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WHICH ABELIAN GROUPS CAN BE FUNDAMENTAL GROUPS OF REGIONS IN EUCLIDEAN SPACES?

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Introduction. It is known that there are a lot of properties of the group of a knot in S^3 which fail to generalize to the group of a knotted sphere in S^4; among them are included Dehn's lemma, Hopf's conjecture, and the aspherity of knots. In this paper, we shall investigate the properties of the fundamental groups of regions in S^3 and in S^4, with examples to show that they are not quite the same. Some special consideration will be given to regions that are the complements in S^3 or in S^4 of a finite number of tamely imbedded manifolds of co-dimension 2, and, more generally, to regions that are the complements of subcomplexes in S^3 or in S^4. We shall obtain a complete classification of those abelian groups that can be fundamental groups of regions in S^n, $n \geqq 3$ as follows: An abelian group G is the fundamental group of a region in S^n for $n = 3$ if and only if $G = 1, Z, Z + Z$, or a subgroup of the additive rationals (Theorem 5), and for $n \geqq 4$ if and only if G is countable (cf. Theorem 6).

1. Which regions of S^3 have abelian fundamental group? Two complexes K_1, K_2 in S^n are said to be of the same type if there exists an autohomeomorphism f of S^n such that $f(K_1) = K_2$. A complex type is called *tame* if it has a polygonal representative. We shall be concerned only with tame complex types. If K is a complex in S^n, we shall also use the same symbol K to denote the complex type represented by K.

By *the group* of a complex K in S^n we mean the fundamental group, $\pi(S^n - K)$, of the complement of K in S^n. In particular, if K is a knot in S^3, it is called the *knot group* and if K is a link in S^3, the *link group*. It is known [**8**] that the trivial knot is the only knot with abelian knot group. R. H. Fox showed that the only link of more than one component with abelian link group is the one shown in Figure 1 (cf. [**5**]).

By a (regular) *handlebody* in S^3 we mean a tubular neighborhood V of a tame imbedding of the complex K consisting of n circles each intersecting the preceding one at only one point, as shown in Figure 2. The integer n is called the *genus* of the handlebody. A handlebody of genus 0 is a 3-*cell* and one of genus 1 is a *solid torus*. We shall call a tubular neighborhood of the trivial knot a *solid torus of trivial type*. A handlebody of genus 2 is called a *double solid torus*.

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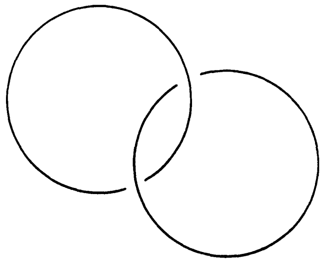

FIGURE 1

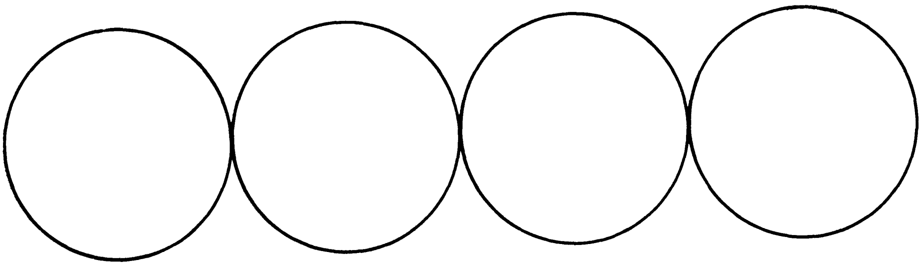

FIGURE 2 ($n = 4$)

LEMMA 1. *The group of a handlebody of genus $n \geqq 2$ in S^3 is never abelian.*

Proof. Since the group of such a handlebody V can be mapped surjectively onto the group of a handlebody of genus 2, obtained from V by cutting all but its first two handles, it is sufficient to prove Lemma 1 for $n = 2$. Let V be the image of the double solid torus H under a tame imbedding $f: H \rightarrow S^3$. Let A, B, B' be circles in H as shown in Figure 3. Then $f(A \cup B)$ and $f(A \cup B')$ are links l_1, l_2 contained in V.

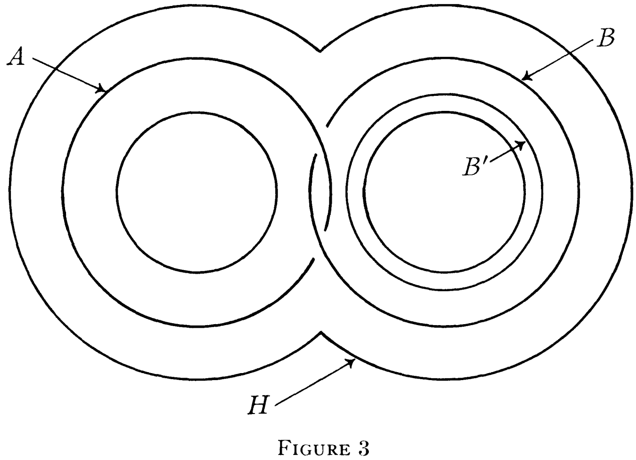

FIGURE 3

Clearly the linking number of l_1 and l_2 in S^3 differ by 1. There are natural maps

$$f_1 : \pi(S^3 - V) \rightarrow \pi(S^3 - l_1)$$
$$f_2 : \pi(S^3 - V) \rightarrow \pi(S^3 - l_2)$$

induced by inclusion. (We shall deal with oriented links only so its linking numbers are well defined.) It is easy to see that both f_1 and f_2 are surjective; this implies that if $\pi(S^3 - V)$ is abelian, then both l_1 and l_2 will have abelian link groups. However, since the only link that has abelian group has its linking number equal to 1, and since l_1 and l_2 have different linking numbers, l_1 and l_2 cannot both have abelian link groups; hence $\pi(S^3 - V)$ is not abelian. This completes the proof of the lemma.

THEOREM 2. *If V is the union of disjoint handleboides $V_1, \ldots, V_m$ and $\pi(S^3 - V)$ is abelian, then each V_i is either a 3-cell or a solid torus of trivial type. Moreover, there are at most two components of the latter type, and if there are exactly two of them, say V_1 and V_2, then V_1 and V_2 must be situated relative to ezch other as shown in Figure* 1.

Proof. Since $\pi(S^3 - V_i)$ is, for each $i = 1, \ldots, m$, the homomorphic image of $\pi(S^3 - V)$, we conclude that $\pi(S^3 - V_i)$ must be abelian. By Lemma 1, we conclude that each V_i is either a 3-cell or a solid torus. We thus know that V is just the tubular neighborhood of a link and a finite number of 3-cells, and our theorem then follows from the well-known theorems about knots and links, quoted above.

COROLLARY 3. *If the group G of a graph $K = K_1 \cup \ldots \cup K_m$ is abelian, then each component K_i of K is either contractible or has an unknotted simple closed curve C_i as a deformation retract (i.e., K_i is the union of the unknotted simple closed curve C_i and a tree T_i for which $C_i \cap T_i$ is a simple arc). Moreover there are at most two components of the latter type, and if there are exactly two of them, say K_1 and K_2, then C_1 and C_2 must be situated relative to each other as shown in Figure* 1.

Proof. A tubular neighborhood of K is the disjoint union of handlebodies V of Theorem 2.

2. Which abelian groups are fundamental groups of regions of S^3? A *region* of S^n is an open subset of S^n which is connected. It is known that the fundamental group of a region of S^2 is always free and countable, and cannot contain an abelian subgroup whose rank is greater than 1. In S^3, not all open regions have free groups as their fundamental groups; but there are other restrictions on G in order that it be the fundamental group of a region of S^3. Papakyriakopoulos' work [**8**, Corollary (31.8)] shows that G cannot contain elements of finite order. (This was known as Hopf's conjecture.)

We shall be mainly interested in regions of S^3 that have abelian fundamental groups. We have seen tame regions of S^3 whose fundamental groups are 1, Z, $Z + Z$ respectively. There are also wild regions of S^3 whose fundamental groups are abelian, though not finitely generated; the best known example is the following one (cf. [**3**, p. 330]).

Let T be a solid torus of trivial type, and let $T^{(p)}$, p a positive integer, be another solid torus that is imbedded in T as shown in Figure 4; thus $T^{(p)}$ is a tubular neighborhood of a torus knot of type $(p, 1)$. We have a canonical

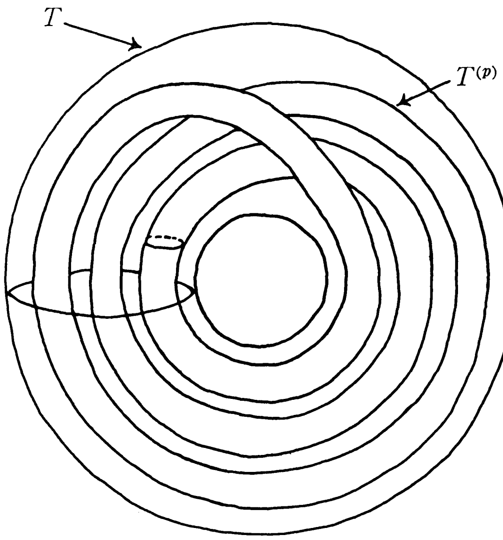

FIGURE 4 ($P = 3$)

homeomorphism f_p of T onto $T^{(p)}$ which maps meridian to meridian and longitude to longitude. Now we denote T by T_1, and define T_n inductively by letting $f_p(T_k) = T_{k+1}$.

Let

$$A_n = S - T_n, \text{ and } A = \bigcup_{n=1}^{\infty} A_n.$$

Since $\pi(A)$ is the direct limit of the groups $\pi(A_r)$, each of which is infinite cyclic and the connecting maps are induced by inclusion, it follows that $\pi(A)$ is the subgroup of the rational numbers of the form q/p^n, where q, n are integers.

In general, for any subgroup G of the rationals, by using a similar construction, we can construct a region in S^3 whose fundamental group is G (cf. [**4**, p. 209] or [**2**]).

THEOREM 4. *If G is the fundamental group of a region Q of S^3, then no abelian subgroup of G has rank greater than two.*

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Proof. We first select a family of open sets $\{Q_n\}$ in S^3 with the following properties:

(1) each Q_n is connected;

(2) $S^3 - Q_n$ is the disjoint union of a finite number of handlebodies with holes, semilinearly imbedded in S^3;

(3) $Q_n \subset Q_m$ if $n < m$;

(4) $\bigcup_{n=1}^{\infty} Q_n = Q$.

If $Q = S^3$, then we take $Q_n = Q$ for each n. In general, $Q \neq S^3$, and we can assume that Q lies in R^3. We divide R^3 by using a brick subdivision T_n of mesh converging to zero, where T_m is a refinement of T_n if $n < m$. Let U_n consist of the complement of those bricks which intersect $R^3 - Q$ or are at a distance $> n$ from a chosen point e in Q, i.e., U_n is the interior of the union of those bricks that lie in Q and are at a distance $\leqq n$ from e. We may assume that $e \in U_1$. Take Q_n to be the component of U_n that contains e; it is easy to see that $\{Q_n\}$ has the properties (1), (2), (3), and (4).

Now consider any three elements a, b, c in G. We want to show that a, b, c are linearly dependent. Use e as the base point for Q and for each Q_r. Since G is the direct limit of $\{\pi(Q_n)\}$, we can find an N such that each of a, b, c has a representative in $\pi(Q_N)$ and that these commute in $\pi(Q_N)$.

Case 1. *$S^3 - Q_N$ is not geometrically splittable.* Papakyriakopoulos' paper [**8**] shows that $\pi_i(Q_N) = 0$ for $i \geqq 2$ (Theorem 26.1). Hence a, b, c, are linearly dependent. (See Conner [**2**, Theorem 2, and replace $S^3 - K$ by Q_N].)

Case 2. *$S^3 - Q_N$ is geometrically splittable.* Since $S^3 - Q_N$ is the union of a finite number of manifolds, we can proceed as follows:

Let $S^3 - Q_N$ be the union of k manifolds and B a 2 sphere in Q_N, semilinearly imbedded in S^3, that separates S^3 into B_1, B_2, with $B_i \cap (S^3 - Q_N) \neq \emptyset$ for each i. Now $B \subset Q_N$, so $\pi(Q_N) = \pi(B_1 \cap Q_N) * \pi(B_2 \cap Q_N)$. Since a, b, c commute with each other, they are represented by commuting elements of $B_i \cap Q_N$ for some i. We can assume that they are all contained in $B_1 \cap Q_N$.

If $B_1 \cap Q_N$ is not geometrically splittable in B_1, then we are done. Otherwise, since $B_1 - B_1 \cap Q_N = S^3 - (Q_N \cup B_2)$, is a union of K_1 manifolds with $K_1 < K$, we can continue our process until we have a region $Q_N \cap W_1$ which is not geometrically splittable in W_1, and such that a, b, c have representatives in it, that commute with each other. We thus reduce this case to Case 1, where we know that a, b, c must be linearly dependent in $\pi(Q_N \cap W_1)$, and hence also in $\pi(Q)$.

We can now state the main theorem of this section.

THEOREM 5. *An abelian group G is the fundamental group of a region in S^3 if and only if $G = 1, Z, Z + Z$, or is a subgroup of the rationals.*

Proof. In [**4**], B. Evan and L. Moser proved that if M is a 3-manifold and if $\pi(M)$ is a non finitely generated abelian group, then $\pi(M)$ is a subgroup of the rationals (Theorem 8.3). Therefore, from Theorem 4 and the statement in front of Theorem 4, we know that an abelian group G is the fundamental group of a region in S^3 if and only if $G = 1, Z, Z + Z$, or a subgroup of the rationals.

3. Which abelian groups are fundamental groups of regions of S^4? Our last theorem gives us a characterization of the groups which are abelian and are the fundamental groups of open regions in S^3: They are torsion free, are of rank $\leqq 2$ when finitely generated and otherwise have rank 1. It is known, however, that there is a region in S^4 whose fundamental group is Z_2. The main reason why the proof of the former theorem cannot carry through is that in S^4 it is no longer true that the complement of a connected surface is aspherical.

We shall now show that given any finitely generated abelian group G, we can find a region A, in S^4 that has G as its fundamental group. We first show that given any $m < \infty$, we can find a region in S^4 that has Z_m as its fundamental group.

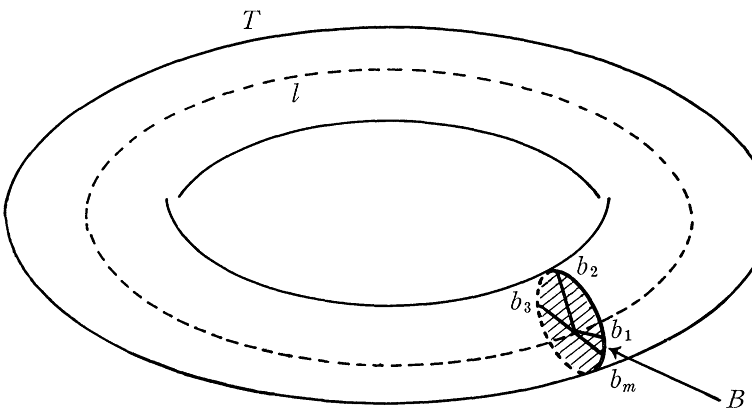
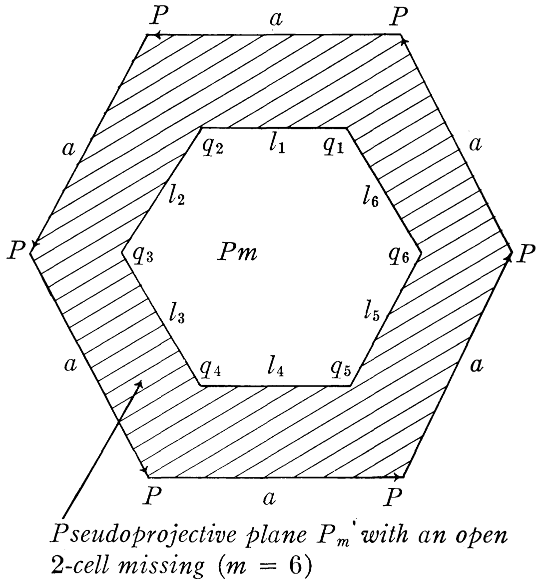


FIGURE 5

Let V be a solid torus of trivial type, T the boundary of V, and l the central line of V. Let P_m be the pseudoprojective plane whose fundamental group is Z_m. We shall construct an imbedding from a neighborhood T_m of the edge a in P_m to R^3 as follows: map a onto l, q_1 to b_1, and the edge q_1q_2 onto a curve on T which is described by a point q that twists through the angle $2\pi/m$ as it goes once around T (so q_2 maps to b_2). We map q_2q_3 and, in general, q_kq_{k+1}, in the same way, so that $q_1q_2 \ldots q_mq_1$ is mapped onto a torus knot of type $(m, 1)$; now we fill in, between this knot and l, a strip in the canonical way.

Now in R^4, let T be imbedded as described above, in the hyperplane $x_4 = 0$, where (x_1, x_2, x_3, x_4) is the coordinate of a point of R^4. We can easily extend this imbedding of T_m in R^4 to an imbedding of P_m in R^4, for instance, by shrinking the loop $q_1 \ldots q_mq_1$ to a point as x_4 increases. This imbedding is not locally



Pseudoprojective plane P_m with an open 2-cell missing ($m = 6$)

FIGURE 6

flat, but it is semilinear. Now a subcomplex of a manifold is a strong deformation retract of some neighborhood N, and N is a region in R^4 that has Z_m as its fundamental group. Of course N is also a region in S^4.

We now want to show that given any $m, n < \infty$, we can find a region in R^4 (hence also in S^4) that has $Z_m + Z_n$ as its fundamental group. (When one or both of m, n is ∞, it is still true; however, we omit the details.)

Let us first imbed P_m and P_n in R^4 as before. We make the central lines of these two solid tori coincide with two disjoint unlinked circles c_1 and c_2. Let W be a complex in R^4 that for $x_4 \geqq 0$ is this imbedded subcomplex $P_m \cup P_n$, and for $-1 \leqq t < 0$, has $c_1 \cup c_2$ as its cross section with the hyperplane $x_4 = t$ (thus W is not connected). Let U be a region of R^4 whose intersection with the hyperplane $x_4 = t$ is empty when $t > -1$, and for $t \leqq -1$, has as its complement the surface described by the hyperplane cross sections given in Figure 8.

We first find a neighborhood N of W which has W as a strong deformation retract. N is not connected, but $N \cup U$ is connected. By using van Kampen's

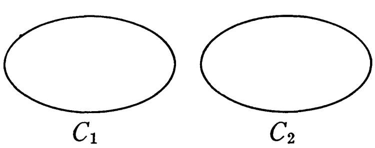


FIGURE 7

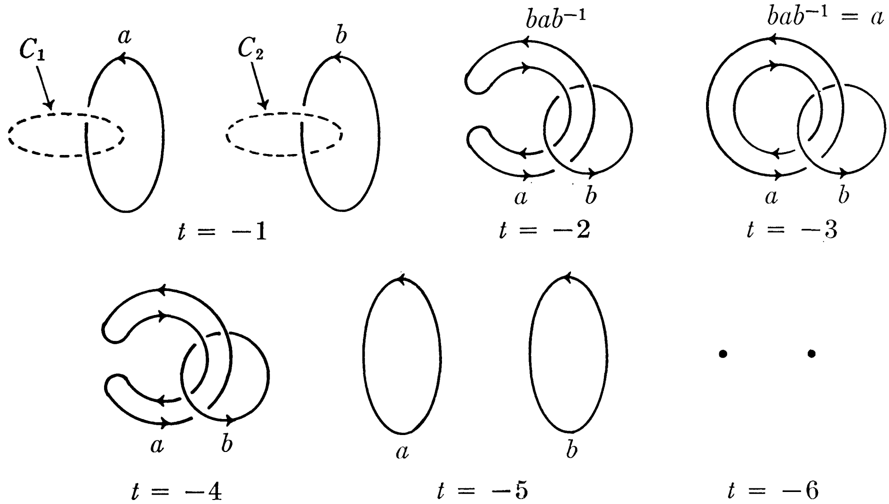


FIGURE 8

theorem, it is easy to see that $N \cup U$ has $Z_m + Z_n$ as its fundamental group (see Figure 8).

It will be seen, by this method, that given any sequences S of positive integers $(P_1 \ldots P_n \ldots)$ (S may eventually be an infinite sequence and P_n may be the symbol ∞. We will interpret Z_∞ as Z.), we can construct a region in R^4 (hence in S^4) that has $Z_{p_1} + Z_{p_2} + \ldots + Z_{p_n} + \ldots$ as its fundamental group. In particular, if G is any finitely generated abelian group, we can, since G is of the form $Z_{p_1} + Z_{p_2} + \ldots + Z_{p_N}$, construct a region in S^4 that has G as its fundamental group. The following construction shows how to construct a region in S^4 whose fundamental group is $Z_l + Z_m + Z_n$, where l, m, n are any three given positive integers: we imbed P_l, P_m and P_n in R^4 as before, and make the central lines of these three solid tori coincide with three unlinked circles c_1, c_2 and c_3.

Let W be a complex in R^4 that for $x_4 \geqq 0$ is just this imbedded subcomplex $P_1 \cup P_m \cup P_n$, and for $-1 < t < 0$, has $c_1 \cup c_2 \cup c_3$ as its cross section with the hyperplane $x_4 = t$. Let U be a region of R^4 whose intersection with the hyperplane $x_4 = t$ is empty when $t > -1$ and, for $t \leqq -1$, has as its complement the surface described by the hyperplane cross sections given in Figure 9.

We now find a neighborhood N of W which has W as a strong deformation retract, and $N \cup U$ has $Z_1 + Z_m + Z_n$ as its fundamental group.

Remarks. 1. If a region Q of S^n is the complement of a compact $(n-2)$-dimensional simplicial manifold M in S^n, and if $\pi(Q)$ is abelian, then

$$\pi(Q) = Z + \ldots + Z + Z_2 + \ldots + Z_2$$

where $p, q < \infty$ (p Z's and q Z_2's).

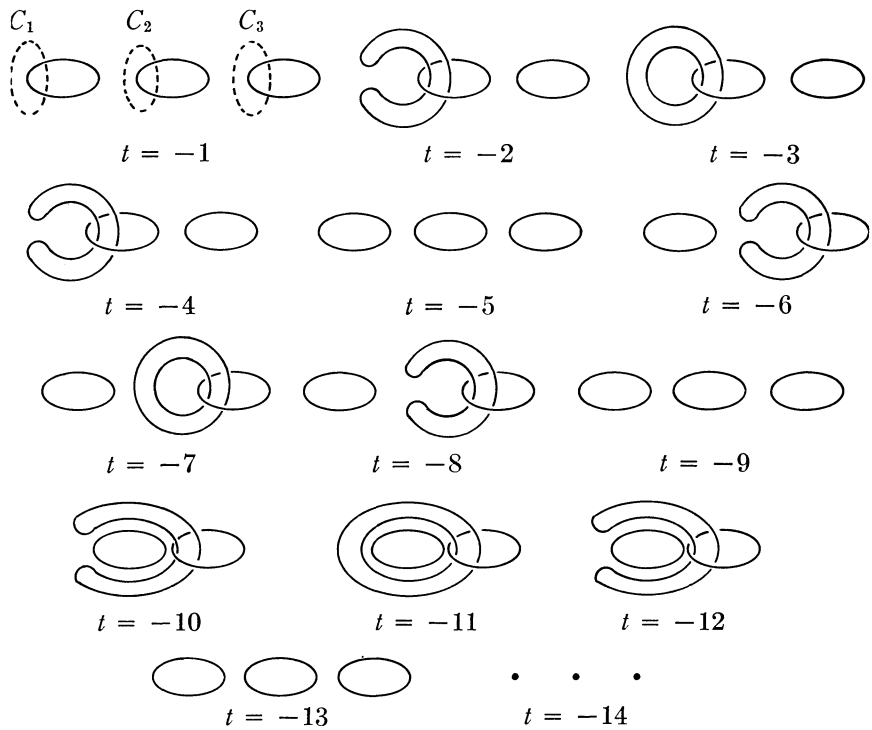


FIGURE 9

2. Let G be any finitely presented group. Since any finitely presented group can be represented as the fundamental group of some 2-dimensional CW complex, and since every 2-dimensional CW complex can be imbedded semilinearly in S^n, for $n \geqq 5$, and hence is a strong deformation retract of some region of S^n, we can always find a region in S^n that has G as its fundamental group.

Now if A is a region in S^n, we can, as we did in the proof of Theorem 4, construct a sequence of connected finite simplicial complexes Q_n such that $\bigcup_{n=1}^{\infty} Q_n = A$. Thus $G = \pi(A)$ is the direct limit of the sequence of finitely generated groups $\{G_n = \pi(Q_n)\}$ (This shows that G is always countable, so in particular, the additive group of real numbers cannot be the fundamental group of a region in S^n.); and if G is abelian, since each G_n is finitely generated, we can replace the sequence $\{G_n\}$ by a sequence $\{K_n\}$ of finitely generated abelian groups. We shall now prove the converse of the above statement, that is: Given any direct sequence of finitely generated abelian groups $\{K_n\}$, we can construct a region A in S^4 whose fundamental group is the direct limit of this sequence $\{K_n\}$.

We mentioned before that given any finitely generated abelian group

$$G = Z_{p_1} + Z_{p_2} + \ldots + Z_{p_N},$$

we can construct a region A in S^4 that has G as its fundamental group. We can even assume that A lies in R^4 in the region given $-1 \leqq x_4 < 1$, and that the cross section with the hyperplane $x_4 = t$ for $-1 \leqq t \leqq 0$ is the complement of the following figure:

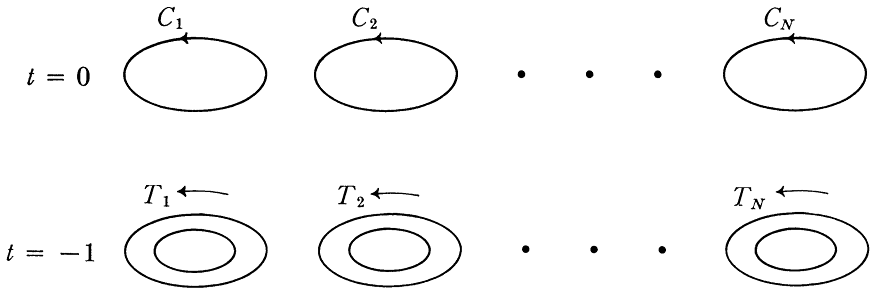


FIGURE 10

where the circles c_i thicken to become the solid torus T_i as t decreases, and where the meridian of T_i is a representative of the generator a_i of Z_{p_i}.

Now, given another group $H = Z_{r_1} + Z_{r_2} + \ldots + Z_{r_M}$ and a homomorphism $f : G \to H$, let b_i denote a generator of Z_{r_i}. Then f is determined by

$$f(a_i) = \sum_{j=1}^{M} \alpha_{ij} b_j.$$

We first construct a region in R^4, given by the restriction $-3 < x_4 < -1$, whose fundamental group is equal to H, and whose cross section with the hyperplane $x_4 = t$ for $-3 < t < -2$ is the complement of the following figure:

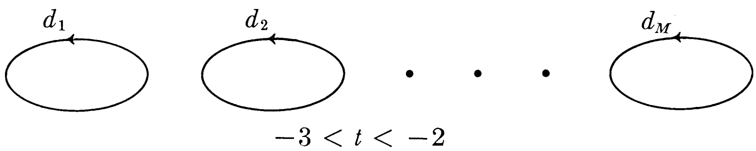


FIGURE 11

and the projection of these circles on the hyperplane $x_4 = -1$ is disjoint from all the T_i's.

Now we imbed M circles in T_i in the following way: we imbed the jth circle so that it goes α_{ij} times in T_i in the trivial way. (When $\alpha_{ij} = 0$, we do not imbed the jth circle; when $\alpha_{ij} < 0$, we imbed it in the opposite direction of the torus.)

Now we put this set of circles in the hyperplane $x_4 = t$ for each $-2 < t < -1$, and as the hyperplane moves from $x_4 = -2$ to $x_4 = -3$, we let the jth circle in each T_i gradually combine with d_j one after the other.

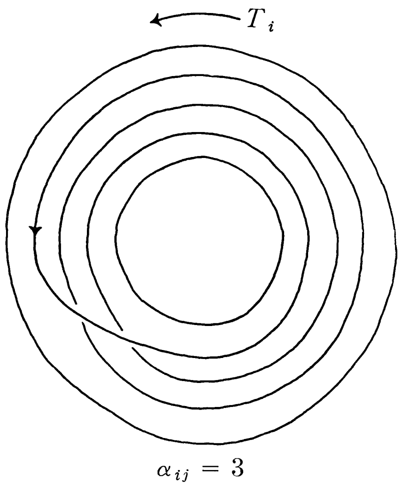


FIGURE 12

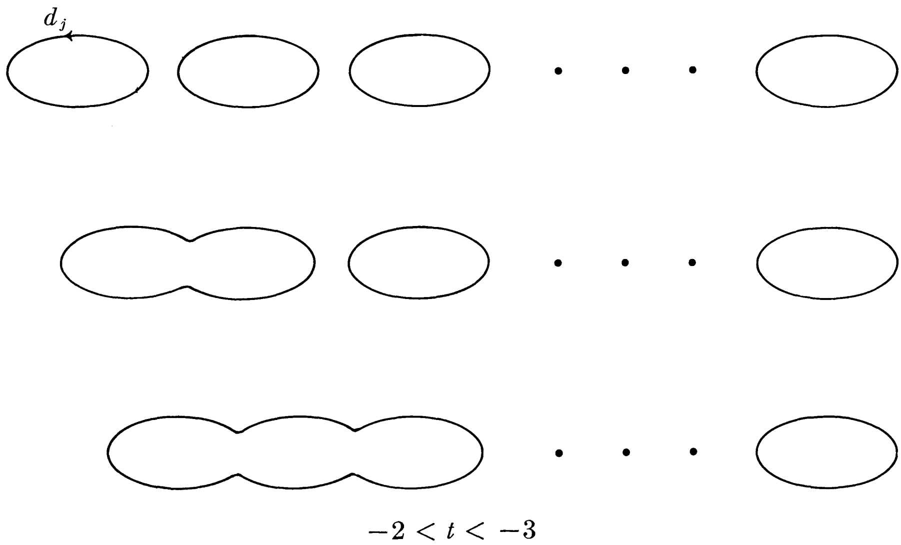


FIGURE 13

It is easy to see now that the region between $-4 < x_4 < -1$ has its fundamental group equal to the direct limit of the direct system $\{G, H, f\}$, and the cross section of the region near $x_4 = -4$ is as shown in Figure 11.

By this method, we can, for any direct sequence of finitely generated abelian groups $\{G_n\}$ construct a region in R^4 that has the direct limit of $\{G_n\}$ as its fundamental group.

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It is trivial to see that an abelian group G is countable if and only if it is the direct limit of a sequence of finitely generated abelian groups. Our construction above and the last remark therefore prove the following theorem:

THEOREM 6. *An abelian group G is the fundamental group of a region in S^4 if and only if it is countable.*

Remarks. 1. This theorem actually says that an abelian group G is the fundamental group of a region in S^n for $n \geqq 4$ if and only if it is countable, since the "only if" part of the theorem holds for any n.

2. It is now known (cf. [**1**] or [**7**]) that for the group G of disjoint 2-spheres in S^4 ($G = \pi(S^4 - S_1^2 \cup S_2^2 \cup \ldots \cup S_n^2)$), G/G_2 is isomorphic to F/F_2 if $n > 1$, where F is the free group of rank n. Therefore, G cannot be abelian.

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