sections 2 to 4. Section 2 investigates the K-theory of the torus from a C^*-algebra point of view. An explicit formula for a nontrivial projection in $M_2(C(\mathbf{T}^2))$ is obtained, leading to the definition of a projection $e(U, V)$ which is defined for any pair of unitaries U and V which commute. In Section 3, this formula is extended to pairs of unitaries with small, but nonzero, commutators for which it defines matrices that are approximately projections. This extended formula is applied to S_n and Ω_n in Section 4. The dimension of the spectral subspace of $e(S_n, \Omega_n)$, corresponding to an interval near one, is found to be one less than would be expected, and so precludes the existence of commuting approximants.

Other types of matrix commutation problems are discussed in Section 5. In particular, the problem of finding commuting approximants in the case of a self-adjoint and a normal is related to a lifting problem for maps from $C(S^2)$ into $\mathscr{A}/\mathscr{I}$. A new proof of the nonexistence of commuting

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approximants in Davidson's example is given that parallels Sections 2 through 4. Actually, it is possible to derive these K-theoretic results from the results of Section 4, or vice versa. Independent proofs are given because the intermediate lemmas in Sections 2 through 4 have proven useful in the study of maps from $C(\mathbf{T}^2)$ to AF algebras, and the results in Section 5 may yet prove similarly useful. The method of calculation is interesting in its own right. The similarities between the proofs of Lemma 4.7 and Proposition 2.1 suggest that there may be a more conceptual way of calculating the K-theory of the map $\varphi:C(\mathbf{T}^2) \to \mathscr{A}/\mathscr{I}$, perhaps involving cyclic cohomology.

The final section contains some examples of solvable matrix commutation problems. In these examples, the K-theory presents no obstruction, and the commuting approximants are found using techniques that are derived from Berg's technique ([**1**]). Thus the evidence seems to indicate that the lifting problem for maps $C(X) \to \mathscr{A}/\mathscr{I}$ for which there is no K-theoretic obstruction is just as difficult for $X = S^2$ and $X = \mathbf{T}^2$ as it is for $X = [0, 1]^2$. It may even be true that these are equivalent problems.

The first four sections are based on the author's dissertation, written under the skillful guidance of Marc Rieffel. Sections 5 and 6 are due in large part to discussions with Ken Davidson and Dan Voiculescu, for whose assitance I am most grateful.

2. The K-theory of the torus. The complex vector bundles over the torus $\mathbf{T}^2$ are classified up to isomorphism by their images in $K^0(\mathbf{T}^2)$, which is isomorphic to $\mathbf{Z}^2$. The first integer corresponds to the dimension of the fibres and the second integer is the first Chern class. This section will describe how to find a projection in $M_2(C(\mathbf{T}^2))$ of a particularly simple form which corresponds to the bundle of dimension one with first Chern class equal to one.

Following along the lines of [**7**], we guess that the desired projection can be taken to be of the form

$$e = \begin{bmatrix} f & g + hU \\ hU^* + g & 1 - f \end{bmatrix}$$

where $U = e^{2\pi i y}$ and $f(x)$, $g(x)$, and $h(x)$ are nonnegative functions on $S^1 = \mathbf{R}/\mathbf{Z}$ (i.e., are functions on the real line of period one). Setting e^2 equal to e imposes the conditions

$$gh = 0,$$

$$g^2 + h^2 = f - f^2.$$

One way to satisfy these equations is to choose any f for which

(1) $\quad 0 \leqq f \leqq 1,$

(2) $f(0) = 1, f(1/2) = 0$ and $f(1) = 1$,

and then define g and h by

(3) $g = \chi_{[0,1/2]}(f - f^2)^{1/2}$,

(4) $h = \chi_{[1/2,1]}(f - f^2)^{1/2}$,

where χ_X denotes the characteristic function of the set X. Furthermore, we assume that f, g, and h are smooth functions.

As defined above, e will have trace one and so represents a bundle with one-dimensional fibres. To calculate the first Chern class $c_1(e)$ we use the formula (see **[3]**)

$$c_1(e) = \tau(e(\delta_1(e)\delta_2(e) - \delta_2(e)\delta_1(e)))/2\pi i$$

where δ_1 and δ_2 are the componentwise extensions to $M_2(C^\infty(\mathbf{T}^2))$ of differentiation with respect to x and y respectively. The symbol τ denotes the trace on $C(\mathbf{T}^2)$ corresponding to Lebesgue measure, extended to matrices in the usual way.

2.1 PROPOSITION. *For any choice of smooth functions f, g, and h satisfying conditions* (1) *to* (4), *the first Chern class $c_1(e)$ is equal to* 1.

Proof. Clearly

$$\delta_1(e) = \begin{bmatrix} f' & g' + h'U \\ h'U^* + g' & -f' \end{bmatrix} \quad \text{and}$$

$$\delta_2(e)/2\pi i = h\begin{bmatrix} 0 & U \\ -U^* & 0 \end{bmatrix},$$

and since $gh = g'h = gh' = 0$,

$$e(\delta_1(e)\delta_2(e) - \delta_2(e)\delta_1(e))/2\pi i$$
$$= 2\begin{bmatrix} -fhh' + h^2f' & ff'hU + h^2h'U \\ -h^2h'U^* + (h - fh)f'U^* & -fhh' + h^2f' + hh' \end{bmatrix}.$$

Applying the trace, we get

$$c_1(e) = 2\tau(-fhh' + h^2f') + 2\tau(-fhh' + h^2f' + hh')$$
$$= \int 2hh' - 4fhh' + 4h^2f'$$

which, by the next lemma, is equal to 1.

It is interesting to note that these integrals will appear again in Section 4 when computing the dimension of certain finite-rank projections.

2.2 LEMMA. *For functions f, g and h which satisfy conditions* (1) *to* (4),

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(i) $\int hh' = 0$

(ii) $\int fhh' = -1/12$

(iii) $\int h^2 f' = 1/6.$

Proof. Integration by parts proves (i) and shows that (ii) follows from (iii). On the interval $[0, 1/2]$, h is equal to zero, while on the interval $[1/2, 1]$ we have $h^2 = f - f^2$. Therefore,

$$\int_0^1 h^2 f' = \int_{1/2}^1 (f - f^2) f' = \int_0^1 (\lambda - \lambda^2) d\lambda = 1/6.$$

3. Approximate projections and almost commuting unitaries. When U and V are unitaries in a C^*-algebra A that commute, we can define a projection $e(U, V)$ in $M_2(A)$ as the image of e under the map of $C(\mathbf{T}^2)$ to A defined by sending $e^{2\pi i x}$ and $e^{2\pi i y}$ to U and V. More generally, when U and V are close to commuting, we can define $e(U, V)$ to be a two by two matrix over A that is almost a projection. We now fix some choice of functions f, g and h which satisfy conditions (1) to (4) of the last section.

3.1 *Definition.* Let U and V be unitaries in a C^*-algebra A. We consider f, g, and h to be functions defined on $\{z \in \mathbf{C} : |z| = 1\}$ so we can use the functional calculus to define $f(V)$, $g(V)$, and $h(V)$. Define, $e(U, V)$ to be the matrix over A

$$e(U, V) = \begin{bmatrix} f(V) & g(V) + h(V)U \\ U^*h(V) + g(V) & 1 - f(V) \end{bmatrix}.$$

For any unitaries, $e(U, V)$ is self-adjoint, and if U and V commute, then $e(U, V)$ is a projection. In order to explore the continuity properties of the function e, we need two well known lemmas. Lemma 3.2 is proven using polynomial approximation, and Lemma 3.3 follows immediately since

$$\begin{aligned} \|f(V)U - Uf(V)\| &= \|U^*f(V)U - f(V)\| \\ &= \|f(U^*VU) - f(V)\|. \end{aligned}$$

3.2 Lemma. *For any $f \in C(S^1)$, the function $U \mapsto f(U)$ is uniformly continuous on the set of unitary operators.*

3.3 Lemma. *Let $f \in C(S^1)$. For unitaries U and V, $\|f(V)U - Uf(V)\|$ tends toward zero uniformly as $\|UV - VU\|$ tends toward zero.*

These lemmas prove

3.4 Proposition. *The function*

$$e:\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H}) \to M_2(\mathcal{B}(\mathcal{H}))$$

is uniformly norm continuous.

3.5. PROPOSITION. *There exists a constant M such that if $\|UV - VU\| < M$, then $1/2$ is not in the spectrum of $e(U, V)$. More generally,*

$$\|e(U, V)^2 - e(U, V)\|$$

tends toward zero uniformly as the commutation error $\|UV - VU\|$ tends to zero.

Proof. Notice that this says nothing about U and V converging. In fact, if U_n and V_n are unitaries in varying C^*-algebras A_n, the lemma says that if

$$\|U_nV_n - V_nU_n\| \to 0$$

then

$$\|e(U, V)^2 - e(U, V)\| \to 0.$$

We begin by calculating the square of $e(U, V)$. Let us identify f, g, and h with $f(V)$, $g(V)$, and $h(V)$. It is easy to see that

$$e(U, V)^2 = e(U, V) + \begin{bmatrix} gU^*h + hUg & fhU - hUf \\ U^*fh - fU^*h & U^*h^2U - h^2 \end{bmatrix}.$$

Working out the norms in each of the entries in the error term we get

$$\|gU^*h + hUg\| \leqq 2\|hUg\| = 2\|hUg - hgU\| = 2\|h(Ug - gU)\|$$
$$\leqq 2\|h\|\,\|Ug - gU\| = 2\|Ug - gU\|,$$
$$\|fhU - hUf\| = \|U^*fh - fU^*h\| \leqq \|fU - Uf\|,$$

and

$$\|U^*h^2U - h^2\| \leqq 2\|Uh - hU\|.$$

Thus by Lemma 3.3 we are done.

Since we will need these estimates again later, we record them as a corollary.

3.6 COROLLARY. *For any unitaries U and V in C^*-algebra A,*

$$\|e(U, V)^2 - e(U, V)\|$$

is less than or equal to

$$2(\|Ug(V)U^* - g(V)\| + \|Uf(V)U^* - f(V)\| + \|Uh(V)U^* - h(V)\|).$$

4. Voiculescu's example. This section provides a proof of Theorem 1.1. The main idea in the proof is to compare the spectral projections of $e(S_n, \Omega_n)$ to the projections $e(U_n, V_n)$ obtained from commuting unitary matrices.

4.1 LEMMA. *If U and V are commuting unitaries in $M_n(\mathbf{C})$, then the projection $e(U, V) \in M_{2n}(\mathbf{C})$ has dimension n.*

Proof. Let τ denote the trace on $M_n(\mathbf{C})$ normalized so that $\tau(1) = n$. Then

$$\tau(e(U, V)) = \tau(f) + \tau(1 - f) = n.$$

For the rest of this section, let χ denote the characteristic function of the interval $[1/2, 2]$. As long as $\|UV - VU\|$ is less than the constant M of Proposition 3.5, the spectrum of $e(U, V)$ will have a gap around $1/2$, and the projection $\chi(e(U, V))$ will be in the C^*-algebra generated by U and V.

4.2 THEOREM. *For large values of n, $\chi(e(S_n, \Omega_n))$ is a projection of dimension $n - 1$.*

Before proving Theorem 4.2, we shall see how it implies Theorem 1.1. Actually, we prove a slightly stronger result:

4.3 THEOREM. *There do not exist paths of unitaries $U_n^{(t)}$ and $V_n^{(t)}$ from S_n, Ω_n to a commuting pair $U_n^{(1)}$, $V_n^{(1)}$ such that*

$$\lim_n \sup_t \|U_n^{(t)}V_n^{(t)} - V_n^{(t)}U_n^{(t)}\| = 0.$$

Proof. Let $e_n^{(t)} = e(U_n^{(t)}, V_n^{(t)})$. This is a continuous path of matrices from $e(S_n, \Omega_n)$ to $e(I, I)$. If

$$\lim_n \sup_t \|U_n^{(t)}V_n^{(t)} - V_n^{(t)}U_n^{(t)}\|$$

were equal to zero then, for large n, $\chi(e_n^{(t)})$ would be a continuous path of projections from $\chi(e(S_n, \Omega_n))$ to

$$\chi(e(U_n^{(1)}, V_n^{(1)})) = e(U_n^{(1)}, V_n^{(1)}).$$

By Lemma 4.1, this would imply that

$$\dim \chi(e(S_n, \Omega_n)) = n \quad \text{for large } n,$$

which would contradict Theorem 4.2.

We now proceed with the proof of Theorem 4.2 It seems odd that $e(S_n, \Omega_n)$ has trace n and yet only has $n - 1$ eigenvalues near one. The reason that this can happen is that, as the next lemma shows, the "spectral errors"

$$\max_{\lambda \in \sigma(e(S_n, \Omega_n))} \min_{s=0,1} |\lambda - s| = \|e(S_n, \Omega_n) - \chi(e(S_n, \Omega_n))\|$$

are tending toward zero only on the order of $1/n$. Therefore the trace does not give an accurate reading of the dimension of $\chi(e(S_n, \Omega_n))$.

To simplify notation, we now write e_n for $e(S_n, \Omega_n)$. Also, S_n implements the shift automorphism on $C^*(\Omega_n)$ which we denote by α. Any element of $C^*(\Omega_n)$ is a diagonal matrix, and so a multiplication operator. If k is any function on $\mathbf{R}/\mathbf{Z}$ then we let k_n (or simply k when this is clear from the context) denote $k(\Omega_n)$, i.e., the n by n matrix

$$\begin{bmatrix} k(\Delta) & & & & & \\ & k(2\Delta) & & & & \\ & & k(3\Delta) & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & & k(1) \end{bmatrix}$$

where Δ (or Δ_n) equals $1/n$. Sometimes we shall write S instead of S_n.

4.4 LEMMA. *As* $n \to \infty$, $\|e_n - \chi(e_n)\| \to 0$ *at least on the order of* $1/n$.

Proof. What we wish to show is that $n\|e_n - \chi(e_n)\|$ is bounded. Since

$$|x - \chi(x)| \leqq 2|x^2 - x|$$

for all real numbers x, we have

$$\|e_n - \chi(e_n)\| \leqq 2\|e_n^2 - e_n\|,$$

so it suffices to show that $n\|e_n^2 - e_n\|$ is bounded.

Let k denote any smooth function on the circle. Then

$$\|S_nk - kS_n\| = \|S_nkS_n^* - k\| = \|\alpha(k) - k\|$$
$$= \sup|k(\rho\Delta) - k((\rho + 1)\Delta)|$$

which is bounded by $\|k'\|_\infty\Delta = \|k'\|_\infty/n$. Hence by Corollary 3.6,

$$n\|e_n^2 - e_n\| \leqq 2(\|g'\|_\infty + \|f'\|_\infty + \|h'\|_\infty).$$

The trace can be used to give an accurate count of the eigenvalues near one if we replace the e_n by matrices whose spectral errors vanish on the order of $1/n^2$. We cannot calculate $\chi(e_n)$ without knowing a priori the spectral decomposition of e_n, but we can apply a polynomial approximation. The polynomial $3x^2 - 2x^3$ is a second degree approximation to χ at zero and one, and so will turn the $1/n$ convergence into $1/n^2$ convergence.

4.5 LEMMA. $\lim(\dim \chi(e_n) - \tau(3e_n^2 - 2e_n^3)) = 0$.

Proof. Let $p(x) = 3x^2 - 2x^3$. Since $p(0) = 0$ and $p'(0) = 0$, if $\lambda_n \to 0$ on the order of $1/n$ then $p(\lambda_n) \to 0$ on the order of $1/n^2$. Similarly, $p(1) = 1$ and $p'(1) = 0$ so if $\lambda_n \to 1$ on the order of $1/n$ then $p(\lambda_n) \to 1$ on the order of $1/n^2$.

By the spectral mapping theorem and Lemma 4.4,

$$\|p(e_n) - \chi(e_n)\| \to 0$$

on the order of $1/n^2$. Therefore the n eigenvalues of $\chi(e_n)$ and those for $p(e_n)$ differ by at most a constant times $1/n^2$, so

$$\lim(\tau(\chi(e_n)) - \tau(p(e_n))) = 0.$$

4.6 LEMMA. $\tau(e_n^2) = n$.

Proof. In Lemma 3.5 we saw that e_n^2 equaled e_n plus

$$M_n = \begin{bmatrix} gS^*h + hSg & h(fS - Sf) \\ (S^*f - fS^*)h & S^*h^2S - h^2 \end{bmatrix}.$$

Thus it suffices to show that $\tau(M_n) = 0$. Since gS^*h and hSg are zero on the diagonal, they have trace zero. Therefore,

$$\begin{aligned}\tau(M_n) &= \tau(gS^*h + hSg) + \tau(S^*h^2S - h^2) \\ &= \tau(h^2SS^*) - \tau(h^2) = 0.\end{aligned}$$

4.7 LEMMA. $\lim(\tau(e_n^3) - n) = 1/2$.

Proof. Using the last lemma, we find that

$$\begin{aligned}\tau(e_n^3) &= \tau(e_n(e_n + M_n)) \\ &= \tau(e_n^2) + \tau(e_nM_n) = n + \tau(e_nM_n).\end{aligned}$$

Thus it suffices to prove that $\lim \tau(e_nM_n) = 1/2$.

The coefficient of S^0 in the top left-hand corner of the matrix e_nM_n is

$$hS(S^*f - fS^*)h = h(f - \alpha(f))h.$$

The coefficient of S^0 in the lower right-hand corner is

$$S^*h^2(fS - Sf) + (1 - f)(S^*h^2S - h^2).$$

Using the fact that $\tau(xy) = \tau(yx)$, we see that

$$\tau(e_nM_n) = 2\tau((f - \alpha(f))h^2) + \tau((1 - f)(\alpha^{-1}(h^2) - h^2)).$$

For any two smooth functions r and s on $\mathbf{R}/\mathbf{Z}$,

$$\tau(r(s - \alpha(s))) = \sum r(\rho\Delta)[s(\rho\Delta) - s((\rho - 1)\Delta)]$$

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$$\to \int r(t)ds(t)$$
$$= \int rs',$$

and similarly,

$$\tau(r(s - \alpha^{-1}(s))) \to -\int rs'.$$

Therefore

$$\tau(e_nM_n) \to 2\int h^2f' + \int(1 - f)(h^2)'$$
$$= 2\int h^2f' + 2\int hh' - 2\int fhh' = 1/2$$

by Lemma 2.2.

This finishes the proof of Theorem 4.2 since

$$\begin{aligned}\lim(\dim \chi(e_n) - n) &= \lim(\tau(3e_n^2 - 2e_n^3) - n)\\ &= \lim(-\tau(2e_n^3) + 2n)\\ &= -2\lim(\tau(e_n^3) - n) = -1.\end{aligned}$$

5. The sphere and almost commuting matrices. In this section, we consider the problem of finding commuting approximants to a sequence of normal matrices and a sequence of self-adjoint matrices that asymptotically commute. In particular, we consider an example that is almost identical to that considered by Davidson in [**4**]. As with Voiculescu's example, we give a K-theoretic approach which demonstrates the dependence of this example on the nonzero second-cohomology of the sphere.

In general, by a matrix commutation problem we shall mean the question of whether, in the notation of Section 1, a *-homomorphism $\varphi:C(X) \to \mathcal{A}/\mathcal{I}$ can be lifted to $\mathcal{A}$. If X is the torus, this is the problem of finding commuting unitary approximants to asymptotically commuting unitaries. If $X = [0, 1]^2$ then the lifting problem is equivalent to the approximation question for a pair of bounded sequences of self-adjoint matrices which commute asymptotically. This case is the most difficult to work with because there seems to be no way to put K-theory to work, principally because $H^2(X) = 0$.

The commutation problem for a normal and a self-adjoint is related to the lifting problem in the case $X = D \times [0, 1]$, where D denotes the closed unit disk. (The bounds on normal and self-adjoint matrices will be left unstated since they can be changed by renormalizing. Generally, normals will be bounded in norm by 1/2 or one, and self-adjoints will be bounded between 0 and 1, or 0 and 1/2.) Given a sequence of normal matrices N_n

which commutes asymptotically with a sequence of self-adjoint matrices h_n, one can define a map of $C(D \times [0, 1])$ to $\mathcal{A}/\mathcal{I}$ which can be lifted only if h_n and N_n can be approximated by commuting normal and self-adjoint sequences. The lifting problem for $D \times [0, 1]$ is somewhat more general since a map from $(D \times [0, 1])$ defines a self-adjoint sequence and an asymptotically normal sequence. It is unknown whether asymptotically normal matrices can be approximated by normal ones.

It appears that there is no way to use K-theory to attack the lifting problem for $X = D \times [0, 1]$ since X is contractible. However, since D is three-dimensional, it has closed subsets, such as the sphere, that have nonzero second-cohomology.

As we shall see, it is convenient from a K-theoretic point of view to consider the sphere as the zero set of the equation $r^2 + \bar{z}z = r$ for $(r, z) \in \mathbf{R} \times \mathbf{C}$. In this way, the two coordinate functions (one real-valued, one complex-valued) give generators h and N of $C(S^2)$ which satisfy the equation $h^2 + N^*N = h$. We shall then think of $C(S^2)$ as the universal C^*-algebra generated by a self-adjoint h and a normal N which commute and satisfy $h^2 + N^*N = h$. If we drop this last equation (and add bounds on the generators), we get a universal description of $C(D \times [0, 1])$. Sending the generators to the generators defines a surjection

$$\rho:C(D \times [0, 1]) \rightarrow C(S^2).$$

Using these universal properties, we are able to show how a non-liftable map from $C(S^2)$ gives a non-liftable map from $C(D \times [0, 1])$.

5.1 LEMMA. *Let* $\varphi:C(S^2) \rightarrow \mathcal{A}/\mathcal{I}$ *be a *-homomorphism. If*

$$\varphi \circ \rho:C(D \times [0, 1]) \rightarrow \mathcal{A}/\mathcal{I}$$

is liftable, then φ *is liftable.*

Proof. The map φ is defined by two sequences of matrices h_n and N_n such that $\| [h_n, N_n] \|$, $\|h_n^* - h_n\|$, $\| [N_n^*, N_n] \|$ and $\| [h_n^2 + N_n^*N_n - h_n\|$ approach zero as $n \rightarrow \infty$. If $\varphi \circ \rho$ is liftable, then there are sequences h_n' and N_n' which commute, approximate h_n and N_n, and for which h_n' is self-adjoint and N_n' is normal. Since

$$\|h_n^2 + N_n^*N_n - h_n\| \rightarrow 0,$$

it is also true that

$$\| (h_n')^2 + N_n'^*N_n' - h_n'\| \rightarrow 0.$$

Any initial portion of h_n' or N_n' can be changed without affecting the image in $\mathcal{A}/\mathcal{I}$, so we may assume that

$$\| (h_n')^2 + N_n'^*N_n' - h_n'\| < 1/4 \quad \text{for all } n.$$

Let

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$$\epsilon_n = \|(h'_n)^2 + N'^*_n N'_n - h'_n\|.$$

The matrices h_n and N_n define a map ν_n of $C(X_n)$ to $M_{m(n)}$ where

$$X_n = \{(r, z) \,|\, r - \epsilon_n \leqq r^2 + \bar{z}z \leqq r + \epsilon_n\}.$$

As long as $\epsilon_n < 1/4$, X_n is a thickening of the sphere

$$S^2 = \{(r, z) \,|\, r^2 + \bar{z}z = r\}.$$

Let η_n denote the obvious surjection of X_n onto S^2. Let η_n also denote the induced map from $C(S^2)$ to $C(X_n)$. The composition $\nu_n \circ \eta_n$ defines a self-adjoint and a normal, h''_n and N''_n, which commute and satisfy the extra relation defining the algebra of the sphere. As $n \to \infty$, the supremum norm of η_n minus the identity approaches zero, so $\|h''_n - h'_n\|$ and $\|N''_n - N'_n\|$ approach zero. These define the lifting of φ.

We now describe a non-liftable map from $C(S^2)$. Choose a smooth function f on $[0, 1]$ such that $f(0) = 0, f(1) = 1$ and $0 \leqq f \leqq 1$. Let f_n denote the diagonal matrix with diagonal elements $f(\Delta), f(2\Delta), \dots, f(n\Delta)$, where $\Delta = 1/n$. Let

$$N_n = (f_n - f_n^2)^{1/2} S_n.$$

Then f_n is self-adjoint, N_n is asymptotically normal and $\|[f_n, N_n]\| \to 0$. Since

$$f_n^2 + N_n N_n^* - f_n = 0,$$

$$\|f_n^2 + N_n^* N_n - f_n\| \to 0.$$

Thus f_n and N_n define a map from $C(S^2)$ to $\mathscr{A}/\mathscr{I}$. It should be remarked that, just as in [**4**], N_n is a weighted shift and so can be approximated by a sequence of normal matrices (see [**1**]).

The matrices A_n and $\widetilde{B}_n$ that Davidson describes ([**4**, Theorem 2.3]) are quite similar, with A_n diagonal and $\widetilde{B}_n$ a weighted shift. Let $B_n = (1/2)\widetilde{B}_n$. Although A_n and B_n do not satisfy the relation $A_n^2 + B_n^* B_n = A_n$ so that the map they define on $C(D \times [0, 1])$ does not drop to a map of $C(S^2)$, it can be shown that the map drops to $C(Z)$ where Z is $D \times [0, 1]$ minus a small open ball in the center. Linear interpolation then defines a homotopy from A_n and B_n to f_n and N_n, defining a homotopy of maps from $C(Z)$ to $\mathscr{A}/\mathscr{I}$. Thus, Davidson's example shares the same K-theory obstruction. From a K-theoretic standpoint, f_n and N_n are preferable since, as elements of $\mathscr{A}/\mathscr{I}$, their joint spectrum is S^2 (as the zero set of $r^2 + \bar{z}z = r$) while A_n and $\widetilde{B}_n$ have joint spectrum the boundary of $D \times [0, 1]$.

5.2 THEOREM. *Let $\varphi: C(S^2) \to \mathscr{A}/\mathscr{I}$ be the map defined by f_n and N_n. Any homomorphism $\psi: C(S^2) \to \mathscr{A}/\mathscr{I}$ which is homotopic to φ cannot be lifted to a homomorphism to $\mathscr{A}$.*

5.3 COROLLARY. *There do not exist commuting sequences* h_n' and N_n' *which approximate* f_n *and* N_n *for which* f_n *is self-adjoint and* N_n' *is normal.*

Proof. This follows from Theorem 5.2 and Lemma 5.1.

The rest of this section is devoted to giving a K-theoretic proof of Theorem 5.2. The methods used are the same as those used with the torus in Sections 2 through 4. Therefore, some of the proofs will be omitted or only sketched.

First we describe generators of $K_0(C(S^2))$. One generator is of course the identity element. The other is easy to describe in terms of the generators h and N of $C(S^2)$. Since $h^2 + N^*N = h$ and everything commutes, it is easy to see that p, defined below, is a (self-adjoint) projection.

$$p = \begin{bmatrix} h & N \\ N^* & 1 - h \end{bmatrix}$$

An easy way to see that $[p]$ and $[1]$ generate $K^0(S^2)$ is to consider p as a function from S^2 to $\mathbf{CP}^1$, where $\mathbf{CP}^1$ is identified with the rank one, two-by-two, self-adjoint matrices. Since p is a bijection, the bundle it defines will have first Chern class equal to one or minus one.

5.4 *Definition.* If h and N are elements of a C^*-algebra A, then define $p(h, N)$ as

$$p(h, N) = \begin{bmatrix} h & N \\ N^* & 1 - h \end{bmatrix}.$$

We want to consider the size of $M(h, N) = p(h, N)^2 - p(h, N)$. Clearly,

$$M(h, N)$$
$$= \begin{bmatrix} (h^2 + N^*N - h) + (NN^* - N^*N) & hN - Nh \\ N^*h - hN^* & h^2 + N^*N - h \end{bmatrix}.$$

5.5 LEMMA. *The norm of* $p(h, N)^2 - p(h, N)$ *is bounded by*

$$\|[N^*, N]\| + \|h^2 + N^*N - h\| + \|[N, h]\|.$$

Now let p_n denote $p(f_n, N_n)$, where f_n and N_n are the matrices in Corollary 5.3. Since $\|[S_n, f_n]\|$ and $\|[S_n, (f_n - f_n^2)^{1/2}]\|$ converge to zero on the order of $1/n$, it follows from Lemma 5.5 that $\|p_n^2 - p_n\|$ converges to zero at least on the order of $1/n$. An argument analogous to the proof of Lemma 4.5 proves the next lemma. Recall that χ is the characteristic function of the interval $[1/2, 2]$.

5.6 LEMMA. $\lim(\dim \chi(p_n) - \tau(3p_n^2 - 2p_n^3)) = 0.$

Let $M_n = M(f_n, N_n)$. Then since $f_n^2 + N_nN_n^* - f_n = 0$,

$$M_n = \begin{bmatrix} 0 & f_nN_n - N_nf_n \\ N_n^*f_n - f_nN_n^* & -N_nN_n^* + N_n^*N_n \end{bmatrix}.$$

5.7 LEMMA. *For large n*, $\dim \chi(p(f_n, N_n)) = n - 1$.

Proof. By the last lemma, it suffices to show that $\tau(p_n^2) = n$ and that

$$\lim(\tau(p_n^3) - n) = 1/2.$$

Now

$$\tau(p_n^2) - n = \tau(p_n^2 - p_n) = \tau(M_n) = \tau(-N_nN_n^* + N_n^*N_n) = 0$$

and $\tau(p_n^3) - n = \tau(p_nM_n)$ since $p_n^3 = p_n^2 + p_nM_n$. Using the property that $\tau(xy) = \tau(yx)$, one sees that

$$\tau(p_nM_n) = 3\tau(f_n(N_nN_n^* - N_n^*N_n)).$$

Since

$$N_nN_n^* - N_n^*N_n = (f_n - f_n^2) - \alpha^{-1}(f_n - f_n^2),$$

we see that

$$\tau(p_nM_n) \to -3\int f_nd(f_n - f_n^2) = 3\int(f_n - f_n^2)f_n'$$

$$= 3\int(\lambda - \lambda^2)d\lambda = 1/2.$$

Proof of Theorem 5.2. If φ lifts, then there exist commuting sequences h_n and N_n', which approximate f_n and N_n, and for which f_n is self-adjoint and N_n' is normal. Since the lifting is a map from $C(S^2)$, we also know that

$$h_n^2 + N_n'^*N_n' - h_n = 0,$$

so that $p(h_n, N_n')$ is a projection. Since the trace of $p(h_n, N_n')$ equals n, this is a projection of dimension n.

The function $h, N \mapsto p(h, N)$ is clearly uniformly continuous, so for large n, the distance from $p(h_n, N_n')$ to $p(f_n, N_n)$ is less than 1/4. By Lemma 5.5, we also know that, for large n, the distance from $p(f_n, N_n)$ to $\chi(p(f_n, N_n))$ is less than 1/4. Since projections that are less than distance 1/2 apart are equivalent, this implies that

$$\dim(\chi(p(f_n, N_n))) = n \quad \text{for large } n.$$

This contradicts Lemma 5.7.

Since it is the K-theory that prevents the lifting, it should be clear that no map which is homotopic to φ can be lifted either.

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6. Examples of liftable mappings. One might conjecture that, given a map $\varphi:C(X) \rightarrow \mathscr{A}/\mathscr{I}$, if $K_0(\varphi)$ kills the higher cohomology in $K_0(C(X))$, that is, if $\varphi_*([p])$ depends only on the zeroth Chern class of p, then φ can be lifted to A. If this were true for $X = \mathbf{T}^2$, it would settle the commutation question for two sequences of self-adjoint matrices. For if H_n and K_n are self-adjoint matrices of norm one which commute asymptotically, then the unitaries

$$U_n = \exp(\pi i H_n) \quad \text{and} \quad V_n = \exp(\pi i K_n)$$

induce a map from $C(\mathbf{T}^2)$ which kills the second-cohomology. A lifting of this map provides commuting approximants to U_n and V_n which have as spectrum a proper subset of the unit circle. Taking logarithms produces commuting approximants for H_n and K_n.

The examples of Voiculescu and Davidson do not seem to lead to a counterexample to this conjecture. Although there are several obvious ways to alter their examples to obtain maps $C(X) \rightarrow \mathscr{A}/\mathscr{I}$ for which there is no K-theoretic obstruction to lifting, in each case, the resulting map can in fact be lifted.

One way to eliminate the K-theory obstruction is to replace the underlying space with a space that has no second cohomology. The obvious surjection $\rho:S^2 \rightarrow D$, where D is the closed unit disk, provides a map

$$\rho:C(D) \rightarrow C(S^2).$$

The algebra $C(D)$ is the universal C^*-algebra generated by a normal operator of norm one. Under the map ρ, this normal is sent to $2N$, where N and h are the generators of $C(S^2)$ described in the last section. Given a map

$$\varphi:C(S^2) \rightarrow \mathscr{A}/\mathscr{I}$$

defined by sequences N_n and h_n, the composition

$$\varphi \circ \rho:C(D) \rightarrow \mathscr{A}/\mathscr{I}$$

can be lifted to $\mathscr{A}$ if, and only if, the essentially normal sequence N_n can be approximated by a sequence of normal matrices. If φ is the map defined in the last section, then the composition is liftable since, as pointed out earlier, Berg [**1**] has shown that an essentially normal sequence of finite-dimensional weighted shifts can be approximated by normals.

The situation of the torus is similar. For this example let

$$\rho:C([0, 1] \times S^1) \rightarrow C(\mathbf{T}^2)$$

be the map induced from the obvious surjection of the torus onto the annulus. The generators for $C([0, 1] \times S^1)$ are a unitary and a self-adjoint. The unitary gets mapped to one of the unitaries on the torus, and the self-adjoint gets mapped to the real part of the other. Given a map

$$\varphi: C(\mathbf{T}^2) \to \mathscr{A}/\mathscr{I}$$

defined by sequences U_n and V_n, the composition

$$\varphi \circ \rho: C([0, 1] \times S^1) \to \mathscr{A}/\mathscr{I}$$

can be lifted if, and only if, U_n and $\mathrm{Re}(V_n)$ can be approximated by commuting pairs of unitaries and self-adjoints. As Voiculescu has pointed out to me, in the case of his example S_n, Ω_n, the resulting composition can be lifted. This is plausible since the product $(\mathrm{Re}\ \Omega_n)S_n$ is an essentially normal weighted shift. The techniques in **[1]** can be modified to construct commuting approximants to S_n and $\mathrm{Re}(\Omega_n)$.

In the case of the torus, there is a second way to eliminate the K-theory obstruction. Given sequences U_n and V_n of asymptotically commuting unitaries in $M_{m(n)}$, the unitaries $U_n \oplus U_n^*$ and $V_n \oplus V_n$ are also asymptotically commuting unitaries, regarded as elements of $M_{2m(n)}$. This eliminates the K-theory obstruction since, for any unitaries U, V in a C^*-algebra A, $e(U \oplus U^*, V \oplus V)$ represents the order unit in $K_0(M_2(A))$. (This is easy to prove for $C(\mathbf{T}^2)$, and the general case follows from this.) Davidson has shown me how to construct commuting approximants to $S_n \oplus S_n^*$ and $\Omega_n \oplus \Omega_n$. The proof is a straightforward modification of the method in **[5]**. In the special case where n is a perfect square, this construction can be written down concisely along the lines of Pimsner's work in **[6]**.

6.1 PROPOSITION. *For any m, there exist commuting unitaries U and V in M_{2m^2} such that*

$$\|U - S_{m^2} \oplus S_{m^2}^*\| \leqq 2\pi/m \quad \text{and} \quad \|V - \Omega_{m^2} \oplus \Omega_{m^2}\| \leqq 4\pi/m.$$

Proof. A perturbation of norm at most $2\pi/m$ makes the diagonal elements of $\Omega_{m^2} \oplus \Omega_{m^2}$ constant on blocks of length m. Let M_{2m^2} act on a vector space with basis indexed by $0, 1, \ldots, 2m^2 - 1$. Then the change of basis corresponding to the permutation

$$bm + c \mapsto 2(cm + b) \quad (0 \leqq b, c < m)$$
$$m^2 + bm + c \mapsto 2((m - c - 1)m + b) + 1$$

sends $S_{m^2} \oplus S_{m^2}^*$, and the perturbation of $\Omega_{m^2} \oplus \Omega_{m^2}$, to the matrices $P, V \in M_{2m^2}$ where

$$P = \begin{bmatrix} 0 & & & & & P_0 \\ I & 0 & & & & \\ & I & 0 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & I & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \Omega'_{2m} & & & & \\ & \Omega'_{2m} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \Omega'_{2m} \end{bmatrix}.$$

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If $e_0, e_1, \ldots, e_{2m-1}$ is a basis on which M_{2m} acts, then P_0 and Ω'_{2m} are defined by

$$P_0(e_{2j}) = e_{2j+2}, \quad \Omega'_2m(e_{2j}) = \omega^j e_{2j},$$
$$P_0(e_{2j+1}) = e_{2j-1}, \quad \Omega'_2m(e_{2j+1}) = \omega^j e_{2j+1}$$

where $\omega = \exp(2\pi i/m)$. (The arithmetic operations on indices of basis elements are to be taken modulo $2m$.) It suffices then to approximate P and V to within $2\pi/m$ by commuting unitaries.

Define Y_0 and Z_0 in M_{2m} by

$$Y_0(e_{2j}) = e_{2j+1}, \quad Z_0(e_{2j}) = e_{2j-1},$$
$$Y_0(e_{2j+1}) = e_{2j}, \quad Z_0(e_{2j+1}) = e_{2j+2}.$$

Use the functional calculus to define an mth root of $Y_0^{1/m}$ of Y such that

$$\|I - Y_0^{1/m}\| \leqq 2\pi/m.$$

Let Z and Y denote the $2m^2$-dimensional unitaries

$$Z = \begin{bmatrix} 0 & & & & Z_0 \\ I & 0 & & & \\ & I & 0 & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & \cdot & \\ & & & I & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} Y^{1/m} & & & & \\ & Y^{2/m} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & Y \end{bmatrix}.$$

Since $P_0 Y_0 = Z_0$,

$$Y^*PY = \begin{bmatrix} 0 & & & & Y_0^{1/m}Z_0 \\ Y_0^{-1/m} & 0 & & & \\ & Y_0^{-1/m} & 0 & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & \cdot & \\ & & & Y_0^{-1/m} & 0 \end{bmatrix}.$$

Therefore $\|Y^*PY - Z\| \leqq 2\pi/m$. Clearly Y_0 commutes with Ω'_{2m}, from which it follows that $Y^*VY = V$. Now define Ω''_{2m} as $S^*_{2m}\Omega'_{2m}S_{2m}$, and let V' be the $2m^2$-dimensional matrix

$$V' = \begin{bmatrix} \Omega''_{2m} & & & & \\ & \Omega''_{2m} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \Omega''_{2m} \end{bmatrix}.$$

Since Z_0 and Ω''_{2m} commute, so do Z and V'. The unitaries YZY^* and $YV'Y^*$ commute, and since

$$\|YZY^* - P\| = \|Z - Y^*PY\| \leqq 2\pi/m$$

and

$$\|YV'Y^* - V\| = \|V' - Y^*VY\| = \|V' - V\| \leqq 2\pi/m$$

these are the desired commuting approximants to P and V.

7. Addendum. After completing the first version of this paper, I received from Choi a preprint ([**2**]) which contains a stronger version of Corollary 5.3. He shows that there do not exist commuting approximants of any type to the matrices in Davidson's example. Choi gives an argument which can be rephrased in terms of K-theory, and which can be modified to show that there are no commuting approximants of any type to Voiculescu's unitaries.

A key lemma [**2**, Lemma 4] in Choi's proof is the following result which he proves with a clever determinant argument.

7.1 LEMMA. *If A, B, and C are n-dimensional matrices over* $\mathbf{C}$ *and* $AB = BA$, *then the spectrum, including multiplicities, of the matrix*

$$\begin{bmatrix} A & B \\ C & -A \end{bmatrix}$$

is symmetric across the imaginary axis.

We will apply this result as follows. Let χ now denote the characteristic function of the half plane $\{z \in \mathbf{C} \mid \operatorname{Im} z > 1/2\}$. If $AB = BA$, and if the matrix

$$P = \begin{bmatrix} A & B \\ C & 1 - A \end{bmatrix}$$

has no eigenvalues on the line $\operatorname{Im} z = 1/2$, then Choi's lemma, applied to $2P - 1$, shows that the (not necessarily self-adjoint) idempotent $\chi(P)$ has dimension n.

7.2 THEOREM. *Let S_n and Ω_n be the unitaries defined in Section* 1. *Then*

$$\lim\| [S_n, \Omega_n] \| = 0,$$

but there do not exist matrices A_n and B_n such that

$$A_nB_n = B_nA_n \quad \textit{and} \quad \lim\|A_n - S_n\| = \lim\|B_n - \Omega_n\| = 0.$$

Proof. The existence of A_n and B_n can be reformulated in terms of a lifting problem as before, except that the C^*-algebra $C(\mathbf{T}^2)$ must be replaced by a Banach algebra. Let X denote a closed annulus around the unit circle, and let $\mathscr{B}$ denote the algebra of continuous functions on $X \times X$ which are analytic on the interior. By choosing X sufficiently thin, we may assume that the restriction map $\pi:\mathscr{B} \rightarrow C(\mathbf{T}^2)$ is surjective on K_0. In fact, we may assume that there exists a 2×2 matrix $q = (q_{ij})$ of Laurent polynomials in two complex variables such that $q_{22} = 1 - q_{11}$, the spectrum of q lies off the line $\mathrm{Im}\, z = 1/2$, and $\pi(\chi(q))$ is equivalent to the projection e defined in Section 2.

Suppose that A_n and B_n exist, and, without loss of generality, assume that they have spectrum contained in the interior of X. Then A_n and B_n determine, via the analytic functional calculus (see for example [**8**]), a homomorphism $\varphi'':\mathscr{B} \rightarrow \mathscr{A}$ which lifts the map $\varphi':\mathscr{B} \rightarrow \mathscr{A}/\mathscr{I}$ determined by Ω_n and S_n. Of course, $\varphi' = \varphi \circ \pi$, where $\varphi:C(\mathbf{T}^2) \rightarrow \mathscr{A}/\mathscr{I}$ is also detemined by S_n and Ω_n. Theorem 4.2 shows that $\varphi'_*([\chi(q)])$ corresponds to the equivalence class of the sequence $(n - 1)$. (Recall the discussion of $K_0(\mathscr{A})$ and $K_0(\mathscr{A}/\mathscr{I})$ in Section 1.) On the other hand, Lemma 7.1 shows that $\chi((q_{ij}(A_n, B_n)))$ has dimension n. This means that $\varphi''_*([\chi(q)])$ corresponds to the sequence (n), a contradiction.

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