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SOME FIXED POINT THEOREMS FOR PARTIALLY ORDERED SETS

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1. Introduction. A partially ordered set P has the *fixed point property* if every order-preserving map $f : P \to P$ has a *fixed point*, i.e. there exists $x \in P$ such that $f(x) = x$. A. Tarski's classical result (see [**4**]), that every complete lattice has the fixed point property, is based on the following two properties of a complete lattice P:

(A) For every order-preserving map $f : P \to P$ there exists $x \in P$ such that $x \leqq f(x)$.

(B) Suprema of subsets of P exist; in particular, the supremum of the set $\{x | x \leqq f(x)\} \subset P$ exists.

Of course, arbitrary posets do not have these properties, and we shall work with the following weakened versions, which seem to be more appropriate for arbitrary posets.

(C) For every order-preserving map $f : P \to P$ there exists $x \in P$ such that x and $f(x)$ are comparable, i.e. either $x \leqq f(x)$ or $f(x) \leqq x$.

(D) Every non-empty chain of P has a supremum and an infimum.

THEOREM 1. *Suppose the partially ordered set P satisfies* (C) *and* (D). *Then P has the fixed point property.*

Proof. Let $f : P \to P$ be order-preserving, and let $a \in P$ be comparable to $f(a)$. Since property (D) is self-dual, we may assume that $a \leqq f(a)$. The system of chains C for which

(*) $\quad x \in C \quad$ implies $\quad f(x) \in C$ and $x \leqq f(x)$

contains the non-empty chain $\{f^k(a) | k = 0, 1, 2, \ldots\}$, and therefore contains a maximal chain M by Zorn's Lemma. By assumption, $m = \sup(M) \in P$ exists. Since M satisfies (*), we have $x \leqq f(x) \leqq f(m)$, for all $x \in M$, so that $m \leqq f(m)$. On the other hand, if $m \notin M$, then the chain $M \cup \{f^k(m) | k = 0, 1, 2, \ldots\}$ properly contains M, and satisfies (*) in contradiction to the maximality of M. Therefore, $m \in M$ and also $f(m) \in M$, hence $f(m) \leqq m$. This makes m a fixed point of f.

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m is even a maximal fixed point of f: Suppose $z = f(z)$ is another fixed point, and suppose $m < z$. Then $M \cup \{z\}$ belongs to the system of chains satisfying (*). Again, this contradicts the maximality of M.

In [**1**] and [**5**], one can find fixed point theorems that are similar to Theorem 1 (e.g. Theorem 2 in [**1**] and Theorem 1 in [**5**]). We can obtain a somewhat stronger theorem if condition (D) is weakened even further: not every non-empty chain, but only chains contained in certain intervals of the poset need to have suprema and infima. For our purposes, condition (D) as stated above is sufficiently general and convenient.

COROLLARY. *Suppose P has property* (D). *Then Property* (C) *and the fixed point property are equivalent.*

One consequence of the Corollary is that every finite poset with a least or a largest element has the fixed point property. More generally, a poset with a least (largest) element that satisfies the ascending (descending) chain condition has the fixed point property.

2. A sufficient condition.

THEOREM 2. *Suppose that the poset P has finitely many minimal elements, that each element of P contains a minimal element, and that the supremum of every non-empty subset of minimal elements exists. Then P has Property* (C).

Proof. Let $f : P \to P$ be order-preserving and let M be the set of all minimal elements. We define a sequence of subsets of M by:

$$M_1 = M$$
$$M_{k+1} = \{x \in M | x \leqq f(\sup(M_k))\}$$

Each set M_k is non-empty, because $f(\sup(M_{k-1}))$ contains at least one minimal element by hypothesis. We show by induction, that the system $S = \{M_k | k = 1, 2, \ldots\}$ is a chain. $M_2 \subset M_1 \neq \emptyset$ is obviously true. Let now $x \in M_{k+1}$; since $y \leqq f(\sup(M_k))$, for every $y \in M_{k+1}$, we get

$$(**) \quad x \leqq \sup(M_{k+1}) \leqq f(\sup(M_k)).$$

The induction hypothesis $M_k \subset M_{k-1}$ implies that $f(\sup(M_k)) \leqq f(\sup(M_{k-1}))$, so that $x \leqq f(\sup(M_{k-1}))$, i.e. $x \in M_k$.

P has only finitely many minimal elements, therefore the descending chain

$$M = M_1 \supset M_2 \supset \ldots\ldots \supset M_k \supset \ldots.$$

terminates. Let $M_n = M_{n+1}$; then $\sup(M_n) = \sup(M_{n+1}) \leqq f(\sup(M_n))$, by (**). Thus, property (C) is verified.

Obviously, the dual of Theorem 2—replacing minimal by maximal and supremum by infimum—is also true.

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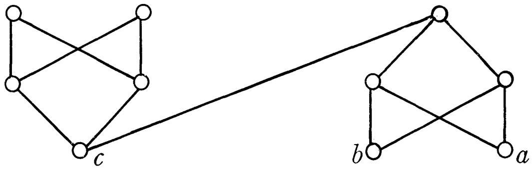
COROLLARY. *Suppose P fulfills the hypotheses of Theorem* 2 (*or of its dual*) *and has Property* (D). *Then P has the fixed point property.*

Specializing to a single extremal element, we see that, for instance, every finite semi-lattice (join or meet) has the fixed point property; or specializing the Property (D), every poset satisfying the hypotheses of Theorem 2 and the ascending chain condition has the fixed point property.

We conclude this section with an example that the conditions of Theorem 2 are not necessary. Consider the following poset of height two:



All suprema of the minimal elements a, b, and c, except for $\sup(a, b)$, exist. Nonetheless, the poset has the fixed point property.

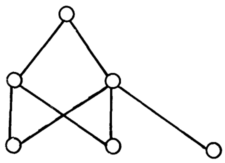
3. An equivalence condition. Let T be a poset, considered as an index set. For a family P_t, $t \in T$, of posets, we define the *lexicographic sum* $P = L\{P_t | t \in T\}$ to be the set $\{(t, x) | t \in T, x \in P_t\}$ with the *lexicographic order*:

$$(s, x) \leqq (t, y) \Leftrightarrow s < t, \text{ or } s = t \text{ and } x \leqq y.$$

The inclusion mappings $i_t : P_t \to P$ defined by $i_t(x) = (t, x)$ are order-embeddings, so that we may identify the posets P_t—we will call them *pieces* of P—with their images $i_t(P_t)$ in P.

Every poset P admits a representation as a lexicographic sum; i.e. there exists an index set T and pieces P_t, $t \in T$, such that $P = L\{P_t | t \in T\}$. For one such representation one may choose $T = P$ and $|P_t| = 1$, for all $t \in T$; for another, one may take $|T| = 1$ and $P_t = P$. These two *trivial* representations always exist; there may, of course, be many more.

The poset



admits, for example, the non-trivial representation $P = L\{P_t | t \in T\}$, where the

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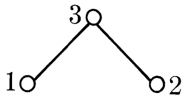
index set

$T =$ 

where the piece P_1 consists of two incomparable elements, P_2 has a single element, and P_3 is a copy of T.

THEOREM 3. *For any poset P, the following two statements are equivalent:*
(1) *P has Property* (C).
(2) *P admits a representation as a lexicographic sum, $P = L\{P_t | t \in T\}$, where the index set T as well as all the pieces P_t have Property* (C).

Proof. The implication (1) $\Rightarrow$ (2) is obvious, since every posets admits the two trivial representations. For the other implication, let $f : P \to P$ be order-preserving. Define a map $\alpha : P \to T$ by $\alpha(t, x) = t$ and let ϕ be a choice function on the family $(P_t)_{t \in T}$. Then the map $\beta : T \to T$ defined by $\beta(t) = \alpha(f(t, \phi(t)))$ is obviously monotone and hence by assumption there exists an element $r \in T$ such that $r < \beta(r)$ or $r > \beta(r)$ or $r = \beta(r)$. In the first case we have $(r, \phi(r)) < f(r, \phi(r))$ and in the second: $(r, \phi(r)) > f(r, \phi(r))$. Hence we may assume that $\beta(r) = r$, i.e. that there exist elements $x, y \in P_r$ with $f(r, x) = (r, y)$. Consider the non-empty subset $Q = \{(r, x) | f(r, x) \in i_r(P_r)\}$ of $i_r(P_r)$. If $Q = i_r(P_r)$, then $f|Q : Q \to Q$ is an order-preserving map on the piece P_r. Since P_r has Property (C) by assumption, we obtain Property (C) for P.

Let from now on $Q \neq i_r(P_r)$.

Case 1: There is $(r, x) \in Q$ and $(r, y) \notin Q$—but still $y \in P_r$—such that x and y are comparable in P_r. If $x < y$, then $f(r, x) \leqq f(r, y)$ and, since $f(r, y) \notin i_r(P_r)$, $f(r, y) = (t, z)$ where $r < t$. Thus, $(r, y) < f(r, y)$, and we have verified Property (C) for P. A similar argument will take care of the case $y < x$.

Case 2: For every $(r, x) \in Q$ and $(r, y) \notin Q$, x and y are incomparable in P_r. Pick any two elements $a, b \in P_r$ such that $(r, a) \notin Q$ and $(r, b) \in Q$. We define a map $g : P_r \to P_r$ by

$$g(z) = \begin{cases} a & \text{if } (r, z) \in Q \\ b & \text{if } (r, z) \notin Q \end{cases}.$$

Our assumption for Case 2 implies that g is order-preserving. Furthermore, there is no element in P_r which is comparable to its image under g. This contradicts the hypothesis that P_r has Property (C). Therefore, this last case cannot occur.

COROLLARY. *If the poset P satisfies Property* (D) *then the following conditions are equivalent:*

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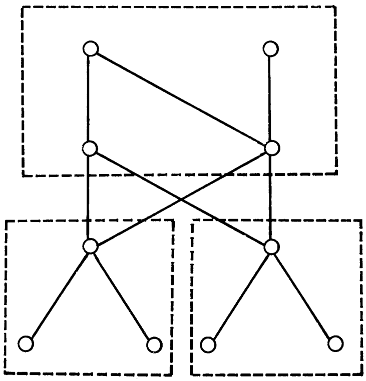
(1) *P has the fixed point property.*

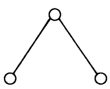
(2) *P admits a lexicographic representation $L\{P_t | t \in T\}$ in which the index set T and each piece P_t has the fixed point property.*

Proof. The implication (1) $\Rightarrow$ (2) is evident again because of the trivial lexicographic representations. The converse follows from Theorem 1 and the implication (2) $\rightarrow$ (1) of Theorem 3.

It should be noted here that if a poset has the fixed point property, it may admit only a trivial representation satisfying (2) of the Corollary. This is the case, for instance, for all posets of length one that have the fixed point property: In any lexicographic representation of P, the pieces must be of length at most one. If the index set T has the fixed point property, then either $|T| = 1$ or T is connected and of length one. If $|T| = 1$, we get a trivial representation. In the other case, if $T \not\cong P$, then $|P_t| \neq 1$, for at least one index. But P_t has to be of length zero, i.e. P_t contains at least 2 incomparable elements, contradicting the fixed point property for P_t. Thus $T \cong P$, and the representation again is trivial. For posets of length one having the fixed point property, Theorem 3 is therefore useless. But these posets—at least the finite ones—have already been characterized in [**3**].

As an example that Theorem 3 and its Corollary can be quite useful, consider the poset P



As the pieces of a lexicographic representation of P we choose the boxes; the index set is then $T =$  . All components evidently have the fixed point property, therefore P has the fixed point property.

In the proof of Theorem 3, we used the fixed point property for the pieces of the lexicographic sum for one single piece only. This suggests that the properties of the index set are crucial for the existence of fixed points.

A subset Q of a poset P is a *retract* of P if there is an onto, order-preserving map $\pi : P \rightarrow Q$ such that $\pi \circ \text{id}_Q = \text{id}_Q$.

THEOREM 4. *If the poset P has the fixed point property and if $Q \subset P$ is a retract of P, then Q has the fixed point property.*

Proof. Let $f : Q \to Q$ be order-preserving and let $\pi : P \to Q$ be the retraction map. Then $f \circ \pi : P \to P$ has a fixed point by assumption, indeed a fixed point in Q. Since $f \circ \pi \circ \mathrm{id}_Q = f$, f also has a fixed point.

A poset P is *fixed point free* if it does not have the fixed point property, i.e. there is an order-preserving map $f : P \to P$ without a fixed point.

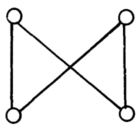
THEOREM 5. *Let P be a lexicographic sum, $P = L\{P_t | t \in T\}$. If T is fixed point free, then so is P.*

Proof. We show that T is order-isomorphic to a retract of P. Then Theorem 4 establishes the assertion. Let χ be a choice function on the set system $\{P_t | t \in T\}$, i.e. $\chi(P_t) \in P_t$, for each $t \in T$; and let $Q = \{\chi(P_t) | t \in T\}$ with the partial order inherited from P, i.e. $\chi(P_s) \leqq \chi(P_t)$ if and only if $(s, \chi(P_s)) \leqq (t, \chi(P_t))$. Then Q and T are order-isomorphic. The map $\pi : P \to Q$ defined by $\pi(t, x) = \chi(P_t)$, for all $t \in T$ and all $x \in P_t$ is the required retraction map.

Let us point out here, that the fixed point property for the index set T alone is not sufficient to force the fixed point property for the whole lexicographic sum.

For example, suppose that T is the two-element chain 

. Let P_1 as well as P_2 consist of two incomparable elements. Their lexicographic sum is the fixed point free poset



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