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k-DISCRETENESS AND k-ANALYTIC SETS

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1. Preliminaries. All spaces considered here are metrizable. k will always denote an infinite cardinal. The successor of k will be denoted by k^+.

Of particular interest will be the Baire spaces $B(k) = \prod_{n=1}^{\infty} T_n$, where each T_n is a discrete space of cardinal k. The product topology on $B(k)$ is the same as the topology given by the (complete) "first-difference" metric, $\rho : \rho(s, t) = 1/n$ if $s_i = t_i$ for $1 \leqq i \leqq n-1$ and $s_n \neq t_n$. A great deal of information about these spaces can be found in [**4**].

A subset A of X is called *k-analytic* (in X) if there exist, for each $t \in B(k)$, closed subsets $F(t_1), \ldots, F(t_1, \ldots, t_n), \ldots$ of X such that

$$A = \cup \{ \cap_{n=1}^{\infty} F(t_1, \ldots, t_n) : t \in B(k)\}.$$

A is called *absolutely k-analytic* if A is homeomorphic to a k-analytic set in some complete metric space. This is equivalent to saying that A is k-analytic in any metric space in which it is embedded. The k-analytic sets of X contain the family of Borel sets of X. Sets k-analytic in this sense were introduced in [**5**], where their basic properties are discussed.

If A is a subset of the metric space (X, d) and if, for some $\epsilon > 0$, $d(x, y) \geqq \epsilon$ whenever $x, y \in A$, we say A is *ϵ-discrete* (in (X, d)). A is called *metrically discrete* if A is ϵ-discrete for some $\epsilon > 0$.

2. k-discrete sets. In this section, we introduce the idea of k-discreteness and some of its elementary properties. Essentially the same concept occurs in a different context in [**3**]. It is designed as a measure of the "thinness" of a space. We precede the definition with the following lemma.

LEMMA 1. *Let $A \subseteq (X, d)$. Then the following are equivalent:*

(1) $A = \cup \{X(\lambda) : \lambda \in \mathfrak{A}\}$, where $|\mathfrak{A}| \leqq k$ and each $X(\lambda)$ is discrete in its relative topology.

(2) $A = \cup \{Y(\lambda) : \lambda \in \mathfrak{B}\}$, where $|\mathfrak{B}| \leqq k$ and each $Y(\lambda)$ is discrete in its relative topology and closed in X.

(3) $A = \cup \{Z(\lambda) : \lambda \in \mathfrak{C}\}$, where $|\mathfrak{C}| \leqq k$ and each $Z(\lambda)$ is metrically discrete.

Proof. Only that (1) implies (3) is not immediate. For each $x \in X(\lambda)$, there is a $\delta(x) > 0$ such that $X(\lambda) \cap S(x; \delta(x)) = \{x\}$. Let

$$X(\lambda, n) = \{x \in X(\lambda) : \delta(x) \geqq 1/n\}.$$

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Then $A = \cup \{ \bigcup_{n=1}^{\infty} X(\lambda, n) : \lambda \in \mathfrak{A}\}$, and each $X(\lambda, n)$ is metrically discrete.

Definition 2. A space A is called *k-discrete* if any one of the equivalent conditions of Lemma 1 holds.

Some elementary properties of k-discreteness are immediate. It is trivial that k-discreteness is a topological invariant. Indeed, though we shall not use the fact, k-discreteness is an invariant of Borel isomorphism among absolute Borel sets. This follows directly from the fact that $\aleph_0$-discreteness ($= \sigma$-discreteness) is such an invariant [**6**].

Any space with $\leqq k$ points is k-discrete, and any subspace of a k-discrete space is k-discrete. If A has weight $\leqq k$ and $A = \cup \{Z(\lambda) : \lambda \in \mathfrak{C}\}$, with $|\mathfrak{C}| \leqq k$ and $Z(\lambda)$ metrically discrete, then each $Z(\lambda)$ must have cardinality $\leqq k$; hence a k-discrete space of weight $\leqq k$ has $\leqq k$ points.

Definition 3. A point $a \in A$ is *k-isolated* if it has an (open) k-discrete neighborhood in A.

We denote $\{a \in A : a$ is k-isolated in $A\}$ by A_k, and $A - A_k$ by A_k^*. Thus A_k^* is closed in A.

PROPOSITION 4. *A is k-discrete if and only if A is locally k-discrete.*

Proof. The latter condition is clearly necessary. So suppose A is locally k-discrete. From a fixed σ-discrete open basis for A, pick a family

$$\{O(\lambda, i) : \lambda \in \Lambda, i = 1, 2, \ldots\}$$

of k-discrete sets covering A so that, for fixed i, $\{O(\lambda, i) : \lambda \in \Lambda\}$ is discrete. Write

$$O(\lambda, i) = \cup \{O(\lambda, i, \alpha) : \alpha < k\}$$

where each $O(\lambda, i, \alpha)$ is metrically discrete, and put

$$B(\alpha, i) = \cup \{O(\lambda, i, \alpha) : \lambda \in \Lambda\}.$$

Given α, i and $x \in \Lambda$, pick a neighborhood N_x of x which meets at most one $O(\lambda, i)$, say $O(\lambda^*, i)$, and then a neighborhood N_x' of x meeting at most one point of $O(\lambda^*, i, \alpha)$. Then $N_x \cap N_x'$ meets $B(\alpha, i)$ in at most one point, so $B(\alpha, i)$ is discrete. Then

$$A = \cup \{B(\alpha, i) : \alpha < k, i = 1, 2, \ldots\}$$

is k-discrete.

The following propositions are easy to check, and, in fact, are special cases of the kernel properties [**7**]; A_k^* is the "nowhere locally k-discrete kernel" of A. In the case $k = 1$, these propositions are familiar properties of discreteness.

PROPOSITION 5. *For any* A,
(1) A_k *is* k-*discrete*,
(2) *if* A_k^* *is* k-*discrete, then so is* A,
(3) *either* $A_k^* = \emptyset$ *or* A_k^* *is not* k-*discrete (and therefore* $|A_k^*| > k$).

PROPOSITION 6. *For any* A,
(1) $(A_k^*)_k^* = A_k^*$,
(2) $(A_k^*)_k = \emptyset$,
(3) $(A_k)_k = A_k$,
(4) $(A_k)_k^* = \emptyset$.

The following simple corollary will be used repeatedly in the next section.

COROLLARY 7. *If* $A \subseteq X$ *and* G *is open in* X, *and if* $M = G \cap A_k^*$, *then* $M_k = \emptyset$. *Hence if* $M \neq \emptyset$, M *is not* k-*discrete.*

Proof. M is open in A_k^*. If $M_k \neq \emptyset$, then $(A_k^*)_k \neq \emptyset$, contrary to Proposition 6. Hence, if $M \neq \emptyset$, the set $M = M_k^*$ is not k-discrete by Proposition 5.

Other generalizations of discreteness have been used in descriptive set theory, for example the properties "$\sigma lw(< k)$" (= σ-locally of weight less than k) and "h-$lw(< k)$" (= h-locally of weight less than k), which occur in [**8**] and in [**9**, **10**] respectively. In the latter two papers, the concept of k-discreteness also occurs.

The following theorem, which was pointed out to the author by the referee, can be used to relate the concepts of k-discreteness and $\sigma lw(<k)$. This is perhaps of special interest since the latter property plays such an important role in the structure theory of absolute Borel sets [**8**].

THEOREM 8. *For any metric space* (X, d), *the following are equivalent:*
(1) X *is* k-*discrete.*
(2) X *is* σ-*locally-of-cardinal* $\leqq k$, *i.e.,* $X = \bigcup_{n=1}^{\infty} Y_n$, *where, for each* n, *each* $y \in Y_n$ *has a neighborhood in* Y_n *of cardinality* $\leqq k$.

Proof. Assume (1). Then $X = \bigcup \{X_\lambda : \lambda \in \Lambda\}$, where $|\Lambda| \leqq k$ and each X_λ is discrete. We may assume, by Lemma 1, that each X_λ is metrically discrete. For each $n = 1, 2, 3, \ldots$, let $\Lambda_n = \{\lambda \in \Lambda : X_\lambda \text{ is } 1/n\text{-discrete}\}$, and

$$Y_n = \bigcup \{X_\lambda : \lambda \in \Lambda_n\}.$$

Then $X = \bigcup_{n=1}^{\infty} Y_n$, and $|Y_n| \leqq k$, since the $(1/2n)$-neighborhood of a point in Y_n contains at most one point from each X_λ.

Conversely, suppose (2) holds. We can assume that X is locally of cardinal $\leqq k$. Cover X by open sets of cardinal $\leqq k$ and let $\{V_{n,\mu} : \mu \in M_n, n = 1, 2 \ldots\}$ be a σ-discrete open refinement of that cover. Index the points of each $V_{n,\mu}$ as $v_{n,\mu,\alpha}$ ($\alpha \leqq$ some ordinal $\alpha_{n,\mu} \leqq k$). Then, for each $\alpha \leqq k$ and each $n = 1, 2, \ldots$, let $D_{\alpha,n} = \{v_{n,\mu,\alpha} : \mu \in M_n\}$. Each $D_{\alpha,n}$ is discrete since the $V_{n,\mu}$'s are open and

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disjoint for fixed n. There are $\leqq k$ $D_{\alpha,n}$'s and

$$X = \bigcup \{D_{\alpha,n} : \alpha \leqq k, n = 1, 2 \ldots\}.$$

It follows that if X is k-discrete, then X is $\sigma lw(\leqq k)$. If X is $\sigma lw(\leqq k)$, then X is $k^{\aleph_0}$-discrete. In particular, if $k^{\aleph_0} = k$, then k-discreteness coincides with $\sigma lw(\leqq k)$; and assuming GCH, if X is $\sigma lw(\leqq k)$, then X is k^+-discrete.

3. k-discreteness and k-analytic sets. In this section we investigate the consequences if an absolutely k-analytic set A is "thick", in the sense that $A_k^* \neq \emptyset$. We first prove the following simple lemma.

LEMMA 9. *Let (A, d) be an absolutely k-analytic metric space with $A_k^* \neq \emptyset$. Then A_k^* contains either a metrically discrete subset of cardinal k^+ or a closed subspace C, of cardinal $k^{\aleph_0}$, and homeomorphic either to the Cantor set or a Baire space $B(p)$.*

Proof. Let m denote the weight of A_k^*. Since A_k^* is not k-discrete, $|A_k^*| > k$. So if $m \leqq k$, then, by a theorem of Stone [**5**], A_k^* contains a closed subset of the form C. On the other hand, if $m > k$, then, letting D_n denote a maximal $1/n$-discrete subset of A_k^*, we get that $|\bigcup_{n=1}^{\infty} D_n| \geqq m$, so some D_n has cardinal $> k$.

COROLLARY 10. *Under the hypotheses of Lemma 9, A_k^* contains either a metrically discrete subset of cardinal k^+, or a copy of $B(p)$ for some p such that $p^{\aleph_0} = k^{\aleph_0}$.*

Proof. If C is the Cantor set, then C contains a copy of $B(\aleph_0)$, and $|B(\aleph_0)| = \aleph_0^{\aleph_0} = c = |C| = k^{\aleph_0}$.

In [**6**], Stone showed that any absolute Borel set is either $\aleph_0$-discrete ($= \sigma$-discrete) or contains a copy of the Cantor set. And in [**1**], El'kin generalized this result to absolutely $\aleph_0$-analytic sets. The next theorem shows that it remains true for absolutely k-analytic sets.

THEOREM 11. *Let (A, d) be an absolutely k-analytic metric space. Then either A is k-discrete or A contains a closed subspace C of cardinal $k^{\aleph_0}$, homeomorphic either to the Cantor set or a Baire space $B(p)$.*

Proof. Assume A is not k-discrete, and write $B(k) = \prod_{n=1}^{\infty} T_n$, where T_n is a discrete space of cardinal k. Let X denote the completion of (A, d). Since A is k-analytic in X, we can find, for each $t \in B(k)$, and for each n, closed subsets $F(t_1, \ldots, t_n)$ of X, with

$$F(t_1, \ldots, t_{n+1}) \subseteq F(t_1, \ldots, t_n), \quad \text{such that} \quad A = \bigcup \{ \bigcap_{n=1}^{\infty} F(t_1, \ldots, t_n) : t \in B(k)\}.$$

Define

$$A(t_1, \ldots, t_n) = \bigcup \{ \bigcap_{n=1}^{\infty} F(t_1, \ldots, t_k) : (t_{n+1}, t_{n+2}, \ldots) \in T_{n+1} \times T_{n+2} \times \ldots\} \subseteq A.$$

Clearly $A = \bigcup \{A(t_1) : t_1 \in T_1\}$, $A(t_1, \ldots, t_n) \subseteq F(t_1, \ldots, t_n)$, and it is easy to check that each $A(t_1, \ldots, t_n)$ is k-analytic in X.

Since A is not k-discrete, neither is its closed subset A_k^*. If A_k^* contains a closed set of form C, we are done. Otherwise, by Lemma 9, for some $\epsilon_1 > 0$, A_k^* contains an ϵ_1-discrete subset $\{a(\lambda_1) : \lambda_1 < k\}$. For each $\lambda_1 < k$, pick an open (in X) sphere $U(\lambda_1)$, centered at $a(\lambda_1)$ and of radius $< \min\{1/2, (\epsilon_1)/3\}$. Note that if $\lambda_1 \neq \lambda_1' < k$, then $\mathrm{cl}_X U(\lambda_1)$ and $\mathrm{cl}_X U(\lambda_1')$ are at distance $> (\epsilon_1)/3$.

For each $\lambda_1 < k$, there is a $t(\lambda_1) \in T_1$ such that $A(t(\lambda_1))_k^* \cap U(\lambda_1) \neq \emptyset$, and therefore, by Corollary 7, is not k-discrete. For if not, then for some $\lambda_1 < k$,

$$A \cap U(\lambda_1) = \bigcup \{[A(t_1)_k \cup A(t_1)_k^*] \cap U(\lambda_1) : t_1 \in T_1\}$$
$$= \bigcup \{A(t_1)_k \cap U(\lambda_1) : t_1 \in T_1\}$$

would be k-discrete. Hence $A_k^* \cap U(\lambda_1)$ would be k-discrete, which, since it contains $a(\lambda_1)$, would contradict Corollary 7.

Now suppose that given n, we have defined for every i, $1 \leqq i \leqq n$, and every i-tuple $(\lambda_1, \ldots, \lambda_i)$ (with $\lambda_s < k$, $1 \leqq s \leqq i$)

(1) points $t(\lambda_1, \ldots, \lambda_i) \in T_i$

(2) positive numbers $\epsilon_i(\lambda_1, \ldots, \lambda_{i-1})$ ($= \epsilon_1$ if $i = 1$)

(3) $\epsilon_i(\lambda_1, \ldots, \lambda_{i-1})$-discrete sets $\{a(\lambda_1, \ldots, \lambda_{i-1}, \lambda_i) : \lambda_i < k\}$

(4) open (in X) spheres $U(\lambda_1, \ldots, \lambda_1)$ of radius $< \min\{1/2^i, \epsilon_i(\lambda_1, \ldots, \lambda_{i-1})/3\}$ centered at $a(\lambda_1, \ldots, \lambda_i)$ in such a way that

(5) for $i > 1$, $\{a(\lambda_1, \ldots, \lambda_i) : \lambda_i < k\} \subseteq A(t(\lambda_1), \ldots, t(\lambda_1, \ldots, \lambda_{i-1}))_k^* \cap U(\lambda_1, \ldots, \lambda_{i-1})$

(6) for $i > 1$, $\mathrm{cl}_X U(\lambda_1, \ldots, \lambda_i) \subseteq U(\lambda_1, \ldots, \lambda_{i-1})$

(7) $A(t(\lambda_1), \ldots, t(\lambda_1, \ldots, \lambda_i))_k^* \cap U(\lambda_1, \ldots, \lambda_i) \neq \emptyset$ (and hence, by Corollary 7, is not k-discrete).

If any of the closed sets $\mathrm{cl}_A(A[t(\lambda_1), \ldots, t(\lambda_1, \ldots, \lambda_n)]_k^* \cap U(\lambda_1, \ldots, \lambda_n))$ contains a closed set of the form C, we are done. So suppose not. The set

$$\mathrm{cl}_A(A[t(\lambda_1), \ldots, t(\lambda_1, \ldots, \lambda_n)]_k^* \cap U(\lambda_1, \ldots, \lambda_n))$$

is not k-discrete or else its subset

$$A[t(\lambda_1), \ldots, t(\lambda_1, \ldots, \lambda_n)]_k^* \cap U(\lambda_1, \ldots, \lambda_n)$$

would be, contrary to (7). Hence it contains a metrically discrete subset of cardinal k, and therefore, for some $\epsilon_{n+1}(\lambda_1, \ldots, \lambda_n) > 0$, $A[t(\lambda_1), \ldots, t(\lambda_1, \ldots, \lambda_n)]_k^* \cap U(\lambda_1, \ldots, \lambda_n)$ contains an $\epsilon_{n+1}(\lambda_1, \ldots, \lambda_n)$-discrete set of cardinal k, say

$$\{a(\lambda_1, \ldots, \lambda_n, \lambda_{n+1}) : \lambda_{n+1} < k\}.$$

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Pick open spheres (in X) $U(\lambda_1, \ldots, \lambda_{n+1})$ of radius $< \min\{1/2^{n+1}, \epsilon_{n+1}(\lambda_1, \ldots, \lambda_n)/3\}$ centered at $a(\lambda_1, \ldots, \lambda_{n+1})$ and so that

$$\mathrm{cl}_X U(\lambda_1, \ldots, \lambda_{n+1}) \subseteq U(\lambda_1, \ldots, \lambda_n).$$

Given now $(\lambda_1, \ldots, \lambda_{n+1})$, there must be a point $t(\lambda_1, \ldots, \lambda_{n+1}) \in T_{n+1}$ such that

$$A[t(\lambda_1), \ldots, t(\lambda_{n+1})]_k^* \cap U(\lambda_1, \ldots, \lambda_{n+1}) \neq \emptyset.$$

Otherwise,

$$\begin{aligned} &A[t(\lambda_1), \ldots, t(\lambda_1, \ldots, \lambda_n)] \cap U(\lambda_1, \ldots, \lambda_{n+1}) \\ &= \cup \{(A[t(\lambda_1), \ldots, t(\lambda_1, \ldots, \lambda_n), t_{n+1}]_k \\ &\quad \cup A[t(\lambda_1), \ldots, t(\lambda_1, \ldots, \lambda_n), t_{n+1})_k^*) \\ &\quad \cap U(\lambda_1, \ldots, \lambda_{n+1}) : t_{n+1} \in T_{n+1}\} \\ &= \cup \{A[t(\lambda_1), \ldots, t(\lambda_1, \ldots, \lambda_n), t_{n+1}]_k \cap U(\lambda_1, \ldots, \lambda_{n+1}) : t_{n+1} \in T_{n+1}\} \end{aligned}$$

which is k-discrete. This would imply $A[t(\lambda_1), \ldots, t(\lambda_1, \ldots, \lambda_n)]_k^* \cap U(\lambda_1, \ldots, \lambda_{n+1})$ is k-discrete, which, since it contains $a(\lambda_1, \ldots, \lambda_{n+1})$, would contradict Corollary 7.

Thus we either produce, at some finite stage of this construction, a closed subset of A of the form C, or else, by induction, we define, for all n, objects satisfying (1)–(7). Assume the latter occurs.

The space of all sequences $\{(\lambda_1, \ldots, \lambda_n, \ldots) : \lambda_n < k\}$, with the "first-difference" metric, is homeomorphic to $B(k)$. Given $\lambda = (\lambda_1, \ldots, \lambda_n, \ldots) \in B(k)$, the sets

$$A[t(\lambda_1)] \cap U(\lambda_1), \ldots, A[t(\lambda_1), \ldots, t(\lambda_1, \ldots, \lambda_n)] \cap U(\lambda_1, \ldots, \lambda_n), \ldots$$

are non-empty. It follows that the decreasing sequence of non-empty closed sets of X,

$$F[t(\lambda_1)] \cap \mathrm{cl}_X U(\lambda_1), \ldots, F[t(\lambda_1), \ldots, t(\lambda_1, \ldots, \lambda_n)] \cap \mathrm{cl}_X U(\lambda_1, \ldots, \lambda_n), \ldots,$$

whose diameters tend to 0, intersect in a single point $f(\lambda)$ of X. In fact, $f(\lambda) \in A$ since

$$\bigcap_{n=1}^{\infty} F[t(\lambda_1), \ldots, t(\lambda_1, \ldots, \lambda_n)] \subseteq A.$$

It is easy to check that the map $f : B(k) \to A$ is continuous and one to one.

f is also an open map of $B(k)$ onto $f(B(k))$. Indeed, if $W(\lambda_1, \ldots, \lambda_n)$ is the basic open set $\{(\lambda_1, \ldots, \lambda_n, \mu_{n+1}, \ldots) : \mu_{n+i} < k\}$ of $B(k)$, then

$$f[W(\lambda_1, \ldots, \lambda_n)] = U(\lambda_1, \ldots, \lambda_n) \cap f[B(k)].$$

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Finally, we claim $f[B(k)]$ is closed in A. We will, in fact, show it is even closed in X, by proving $(f(B(k)), d)$ is complete. Let $\{y_n : n = 1, 2, \ldots\}$ be a Cauchy sequence in $(f[B(k)], d)$. Pick a positive integer N_1 so that if $n \geqq N_1$, then $d(y_n, y_{N_1}) < (\epsilon_1)/3$, where ϵ_1 is as above. Since $y_{N_1} \in f[B(k)]$, it is in some (unique) set of the form

$$F[t(\mu_1)] \cap \mathrm{cl}_X U(\mu_1), \quad \mu_1 < k,$$

and since, if $\mu_1 \neq \mu_1' < k$, the sets $\mathrm{cl}_X U(\mu_1)$ and $\mathrm{cl}_X U(\mu_1')$ are at distance $> (\epsilon_1)/3$, we get that if $n \geqq N_1$,

$$y_n \in F[t(\mu_1)] \cap \mathrm{cl}_X U(\mu_1).$$

Now assume that positive integers $N_s > \ldots > N_1$ have been chosen, and ordinals $\mu_1, \ldots, \mu_s < k$ so that, if $n \geqq N_i$,

$$y_n \in F[t(\mu_1), \ldots, t(\mu_1, \ldots, \mu_i)] \cap \mathrm{cl}_X U(\mu_1, \ldots, \mu_i).$$

Then choose $N_{s+1} > N_s$ so $n \geqq N_{s+1}$ implies

$$d(y_n, y_{N_{s+1}}) < \epsilon_{s+1}(\mu_1, \ldots, \mu_s)/3.$$

Again, $y_{N_{s+1}}$ is in a unique set of the form

$$F[t(\lambda_1), \ldots, t(\lambda_1, \ldots, \lambda_{s+1})] \cap \mathrm{cl}_X U(\lambda_1, \ldots, \lambda_{s+1}),$$

and since $N_{s+1} > \ldots > N_1$, we get $\lambda_i = \mu_i$, $1 \leqq i \leqq s$. Let $\lambda_{s+1} = \mu_{s+1}$. As before, if $n \geqq N_{s+1}$, we have

$$y_n \in F[t(\mu_1), \ldots, t(\mu_1, \ldots, \mu_{s+1})] \cap \mathrm{cl}_X U(\mu_1, \ldots, \mu_{s+1}).$$

Let $y = f(\mu_1, \ldots, \mu_s, \ldots,) \in f[B(k)]$. Since y and y_{N_s} are both in $\mathrm{cl}_X U(\mu_1, \ldots, \mu_s)$, which has diameter $< 1/2^{s-1}$, the sequence $\{y_{N_s} : s = 1, 2, \ldots\} \to y$, and therefore the Cauchy sequence $\{y_n : n = 1, 2, \ldots\}$ converges to y as well.

COROLLARY 12. *Under the hypotheses of Theorem* 11, *A is either k-discrete or contains a copy of $B(p)$ for some p such that $p^{\aleph_0} = k^{\aleph_0}$.*

COROLLARY 13. (El'kin) *If A is absolutely $\aleph_0$-analytic, then A is either $\aleph_0$-discrete or A contains a Cantor set.*

Proof. By Corollary 12, A is either $\aleph_0$-discrete or contains a copy of a Baire space $B(p)$, which in turn contains a Cantor set.

The alternatives of Theorem 11 are not mutually exclusive. For example, the space $B(k)$ itself, having weight k, is k-discrete precisely when $k^{\aleph_0} = k$. We shall show, however, that the alternatives are mutually exclusive if $k^{\aleph_0} > k$. We begin by examining $B(k)$ again.

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PROPOSITION 14. *The smallest k for which $B(m)$ is k-discrete satisfies $m \leqq k \leqq m^{\aleph_0}$, and $m < k$ unless $m = m^{\aleph_0}$.*

Proof. We first show $B(m)$ is not k-discrete if $k < m$. It suffices to show $B(k^+)$ is not k-discrete. We assume $\prod_{n=1}^{\infty} T_n = B(k^+)$ has the "first-difference" metric, ρ.

Suppose $\mathfrak{B} = \{B(\lambda) : \lambda < k\}$ is a family of metrically discrete subsets of $B(k^+)$. We shall show $\bigcup \mathfrak{B} \neq B(k^+)$. For each n, let

$$\Lambda_n = \{\lambda < k : B(\lambda) \text{ is } 1/n\text{-discrete}\}.$$

Then $|\Lambda| \leqq k$ and $\Lambda_n \subseteq \Lambda_{n+1}$.

Pick $x_1^* \in T_1$. Any two points of the form $(x_1^*, x_2, \ldots)$ are at distance $\leqq 1/2$, so no two of them are in one $B(\lambda)(\lambda \in \Lambda_1)$. Thus $\bigcup \{B(\lambda) : \lambda \in \Lambda_1\}$ contains $\leqq k$ points of that form. Therefore we can choose $x_2^* \in T_2$ so no point of the form $(x_1^*, x_2^*, \ldots)$ is in $\bigcup \{B(\lambda) : \lambda \in \Lambda_1\}$.

Continuing in this way, suppose, for $1 \leqq i \leqq n$, $x_i^* \in T_i$ are chosen so that no two points of the form $(x_1^*, \ldots, x_n^*, x_{n+1}, \ldots)$ are in $\bigcup \{B(\lambda) : \lambda \in \Lambda_{n-1}\}$. Since any two such points are at distance $\leqq 1/(n+1)$ no $B(\lambda)(\lambda \in \Lambda_n)$ can contain two of them. Hence, as before, we can choose $x_{n+1}^* \in T_{n+1}$ so that no point of the form $(x_1^*, \ldots, x_{n+1}^*, x_{n+2}, \ldots)$ is in $\bigcup \{B(\lambda) : \lambda \in \Lambda_n\}$. The point $(x_1^*, \ldots, x_n^*, \ldots)$ whose coordinates have been inductively defined in this way is clearly not in $\bigcup \mathfrak{B}$.

The second inequality of the theorem follows from the fact that $B(m)$, with $m^{\aleph_0}$ points, is $m^{\aleph_0}$-discrete. The last assertion follows from the remark following Corollary 13.

THEOREM 15. *If A is absolutely k-analytic and $k^{\aleph_0} > k$, then one and only one of the following holds:*

(1) A is k-discrete

(2) A contains a closed subset C, of cardinal $k^{\aleph_0}$, and homeomorphic to either the Cantor set or a Baire space $B(p)$.

Proof. It only remains to show (1) and (2) are mutually exclusive. So suppose $k^{\aleph_0} > k$ and A is k-discrete. If (2) also holds, then A contains a copy of $B(p)$ with $p^{\aleph_0} = k^{\aleph_0}$. This copy of $B(p)$ is k-discrete, so $p \leqq k$ by Proposition 14; since a k-discrete space of weight $\leqq k$ has $< k$ points, it follows that $p^{\aleph_0} \leqq k < k^{\aleph_0}$, while $p^{\aleph_0} = k^{\aleph_0}$, a contradiction.

Corollary 12 also produces a different proof of the following result due to Stone [**5**].

THEOREM 16. *Let k be an infinite cardinal such that* (i) *$k < k^{\aleph_0}$ and* (ii) *$p^{\aleph_0} < k$ whenever $\aleph_0 \leqq p < k$. Then the following statements about an absolute Borel set X are equivalent:*

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(1) *X has weight $\leqq k$ and $|X| > k$.*
(2) *X is Borel isomorphic to $B(k)$.*
(3) *X is generalized homeomorphic to $B(k)$.*

Proof. That (3) implies (2) is trivial. If (2) holds, then $|X| = k^{\aleph_0} > k$, and the weight of X is $\leqq k$ (since weight is an invariant of Borel isomorphism among absolute Borel sets [**5**]). If (1) holds, then X is not k-discrete and so, by Corollary 12, X contains a Baire space $B(p)$ with $p^{\aleph_0} = k^{\aleph_0} > k$. Then $p \geqq k$, so X contains a copy of $B(k)$. Hence, X is generalized homeomorphic to $B(k)$ [**5**].

We remark that on the generalized continuum hypothesis, any infinite cardinal k satisfying (i) in Theorem 16 also satisfies (ii). Also, (2) and (3) are known to be equivalent (for absolutely $\aleph_0$-analytic metric spaces) without any cardinal assumptions. This follows from theorems of Preiss [**11**] and Hansell [**2**].

4. Results using the generalized continuum hypothesis. If we assume the generalized continuum hypothesis ([GCH]), then the results of the previous section can be somewhat sharpened.

LEMMA 9* [GCH]. *Let (A, d) be an absolutely k-analytic metric space with $A_k^* \neq \emptyset$. Then A_k^* contains either a metrically discrete subset of cardinal k^+ or a closed subspace C, of cardinal $k^{\aleph_0}$, homeomorphic either to the Cantor set or $B(k)$.*

Proof. If $k^{\aleph_0} = k$, then the weight of A_k^* must be $>k$, or else A_k^* would be k-discrete. Then it follows, as in the proof of Lemma 9, that A_k^* contains a metrically discrete subset of cardinal k^+.

If $k^{\aleph_0} > k$, and A_k^* contains a closed set C homeomorphic to $B(p)$, with $p^{\aleph_0} = k^{\aleph_0}$, then either $p = k$ or $p = k^+$, and so A_k^* contains a closed copy of $B(k)$.

THEOREM 11* [GCH]. *Let (A, d) be an absolutely k-analytic metric space. Then either A is k-discrete or A contains a closed subspace C of cardinal $k^{\aleph_0}$, homeomorphic either to the Cantor set or $B(k)$.*

Proof. The proof is virtually identical to that of Theorem 11, replacing the uses of Lemma 9 by Lemma 9*.

Remark. It is not possible to conclude that the set A of Theorem 11* is either k-discrete or contains a closed subset homeomorphic to $B(k)$. For example, the Cantor set is not $\aleph_0$-discrete and contains no closed copy of $B(\aleph_0)$ (or, for that matter, of any Baire space $B(p)$). However, it is easy to see that this stronger conclusion can be drawn, on the generalized continuum hypothesis, if $k > \aleph_0$.

Our next result generalizes, under the generalized continuum hypothesis, the classical theorem that every uncountable, complete, separable, zero-dimen-

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sional space is, after the deletion of an appropriate countable set, homeomorphic to $B(\aleph_0)$ [**4**, p. 443].

COROLLARY 17 [GCH]. *Every complete space with (covering) dimension* 0 *and weight* $\leqq k$ *is the union of two disjoint subspaces* A *and* B *where*

(1) A *is open and has cardinal* $\leqq k$;

(2) B *is either empty or homeomorphic to* $B(k)$.

Proof. Since the proof of the classical result covers the case $k = \aleph_0$, we assume $k > \aleph_0$. Let $A = X_k$ and $B = X_k^*$. Then A is open, and since A is k-discrete and has weight $\leqq k$, $|A| \leqq k$.

If $B \neq \emptyset$, then it is a completely metrizable, zero-dimensional space of weight $\leqq k$ which, by Proposition 6, has no k-isolated points. Hence no open subset of B has k-isolated points. Since each non-empty subset of B is absolutely k-analytic, each contains, by the remarks following Theorem 10*, a closed copy of $B(k)$, and hence a discrete subset of cardinal k. It follows that B is homeomorphic to $B(k)$ [**5**].

In [**5**], Stone has shown, under the generalized continuum hypothesis, that the space X of weight $\leqq k$ has every subset absolutely k-analytic if and only if $|X| \leqq k$, and raised the question of a similar theorem for spaces of arbitrary weight. Our next result provides a partial answer.

THEOREM 17 [GCH]. *Let* A *be absolutely* k*-analytic and assume* $k^{\aleph_0} > k$. *Then the following are equivalent:*

(1) A *is* k*-discrete.*

(2) *Every subset of* A *is (absolutely)* k*-analytic.*

Proof. If A is k-discrete, then every subset of A is the union of $\leqq k$ closed sets and is therefore (absolutely) k-analytic. Now assume (2) holds and A is not k-discrete. Then A contains a copy of $B(k)$. Therefore all subsets of $B(k)$ are absolutely k-analytic, and hence each subset of $B(k)$ is a continuous image of $B(k)$ [**5**]. But the number of continuous images of $B(k)$ in $B(k)$ is $\leqq (k^{\aleph_0})^k = k^k = 2^k$, while $B(k)$ has $2^{k^{\aleph_0}} = 2^{2^k}$ subsets.

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