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SETS OF RANGE UNIQUENESS IN p-ADIC FIELDS

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Abstract We study sets of range uniqueness (SRUs) for analytic functions inside a disc of an algebraically closed field K complete with respect to an ultrametric absolute value. The SRUs we obtain are converging sequences. We first obtain results that look like those known in $\mathbb{C}$ but involve a weaker hypothesis than in $\mathbb{C}$: let (a_n) be a sequence of limit a in a disc $d(a, r^-)$ such that $|a_n - a|$ is a strictly decreasing sequence. If the sequence (a_n) does not make an SRU for the set $\mathcal{A}(d(a, r^-))$ of analytic functions inside $d(a, r^-)$, then, for a certain integer $k \in \mathbb{Z}$, the sequence

$$\left(\frac{a_{n+k} - a}{a_n - a}\right)$$

has a finite limit in K and the sequence

$$\left(\frac{\log|a_{n+k} - a|}{\log|a_n - a|}\right)$$

has a finite rational limit. Next, we show that if the sequence

$$\frac{\log(a_{n+1} - a)}{\log(a_n - a)}$$

converges to a limit $b \geqslant 1$ in such a way that $-b\log|a_n - a| < -b\log|a_{n+1} - a|$ and if $\log|a_n - a| - b\log|a_{n+1} - a|$ has limit 0 or $+\infty$ and if $b^k \notin \mathbb{Q}$ whenever $b > 1$ and $k \in \mathbb{N}^*$, then the sequence (a_n) is an SRU for $\mathcal{A}(d(a, r^-))$. In particular, for every $\gamma \in]0, 1[\cup]1, +\infty[$, $L \in \mathbb{Q} \cap]0, +\infty[$ and $b \geqslant 1$, there exist SRUs for $\mathcal{A}(d(a, r^-))$ of the form $\{a_n \mid n \in \mathbb{N}\}$ such that

$$\lim_{n\to+\infty} \frac{-\log|a_n - a|}{b^n n^\gamma} = L.$$

For example, if $\gamma \in \mathbb{N}$ with $\gamma \neq 0, 1$, there exist SRUs of the form $\{a_n \mid n \in \mathbb{N}\}$ such that $-\log|a_n - a| = Ln^\gamma$ for all $n \in \mathbb{N}^*$. The latter result ceases to hold when $\gamma = 1$. Many examples and counterexamples are provided.

Keywords: sets of range uniqueness (SRUs); uniqueness; p-adic functions

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1. Introduction and results

The concept of *sets of range uniqueness* (SRUs) was introduced by Diamond *et al*. [**3**] for complex analytic functions. It is a generalization of the identity theorem. Several other papers on this topic have appeared over the last 20 years [**1**, **5**, **7**].

Definition 1.1. Consider a family of functions $\mathcal{F}$ defined in a set D. A subset S of D is called *a set of range uniqueness* for $\mathcal{F}$ if, given any two functions $f, g \in \mathcal{F}$ such that $f(S) = g(S)$, we have $f = g$.

In this paper, we will examine the problem in an ultrametric field and we will essentially state some sufficient conditions for a bounded subset to be an SRU or not to be an SRU. We will also give some examples. (Characterization of the SRUs seems to be a very difficult problem.) The proofs that are not very short are given in the second part of the paper.

Notation. We shall denote by F an algebraically closed field of characteristic 0 and by K an algebraically closed field complete for a non-trivial ultrametric absolute value denoted by $|\cdot|$. For all sets S in F or in K, we put $S^* = S \setminus \{0\}$.

We shall denote by 'log' a real logarithm function of base $p > 1$ and by v the valuation function of K defined as $x \mapsto v(x) = -\log|x|$. We put $v(K) = \{v(x) \mid x \in K^*\}$.

Given $r > 0$, we denote by $d(a, r^-)$ the disc $\{x \in K \mid |x - a| < r\}$ and by $K[x]$ the K-algebra of polynomials in one variable, with coefficients in K. We denote by $\mathcal{A}(K)$ (respectively, $\mathcal{A}(d(a, r^-))$) the ring of entire functions in K (respectively, analytic functions in $d(a, r^-)$, i.e. power series converging in $d(a, r^-)$ [**4**]).

Remark 1.2. A subset A of K is an SRU for $\mathcal{A}(K)$ if and only if, for every non-constant affine application σ, the subset $\sigma(A)$ is an SRU for $\mathcal{A}(K)$.

Remark 1.3. A subset S of $d(a, r^-)$ is an SRU for $\mathcal{A}(d(a, r^-))$ if and only if, for every bianalytic bijection Φ from $d(a, r^-)$ onto $d(a, r^-)$, the subset $\Phi(S)$ is an SRU for $\mathcal{A}(d(a, r^-))$.

Example 1.4. The set of zeros S of a function $f \in \mathcal{A}(d(a, r^-))$ is not an SRU for $\mathcal{A}(d(a, r^-))$ because $f(S) = \lambda f(S)$. For example, let $(a_n)_{n\in\mathbb{N}}$ be a sequence in K satisfying $\lim_{n\to\infty} |a_n| = \infty$. The set $S = \{a_n \mid n \in \mathbb{N}\}$ is not an SRU for $\mathcal{A}(K)$ because there exists

$$f(x) = \prod_{n=0}^{\infty} \left(1 - \frac{x}{a_n}\right)$$

satisfying $f(a_n) = 0$ for all $n \in \mathbb{N}$ [**6**].

Example 1.5. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence inside a disc $d(a, r^-)$ satisfying

$$\lim_{n\to\infty} |a_n - a| = r.$$

According to [**6**] there exists $f \in \mathcal{A}(d(a, r^-))$ such that $f(a_n) = 0$ for all $n \in \mathbb{N}$. Hence, the set $S = \{a_n \mid n \in \mathbb{N}\}$ is not an SRU for $\mathcal{A}(d(a, r^-))$.

Remark 1.6. Given a family of functions $\mathcal{F}$ such that $K\mathcal{F} \subset \mathcal{F}$ or $\mathcal{F}\mathcal{F} \subset \mathcal{F}$, if a set S is included in the set of zeros of a function $f \in \mathcal{F}$, it is not an SRU for $\mathcal{F}$. As a consequence, if $K[x] \subset \mathcal{F}$, an SRU for $\mathcal{F}$ is always infinite.

Remark 1.7. In the same way, given a set $S \subset K$ and a K-algebra of functions $\mathcal{F}$, if there exists $f \in \mathcal{F}$ such that $f(S)$ is a finite set, then S is not an SRU for $\mathcal{F}$ because there exists a polynomial P (whose zeros are the points of $f(S)$) such that $P \circ f(S) = \{0\}$.

We observe that this property is shown in [**3**].

Proposition 1.8. *Let a subset S of $d(a,r^-)$ be an SRU for $\mathcal{A}(d(a,r^-))$ and let $b \in d(a,r^-)$. Then the subsets $S \cup \{b\}$ and $S \setminus \{b\}$ are also SRUs for $\mathcal{A}(d(a,r^-))$.*

Remark 1.9. Adding or removing a finite number of points to or from a set does not change the property that this set is an SRU or a non-SRU.

Remark 1.10. On the contrary, adding or removing infinitely many points can deteriorate the property of range uniqueness (see Examples 1.18 and 1.19 below).

Remark 1.11. A set S that is preserved by an affine mapping ϕ is not an SRU for polynomials (and therefore for any family of function containing polynomials) because any polynomial P satisfies $P(S) = P \circ \phi(S)$. For instance, if $\mathbb{Z}$ is included in K, it is not an SRU for polynomials.

Example 1.12. Let A be a subset of K and let σ be a non-constant affine application different from the identity. For an integer $n \geqslant 1$ we put $\sigma^{[n]} = \sigma \circ \cdots \circ \sigma$ (n times). If $n < 0$, we put $\sigma^{[n]} = \sigma^{-1} \circ \cdots \circ \sigma^{-1}$ ($-n$ times) and $\sigma^{[0]} =$ identity. Then it is easy to see that $A_\sigma = \bigcup_{n\in\mathbb{Z}} \sigma^{[n]}(A)$ is not an SRU for $K[x]$.

In particular, let A be a subset of K, let $n \in \mathbb{N}$ and let $\zeta \in K$, $\zeta \neq 1$, be such that $\zeta^n = 1$. Then the set $A_\zeta = \bigcup_{i=0}^{n-1} \zeta^i A$ is not an SRU for $K[x]$.

Proposition 1.13. *Let p be a prime integer, consider that $\mathbb{Q}$ is a subfield of F and let $S \subset \mathbb{Q}$ be a set included in a disc $d(a,r^-)$ in $\mathbb{C}_p$ that is an SRU for the $\mathbb{C}_p$-algebra $\mathcal{A}(d(a,r^-))$. Then S is an SRU for $F[x]$.*

Proof. Let $f, g \in F[x]$ satisfy $f(S) = g(S)$ and let E be a finite extension of $\mathbb{Q}$ containing all coefficients of f and g. There exists a $\mathbb{Q}$-isomorphism from E into $\mathbb{C}_p$; hence, f and g belong to $\mathbb{C}_p[x]$, and therefore $f = g$. □

Proposition 1.13 will be applied in Examples 1.19, 1.25 and 1.29. Now, Proposition 1.14 lets us obtain a bounded sequence that is not an SRU for polynomials, and therefore not an SRU for every class of functions containing them.

Proposition 1.14. *Let $q \in \mathbb{N}$, $q \geqslant 3$. Then the subset $S = \{\zeta \in F \setminus \{1\} \mid \exists j \in \mathbb{N}^*,\ \zeta^{q^j} = 1\}$ is not an SRU for $F[X]$.*

Remark 1.15. In particular, Proposition 1.14 applies to $\mathbb{C}[x]$.

Following the same kind of method as in [**3**], but using specific ultrametric properties of analytic functions, we can obtain the following theorem, which looks like [**3**, Theorem 3], but is a little more general.

Theorem 1.16. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of limit a in the disc $d(a,r^-)$ satisfying $|a_{n+1} - a| < |a_n - a|$ for all $n \in \mathbb{N}$ and suppose that the set $\{a_n \mid n \geqslant 0\}$ is not an SRU for $\mathcal{A}(d(a,r^-))$. There then exist $k \in \mathbb{Z}^*$ and $d \in \mathbb{N}^*$ such that the sequence*

$$\left(\frac{a_{n+k} - a}{a_n - a}\right)^d$$

has a limit in K and the sequence

$$\left(\frac{\log|a_{n+k}-a|}{\log|a_n-a|}\right)$$

converges to a limit in $\mathbb{Q}$.

Corollary 1.17. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of limit a in $d(a,r^-)$ satisfying $|a_{n+1}-a|<|a_n-a|$ for all $n\in\mathbb{N}$, such that the sequence*

$$\left|\frac{a_{n+k}-a}{a_n-a}\right|$$

has no limit, for any fixed $k\in\mathbb{N}^*$. Then $\{a_n \mid n\in\mathbb{N}\}$ is an SRU for $\mathcal{A}(d(a,r^-))$.

Example 1.18. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}_p$ such that

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{1}{p}$$

when n is not of the form p^s and

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{1}{p^2}$$

when n is of the form p^s.

Let k be fixed in $\mathbb{N}^*$ and let $n>k+2$. As $p^s>k+1$ and $p\geqslant 2$, we have

$$\left|\frac{a_{n+k}}{a_n}\right|=\left|\frac{a_{n+1}}{a_n}\right|\left|\frac{a_{n+2}}{a_{n+1}}\right|\cdots\left|\frac{a_{n+k}}{a_{n+k-1}}\right|.$$

First let $n=p^s+1$. For every $j=0,\dots,k-1$ we have $p^s+1\leqslant n+j<p^{s+1}$. Indeed, as $p^s>k+1$ and $p\geqslant 2$, we can check that

$$n+j<n+k=p^s+1+k<p^s+p^s=2p^s\leqslant p^{s+1}.$$

Hence,

$$\left|\frac{a_{n+j+1}}{a_{n+j}}\right|=\frac{1}{p}$$

for each $j=0,\dots,k-1$ and, consequently,

$$\left|\frac{a_{n+k}}{a_n}\right|=\frac{1}{p^k}.$$

Now, let $n=p^s$. We see that $n+1=p^s+1$ and then

$$\left|\frac{a_{n+2}}{a_{n+1}}\right|\left|\frac{a_{n+3}}{a_{n+2}}\right|\cdots\left|\frac{a_{n+k}}{a_{n+k-1}}\right|=\frac{1}{p^{k-1}}.$$

Since

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{1}{p^2},$$

we have

$$\left|\frac{a_{n+k}}{a_n}\right| = \frac{1}{p^{k+1}}.$$

Thus, the sequence $|(a_{n+k})/a_n|$ has no limit. Hence, by Theorem 1.16, the set $S = \{a_n \mid n \geqslant 0\}$ is an SRU for $\mathcal{A}(d(0, r^-))$ with $r > |a_0|$.

In particular, let r be > 1 and

$$S = \{p^n \mid n \in \mathbb{N} \setminus (p\mathbb{N}^*)\} = \{1, p, p^2, \dots, p^{2p-1}, p^{2p+1}, p^{2p+2}, \dots\}.$$

Then S is an SRU for $\mathcal{A}(d(0, r^-))$.

Now, owing to Proposition 1.13, we obtain the following example.

Example 1.19. For every prime integer p, the set $S = \{p^n \mid n \in \mathbb{N} \setminus (p\mathbb{N}^*)\} = \{1, p, p^2, \dots, p^{2p-1}, p^{2p+1}, p^{2p+2}, \dots\}$ is an SRU for $F[x]$. Now, considering S as a subset of $\mathbb{C}$, we observe that it is an SRU for $\mathbb{C}[x]$.

Remark 1.20. It is natural to ask whether an SRU for polynomials is also an SRU for analytic functions either in $\mathbb{C}$ or in a p-adic field. The set S of Example 1.19 shows that it is not an SRU for the algebra of complex entire functions $\mathcal{A}(\mathbb{C})$ because there do exist non-zero $f \in \mathcal{A}(\mathbb{C})$ satisfying $f(S) = \{0\}$.

Also, given a prime number p, consider the set

$$T_p = \left\{\frac{1}{p^{(n!)}},\ n \in \mathbb{N}^*\right\}.$$

By [**3**, Theorem 3] we can check that T_p is an SRU for the $\mathbb{C}$-algebra of analytic functions in a neighbourhood of zero and, therefore, that it is an SRU for $\mathbb{C}_p[x]$. But, in the field $\mathbb{C}_p$, we have

$$\lim_{n\to\infty}\left|\frac{1}{p^{n!}}\right| = +\infty.$$

Hence, there exist non-zero functions $f \in \mathcal{A}(\mathbb{C}_p)$ such that $f(T_p) = \{0\}$ and therefore T_p is not an SRU for $\mathcal{A}(\mathbb{C}_p)$.

Corollary 1.21. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of limit a in $d(a, r^-)$ satisfying $|a_{n+1} - a| < |a_n - a|$ for all $n \in \mathbb{N}$ such that the sequence*

$$\frac{\log|a_{n+1} - a|}{\log|a_n - a|}$$

admits a limit l that is transcendental over $\mathbb{Q}$. Then the set $\{a_n \mid n \in \mathbb{N}\}$ is an SRU for $\mathcal{A}(d(a, r^-))$.

Proof. For all $k \in \mathbb{N}^*$,

$$\lim_{n\to\infty}\frac{\log|a_{n+k} - a|}{\log|a_n - a|} = l^k.$$

Since $l^k \notin \mathbb{Q}$, by Theorem 1.16, $\{a_n \mid n \geqslant 0\}$ is an SRU for $\mathcal{A}(d(a, r^-))$. □

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Example 1.22. Let $(u_n)_{n\in\mathbb{N}}$ be the sequence of decimal approximations of $1/\pi$. After choosing $a_0 \in \mathbb{C}_p$, with $|a_0| < 1$, we can define a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}_p$ such that $v(a_{n+1}) = u_n v(a_n)$. Therefore, all terms a_n lie in the disc $d(0, 1^-)$ of $\mathbb{C}_p$ and satisfy $\log|a_{n+1}|/\log|a_n| = u_n$. Hence,

$$\lim_{n\to\infty} \frac{\log|a_{n+1}|}{\log|a_n|} = \frac{1}{\pi},$$

and therefore $\{a_n \mid n \geqslant 0\}$ is an SRU.

Corollary 1.23. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of limit a in $d(a, r^-)$ satisfying $|a_{n+1}-a| < |a_n - a|$ for all $n \in \mathbb{N}$, such that the sequence $\log|a_{n+1} - a|/\log|a_n - a|$ is unbounded. Then $\{a_n \mid n \geqslant 0\}$ is an SRU for $\mathcal{A}(d(a, r^-))$.*

Example 1.24. Let $(a_n)_{n\geqslant 0}$ be a sequence of $d(0, r^-)$ such that, for all n, $|a_{n+1}| < |a_n|$ and $\lim_{n\to+\infty} a_n = 0$. Suppose that $(\lambda_n)_{n\geqslant 0}$ is a sequence of $\mathbb{R}$ such that $\lim_{n\to+\infty} \lambda_n = +\infty$ and, for all n, $|a_{n+1}| < |a_n|^{\lambda_n}$. Then the subset $S = \{a_n \mid n \geqslant 0\}$ of $d(0, r^-)$ is an SRU for $\mathcal{A}(d(0, r^-))$.

In particular, when $K = \mathbb{C}_p$, for every $q \in \mathbb{N} \setminus \{0; 1\}$, the set $S_q = \{p^{p^{n^q}} \mid n \geqslant 0\}$ is an SRU for any K-algebra $\mathcal{A}(d(0, r^-))$.

Example 1.25. Let p be a prime integer and let q be an integer greater than or equal to 2. Then the set $S_q = \{p^{p^{n^q}} \mid n \geqslant 0\}$ is an SRU for $F[x]$.

Example 1.26. Let (a_n) be a sequence of limit a in a disc $d(a, r^-)$ with $r < 1$, such that $|a_{n+1} - a| = |a_n - a|^2$ whenever n is of the form q^d, with $q, d \in \mathbb{N}^*$ and $|a_{n+1} - a| = |a_n - a|^3$, otherwise. Then the set $S = \{a_n \mid n \in \mathbb{N}\}$ is an SRU for $\mathcal{A}(d(a, r^-))$.

Indeed, suppose S is not an SRU for $\mathcal{A}(d(a, r^-))$. There exists $k \in \mathbb{Z}^*$ such that the sequence $(\log|a_{n+k} - a|/\log|a_n - a|)$ has a rational limit l. Let $d \in \mathbb{N}$ be such that $q^d > |k|$. Suppose first that $k > 0$. Let n be of the form q^m with $m > d$. Since all integers $n+1, \dots, n+k$ lie in $]q^m, q^{m+1}[$, we have $\log|a_{n+k} - a|/\log|a_n - a| = 2.3^{k-1}$ and, hence, $l = 2.3^{k-1}$. Now, however, we check that all integers $n+1, \dots, n+k+1$ also lie in $]q^m, q^{m+1}[$; hence,

$$\frac{\log|a_{n+k+1} - a|}{\log|a_{n+1} - a|} = 3^k,$$

and $l = 3^k$, which is a contradiction. A similar proof applies when $k < 0$.

Theorem 1.16 suggests that a converging sequence (a_n) of limit a which is an SRU for analytic functions inside a disc $d(a, r^-)$ should be such that $(\log|a_{n+k} - a|/\log|a_n - a|)$ admits no rational limit. Actually, this sufficient condition is far from necessary, as is shown in Theorem 1.27. Recall that the values group of K is a $\mathbb{Q}$-vector space.

Theorem 1.27. *Let $b \in [1, +\infty[$ and let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers satisfying the following conditions.*

(i) *$\lambda_n \in v(K)$ for all $n \in \mathbb{N}$.*

(ii) *$\lim_{n\to+\infty}(\lambda_n) = +\infty$.*

(iii) *There exists an integer $m \in \mathbb{N}$ such that $\lambda_{n+1} > b\lambda_n$ for all $n > m$.*

(iv) $\lim_{n\to+\infty}(\lambda_{n+1} - b\lambda_n) = \Omega$, *where $\Omega = 0$ or $\Omega = +\infty$. Moreover, if $\Omega = +\infty$, then either $b = 1$ or $b^k \notin \mathbb{Q}$ for all $k \in \mathbb{N}^*$.*

(v) $\lim\limits_{n\to+\infty}(\lambda_{n+1}/\lambda_n) = b$.

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of $d(a, r^-)$ such that $\log|a_n - a| = -\lambda_n$ for every $n > m$. Then the subset $S = \{a_n \mid n \in \mathbb{N}\}$ is an SRU for $\mathcal{A}(d(a, r^-))$.

Corollary 1.28. *Let $L \in v(K)$ be such that $L > 0$ and let $(a_n)_{n\geqslant 0}$ be a sequence of $d(a, r^-)$ such that, for all n, $\log|a_n - a| = -Ln^\gamma$, with γ an integer greater than or equal to 2. Then the subset $S = \{a_n \mid n \geqslant 0\}$ of $d(a, r^-)$ is an SRU for $\mathcal{A}(d(a, r^-))$.*

Example 1.29. Let $r > 0$ and $c \in d(0, r^-)$ such that $|c| < 1$. For all integers $q \geqslant 2$, the sets $S_q(c) = \{c^{n^q} \mid n \geqslant 1\}$ are SRUs for $\mathcal{A}(d(0, r^-))$. In particular, if $K = \mathbb{C}_p$, we can take $c = p$ (with $r > 1/p$).

Example 1.30. As an application of Proposition 1.13 and Corollary 1.28, assuming that the characteristic of the field F is zero, we can see that in F, for every prime integer p and for every integer $q \geqslant 2$, the set $\{p^{n^q} \mid n \in \mathbb{N}\}$ is an SRU for $F[x]$.

In the same way, the same set in $\mathbb{C}_p$ is an SRU for any K-algebra $\mathcal{A}(d(0, r^-))$ with $r > 1/p$.

The following proposition shows how to construct a sequence $(\lambda_n)_{n\in\mathbb{N}}$ satisfying the hypotheses of Theorem 1.27.

Proposition 1.31. *Let $\gamma \in\,]0, 1[\, \cup\,]1, +\infty[$. Let L be > 0 and, furthermore, let $b \in [1, +\infty[$ satisfy $b^k \notin \mathbb{Q}$ for all $k \in \mathbb{N}^*$ whenever $\gamma > 1$ and $b > 1$. For every $n \in \mathbb{N}$, let*

$$\lambda_n \in v(K) \cap \left[b^n L n^\gamma,\ b^n L n^\gamma + \frac{1}{n+1}\right].$$

Then the sequence λ_n satisfies the hypotheses of Theorem 1.27.

Corollary 1.32. *Let $a \in K$, $r \in\,]0, +\infty[$, $\gamma \in\,]0, 1[\, \cup\,]1, +\infty[$ and $L \in \mathbb{R}_+^*$. Let $b \in [1, +\infty[$ satisfy $b^k \notin \mathbb{Q}$ for all $k \in \mathbb{N}^*$ whenever $b > 1$.*

There exist sequences $(a_n)_{n\in\mathbb{N}}$ in $d(a, r^-)$ such that $\{a_n \mid n \in \mathbb{N}\}$ is an SRU for $\mathcal{A}(d(a, r^-))$, satisfying

$$\lim_{n\to\infty} \frac{v(a_n - a)}{b^n n^\gamma} = L.$$

Remark 1.33. We note that, in the hypothesis of Corollary 1.32, for every fixed $k \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \frac{\log|a_{n+k} - a|}{\log|a_n - a|} = 1.$$

Remark 1.34. In Proposition 1.31, when the valuation group of K is equal to $\mathbb{R}$, we can take $\lambda_n = n^\gamma$ whenever $n \in \mathbb{N}$. But in the most usual case when the valuation group is isomorphic to $\mathbb{Q}$, as it is when $K = \mathbb{C}_p$, if γ is not an integer, we have to choose

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the λ_n different from n^γ, and, more precisely, such that we can find points a_n satisfying $v(a_n - a) = \lambda_n$. Thus, we can take for λ_n a suitable upper rational approximation of Ln^γ and then define a sequence (a_n).

Remark 1.35. In Proposition 1.31 and Corollaries 1.28 and 1.32, the hypothesis $\gamma \neq 1$ is necessary, as shown in the following example.

Example 1.36. Let $a \in K$. Then the set $S_1(a) = \{a^n \mid n \geqslant 0\}$ is not an SRU for $K[X]$. Indeed, if we consider the $f(x) = (1-x)(a-x)$ and $g(x) = f(ax) = a(1-x)(1-ax)$, we have $f(1) = f(a) = 0$, $f(a^n) = a(1-a^{n-1})(1-a^n)$, $n \geqslant 2$ and $g(1) = 0$, $g(a^n) = a(1-a^n)(1-a^{n+1})$, $n \geqslant 1$. Hence, $f(S_1(a)) = g(S_1(a)) = \{0, a(1-a^{n-1})(1-a^n) \mid n \geqslant 2\}$.

In particular, in the field $\mathbb{C}_p$, the subset $S_1(p) = \{p^n \mid n \geqslant 0\}$ is not an SRU for $\mathbb{C}_p[X]$.

Now, we can ask whether a closed open set might be an SRU. Without answering the question, we give some immediate remarks.

1.1. Definitions and notation

Given $A, B \subset K$, we denote by $\delta(A, B)$ the distance from A to B.

Let D be an infinite set in K and let $a \in D$. If D is bounded of diameter r, we denote by $\tilde{D}$ the disc $d(a, r) = \{x \in K \mid |x-a| \leqslant r\}$ and, if D is not bounded, we set $\tilde{D} = K$. It is known that $\tilde{D} \setminus \bar{D}$ admits a unique partition of the form $(d(a_i, r_i^-))_{i \in I}$, with $r_i = \delta(a_i, D)$ for each $i \in I$. The discs $d(a_i, r_i^-)$, for all $i \in I$ are called *the holes of* D [**4**].

Let D be a subset of K. We call the number $\delta(D, K \setminus D)$ the *codiameter* of D (denoted by $\operatorname{codiam}(D)$).

We can now describe a large class of sets D that are not SRUs for $K[X]$ and therefore are not SRUs for any larger class of functions.

Theorem 1.37. *Let D be a set such that* $\operatorname{codiam}(D) > 0$. *Then D is not an SRU for $K[X]$.*

Remark 1.38. If $\operatorname{codiam}(D) > 0$, then D is a closed and open set. The converse is not true: there exist closed open subsets of K with a codiameter equal to 0.

Corollary 1.39. *An affinoid set of K is not an SRU for $K[X]$. In particular, a disc or an annulus is not an SRU for $K[X]$.*

Among the questions which remain, we can consider the following.

(1) Having shown that a set D of codiameter greater than 0 cannot be an SRU for $K[X]$, is it possible to show this for other sets by considering the family $R(D)$ of rational functions $h \in K(x)$ with no pole in D (respectively, the family $H(D)$ of analytic elements on D [**4**])?

(2) All SRUs we have found are countable sets. This leads us to wonder whether an SRU might be uncountable.

(3) Might an SRU for $\mathcal{A}(K)$ included inside a disc $d(a, r^-)$ be a non-SRU for $\mathcal{A}(d(a, r^-))$?

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2. The proofs

Proof of Proposition 1.14. Consider an integer $\ell \geqslant 2$ prime to q. Let us show that the function $f_\ell : x \mapsto x^\ell$ is a permutation of S. Indeed, if x and y are two distinct elements of S such that $x^\ell = y^\ell$, then $x = \xi y$, with $\xi \neq 1$ and $\xi^\ell = 1$. But then there exist $i, j \in \mathbb{N}^*$ such that $x^{q^i} = y^{q^j} = 1$. Without loss of generality we can suppose that $j \leqslant i$. Then $x^{q^j} = y^{q^j} = 1$ and, thus, $\xi^{q^j} = 1$: a contradiction to the fact that $(q^j, \ell) = 1$ and $\xi^\ell = 1$. Thus, f_ℓ is injective. On the other hand, if ζ is an element of E, there exists $j \in \mathbb{N}^*$ such that $\zeta^{q^j} = 1$. Let $u, v \in \mathbb{Z}$ be such that $u\ell + vq^j = 1$. Then we have $\zeta^{1-vq^j} = (\zeta^u)^\ell$. If we set $\eta = \zeta^u$, we can easily check that $\eta \in E$ and $\zeta = f_\ell(\eta)$. Hence, f_ℓ is surjective and is therefore bijective. Thus, we see that, if ℓ and ℓ' are two distinct integers both prime to q, then f_ℓ and $f_{\ell'}$ are two distinct polynomials satisfying $f_\ell(S) = f_{\ell'}(S) = S$. This means that S is not an SRU for $F[x]$. $\square$

Notation. Let $f \in \mathcal{A}(d(a, r^-))$ be such that $f(x) = \sum_{n \geqslant 0} \alpha_n (x-a)^n$. For every $\rho \in]0, r[$ we set $|f|_a(\rho) = \sup_{n \geqslant 0} |\alpha_n| \rho^n$. In order to write this relation additively, we set $v_a(f, \mu) = \inf_{n \geqslant 0}(v(\alpha_n) + n\mu)$, where $\mu = -\log \rho$.

To learn more about the properties of the functions $\rho \mapsto |f|_a(\rho)$ and $\mu \mapsto v_a(f, \mu)$, see [**2**, **4**].

We shall need the following lemma, whose proof is based on the classical properties of analytic functions over ultrametric fields [**4**].

Lemma 2.1. *Let $f(x) = \sum_{m=d}^{\infty} \alpha_m x^m \in \mathcal{A}(d(0, r^-))$ with $\alpha_d \neq 0$ and let $(a_n)_{n \in \mathbb{N}}$ be a sequence in $d(0, r^-)$ such that $\lim_{n \to \infty} a_n = 0$. There then exists $q \in \mathbb{N}$ such that $|f(a_n)| = |\alpha_d|\,|a_n|^d$ for all $n \geqslant q$.*

The next lemma is the main tool to use when starting the proofs of Theorems 1.16 and 1.27.

Lemma 2.2. *Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in $d(0, r^-)$ of limit 0 and such that $|a_{n+1}| < |a_n|$ for all $n \in \mathbb{N}$. Let $f, g \in \mathcal{A}(d(0, r^-))$, $f \neq g$, satisfy $\{f(a_n)\} = \{g(a_n)\}$ and $f(0) = g(0)$. There then exist $k \in \mathbb{Z}^*$ and $q \in \mathbb{N}$ such that $f(a_n) = g(a_{n+k})$ for all $n \geqslant q$.*

Proof. For every $n \in \mathbb{N}$ we denote by $k(n) \in \mathbb{Z}$ an integer such that $f(a_n) = g(a_{n+k(n)})$. Let $f(x) = \sum_{m=c}^{\infty} \alpha_m x^m$ and let $g(x) = \sum_{m=d}^{\infty} \beta_m x^m$ with $\alpha_c \beta_d \neq 0$ and $c, d \in \mathbb{N}^*$. Without loss of generality we may obviously assume that $f(0) = g(0) = 0$. Consequently, we have $cd \neq 0$.

We first note that $\lim_{n \to \infty}(n + k(n)) = +\infty$. However, suppose this is not true. There then exist $A > 0$ and a strictly increasing sequence $(n_s)_{s \in \mathbb{N}}$ of $\mathbb{N}$ such that $n_s + k(n_s) < A$ for all $s \in \mathbb{N}$. Since the set of integers $n_s + k(n_s)$ such that $n_s + k(n_s) < A$ is finite, we see that $\{f(a_{n_s}),\ s \in \mathbb{N}\}$ is a finite set. Since $\lim_{s \to \infty} a_{n_s} = 0$ and since $f(0) = 0$, we see that $f(a_{n_s}) = 0$ has an infinity of solutions converging to zero: a contradiction to the properties of analytic functions stating that zeros are isolated. Consequently, we have $\lim_{n \to \infty} n + k(n) = +\infty$. Therefore, by Lemma 2.1, there exists $t \in \mathbb{N}$ such that $|f(a_n)| = |\alpha_c|\,|a_n|^c$ and $|g(a_n)| = |\beta_d|\,|a_n|^d$ for all $n \geqslant t$.

Consequently, since $c, d > 0$, the sequences $(|f(a_n)|)_{n\geqslant t}$ and $(|g(a_n)|)_{n\geqslant t}$ are strictly decreasing. Also, we can find $s \geqslant t$ with the following property: if $m, l \in \mathbb{N}$ are such that $m < t$ and $l \geqslant s$, then $f(a_m) \neq g(a_l)$. Moreover, since $\lim_{n\to\infty} n + k(n) = +\infty$, there exists $q \geqslant t$ such that $n + k(n) \geqslant s$ for all $n \geqslant q$.

Now, take $n \geqslant q$. We have $|g(a_{n+1+k(n+1)})| = |f(a_{n+1})| < |f(a_n)| = |g(a_{n+k(n)})|$. Since $n + k(n) \geqslant s \geqslant t$ and $n + 1 + k(n + 1) \geqslant s \geqslant t$, this implies that $|a_{n+1+k(n)}| < |a_{n+k(n)}|$, and hence $n + 1 + k(n + 1) > n + k(n)$.

On the other hand, by hypothesis there exists $j \in \mathbb{N}$ such that $f(a_j) = g(a_{n+1+k(n)})$. Taking into account the definition of s, this j must satisfy $j \geqslant t$ because $n+1+k(n) \geqslant s$. Now, since

$$|f(a_j)| = |g(a_{n+1+k(n)})| < |g(a_{n+k(n)})| = |f(a_n)|$$

and since $(|f(a_n)|)_{n\geqslant t}$ is strictly decreasing, we must have $j > n$. Hence, we obtain

$$|f(a_j)| \leqslant |f(a_{n+1})| = |g(a_{n+1+k(n+1)})| \leqslant |g(a_{n+1+k(n)})| = |f(a_j)|.$$

Thus, the above inequality is actually an equality. Consequently, $n + 1 + k(n + 1) = n + 1 + k(n)$, which proves that $k(n + 1) = k(n) = k \in \mathbb{Z}$ for every $n \geqslant q$. Obviously, $k \neq 0$ because otherwise we would have $f(a_n) = g(a_n)$ for all $n \geqslant q$ and then $f = g$. □

Proof of Theorem 1.16. Without loss of generality, we can obviously assume that $a = 0$. Suppose that $\{(a_n) \mid n \in \mathbb{N}\}$ is not an SRU for $\mathcal{A}(d(0, r^-))$ and let $f, g \in \mathcal{A}(d(0, r^-))$ satisfy $f \neq g$ and $\{f(a_n)\} = \{g(a_n)\}$. By extracting subsequences of $\{(a_n)\}$, we can see that $f(0) = g(0)$. Hence, we can also assume that $f(0) = g(0) = 0$.

Let

$$f(x) = \sum_{j=c}^{\infty} \alpha_j x^j, \qquad g(x) = \sum_{j=d}^{\infty} \beta_j x^j$$

with $\alpha_c\beta_d \neq 0$ and $c, d > 0$.

By Lemmas 2.1 and 2.2 there exist $q \in \mathbb{N}$ and $k \in \mathbb{Z}^*$ such that $|f(a_n)| = |\alpha_c|\,|a_n|^c$, $|g(a_n)| = |\beta_d|\,|a_n|^d$ and $f(a_n) = g(a_{n+k})$ for all $n \geqslant q$. Moreover, without loss of generality we can assume that $k > 0$ because f and g play the same role. Thus, we have

$$|\alpha_c|\,|a_n|^c = |\beta_d|\,|a_{n+k}|^d \quad \text{for all } n \geqslant q. \tag{2.1}$$

Consequently, by (2.1) we obtain

$$\frac{\log|a_{n+k}|}{\log|a_n|} = \frac{\log|\alpha_c| - \log|\beta_d|}{d\log|a_n|} + \frac{c}{d}$$

and, since $\lim_{n\to\infty} a_n = 0$, we see that

$$\lim_{n\to\infty} \frac{\log|a_{n+k}|}{\log|a_n|} = \frac{c}{d}.$$

Next, we can write $f(x) = x^c(\alpha_c + \varepsilon(x))$, $g(x) = x^d(\beta_d + \mu(x))$ with $\lim_{x\to 0} \varepsilon(x) = \lim_{x\to 0} \mu(x) = 0$. Take $n \geqslant q$. We have

$$f(a_n) = a_n^c(\alpha_c + \varepsilon(a_n)), \qquad g(a_{n+k}) = (a_{n+k})^d(\beta_d + \mu(a_{n+k})).$$

Since $f(a_n) = g(a_{n+k})$, we see that $a_n^c(\alpha_c + \varepsilon(a_n)) = (a_{n+k})^d(\beta_d + \mu(a_{n+k}))$. Therefore, we obtain

$$\left(\frac{a_{n+k}}{a_n}\right)^d = (a_n)^{c-d}\left(\frac{\alpha_c + \varepsilon(a_n)}{\beta_d + \mu(a_{n+k})}\right). \tag{2.2}$$

On the other hand, since $\lim_{n\to\infty} \varepsilon(a_n) = \lim_{n\to\infty} \mu(a_n) = \lim_{n\to\infty} a_n = 0$ and since $|a_{n+k}/a_n| < 1$, we have $c \geqslant d$. If $d < c$, then, by (2.2), $a_{n+k}/a_n \to 0$. And if $c = d$, then

$$\lim_{n\to\infty}\left(\frac{a_{n+k}}{a_n}\right)^d = \frac{\alpha_c}{\beta_d}.$$

This completes the proof of Theorem 1.16. □

Proof of Theorem 1.27. By hypothesis (iv) we first observe that

$$\lim_{n\to\infty}(\lambda_{n+k} - b^k\lambda_n) = \left(\sum_{j=0}^{k-1} b^j\right)\Omega. \tag{2.3}$$

Without loss of generality, we may assume that $a = 0$. Let $f, g \in \mathcal{A}(d(0, r^-))$ be two non-constant functions such that $f(S) = g(S)$. By property (ii), obviously $\lim_{n\to\infty} a_n = 0$, so we may also assume that $f(0) = g(0) = 0$. Since $\lambda_{n+1} > b\lambda_n$ for $n > m$, we may observe that the sequence $(|a_n|)_{n>m}$ is strictly decreasing. Let

$$f(x) = \sum_{j=c}^{\infty} \alpha_j x^j, \qquad g(x) = \sum_{j=d}^{\infty} \beta_j x^j, \qquad \alpha_c\beta_d \neq 0.$$

By Lemma 2.2, there exists an integer $q \geqslant m$ and there exists $k \in \mathbb{Z}$ such that $f(a_n) = g(a_{n+k})$ for all $n \geqslant q$. Since f and g play the same role, we may assume that $k \geqslant 0$ without loss of generality. We want to show that $k = 0$, and hence $f = g$.

There exists $s \in \,]0, r[$ such that f and g have no zero inside $d(0, s)$ except 0. Therefore, by Lemma 2.1, we have $|f(x)| = |\alpha_c|\,|x|^c$, $|g(x)| = |\beta_d|\,|x|^d$ for all $x \in d(0, s)$. Consequently, there exists $t \in \mathbb{N}$ such that $t \geqslant q + k$ and

$$|f(a_n)| = |\alpha_c|\,|a_n|^c, \qquad |g(a_n)| = |\beta_d|\,|a_n|^d \quad \text{for all } n > t.$$

Thus, we obtain $|\alpha_c|\,|a_n|^c = |\beta_d|\,|a_{n+k}|^d$, and hence $c\log|a_n| = d\log|a_{n+k}| - h$ for all $n > t$, with $h = \log|\alpha_c| - \log|\beta_d|$. Now, by hypothesis we have $\log|a_n| = -\lambda_n$ for all $n > m$. Hence,

$$c\lambda_n = d\lambda_{n+k} + h \quad \text{for all } n > t. \tag{2.4}$$

Suppose that $k > 0$. Assume first that $c \neq d$. By (v) we can check that each sequence $(u_{n,j})_{n\in\mathbb{N}}$, $j = 0, \dots, k-1$, defined as

$$u_{n,j} = \left(\frac{\lambda_{n+j}}{\lambda_n}\right),$$

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has limit b^j and, therefore, by (v) again, each sequence $(\theta_{n,j})_{n\in\mathbb{N}}$, $j=0,\dots,k-1$, defined as

$$\theta_{n,j} = b^{k-j-1}\left(\frac{\lambda_{n+j+1}-b\lambda_{n+j}}{\lambda_{n+j}}\right)\left(\frac{\lambda_{n+j}}{\lambda_n}\right), \quad j=0,\dots,k-1,$$

has limit 0. Consequently, we can check that

$$\lim_{n\to\infty}\left[\frac{\lambda_{n+k}-b^k\lambda_n}{\lambda_n}\right]=0$$

and therefore

$$\lim_{n\to\infty}\frac{c(\lambda_n-b^{-k}\lambda_{n+k})}{\lambda_{n+k}}=0. \tag{2.5}$$

We will show that

$$\lim_{n\to\infty}|c\lambda_n-d\lambda_{n+k}-h|=+\infty. \tag{2.6}$$

Let us write

$$|c\lambda_n-d\lambda_{n+k}-h|=|c(\lambda_n-b^{-k}\lambda_{n+k})+\lambda_{n+k}(cb^{-k}-d)-h|. \tag{2.7}$$

Suppose first that $cb^{-k}\neq d$. By (2.5) and (ii), we have

$$\lim_{n\to\infty}|c(\lambda_n-b^{-k}\lambda_{n+k})+\lambda_{n+k}(cb^{-k}-d)-h|=+\infty,$$

which shows that (2.6) holds.

Suppose now that $cb^{-k}=d$. By (2.7) we see that

$$|c\lambda_n-d\lambda_{n+k}-h|=|c(\lambda_n-b^{-k}\lambda_{n+k})-h|. \tag{2.8}$$

However, since we have supposed that $c\neq d$, we then have $b\neq 1$. Hence, by (iv) we have $\Omega=+\infty$. So, by (2.1), we see that relation (2.6) is clearly satisfied again. Thus, (2.6) is satisfied anyway: a contradiction to (2.4). Consequently, $c=d$.

Thus, by (2.4), we arrive at

$$c(\lambda_n-\lambda_{n+k})=h. \tag{2.9}$$

But $\lambda_{n+k}>b^k\lambda_n$, and hence $c(\lambda_{n+k}-\lambda_n)>c\lambda_n(b^k-1)$. If $b>1$, by (ii) we see that $\lim_{n\to\infty}\lambda_{n+k}-\lambda_n=+\infty$: a contradiction to (2.9). Consequently, we are now led to assume that $b=1$.

By (2.3) we have $\lim_{n\to\infty}[\lambda_{n+k}-\lambda_n]=k\Omega$. But since $\Omega=0$ or $+\infty$, and h is finite, by (2.8) we actually see that $\Omega=h=0$.

Consequently, $\lambda_n=\lambda_{n+k}$ for all $n>t$: a contradiction of (iii). Therefore, we have $k=0$ in every case, and hence $f=g$. This finishes the proof that $\{a_n \mid n\in\mathbb{N}\}$ is an SRU for $\mathcal{A}(d(a,r^-))$. □

Proof of Proposition 1.31. Without loss of generality, we may clearly assume that $L = 1$. First, (i) and (ii) are obviously satisfied. In order to check the three last conditions, we first observe the following inequalities:

$$\lambda_{n+1} - b\lambda_n \geqslant b^{n+1}((n+1)^\gamma - n^\gamma) - \frac{b}{n+1}, \tag{2.10 a}$$

$$\lambda_{n+1} - b\lambda_n \leqslant b^{n+1}((n+1)^\gamma - n^\gamma). \tag{2.10 b}$$

Suppose that $\gamma \in\ ]0,1[$. With the help of the finite increasing theorem, by (2.10), we obtain

$$\lambda_{n+1} - b\lambda_n \geqslant \gamma b^{n+1}(n+1)^{\gamma-1} - \frac{b}{n+1}, \tag{2.11 a}$$

$$\lambda_{n+1} - b\lambda_n \leqslant \gamma b^{n+1} n^{\gamma-1}, \tag{2.11 b}$$

respectively.

Thus, by (2.11 a), (iii) may easily be checked when n is sufficiently large. Moreover, if $b > 1$, then by (2.11 a) we can see that

$$\lim_{n\to\infty} b^{n+1}((n+1)^\gamma - n^\gamma) - \frac{b}{n+1} = +\infty,$$

so (iv) is satisfied. Conversely, if $b = 1$, by (2.11 b) we see that

$$\lim_{n\to\infty} ((n+1)^\gamma - n^\gamma) - \frac{1}{n+1} = 0,$$

which shows (iv) again.

Finally, for all $b \geqslant 1$, we have

$$\frac{\lambda_{n+1} - b\lambda_n}{\lambda_n} \leqslant \frac{b\gamma((n+1)^\gamma - n^\gamma)}{n^\gamma}.$$

Hence by (2.11 b) we obtain

$$\frac{\lambda_{n+1} - b\lambda_n}{\lambda_n} \leqslant \frac{b\gamma((n+1)^{\gamma-1})}{n^\gamma},$$

which shows (v).

Now suppose $\gamma > 1$. By (2.10) we have

$$\lambda_{n+1} - b\lambda_n \geqslant \gamma b^{n+1} n^{\gamma-1} - \frac{b}{n+1}, \tag{2.12 a}$$

$$\lambda_{n+1} - b\lambda_n \leqslant \gamma b^{n+1}(n+1)^{\gamma-1}. \tag{2.12 b}$$

By (2.12 a), (iii) is obviously satisfied and so is (iv) because $\lim_{n\to\infty} \lambda_{n+1} - b\lambda_n = +\infty$, whereas we have assumed that $b^k \notin \mathbb{Q}$ for all $k \in \mathbb{N}$ whenever $b > 1$. And by (2.12 b) we can also check that

$$\frac{\lambda_{n+1} - b\lambda_n}{\lambda_n} \leqslant \frac{b\gamma n^{\gamma-1}}{n^\gamma},$$

which shows (v) again. □

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Proof of Theorem 1.37. Let $r = \mathrm{codiam}(D)$ and $\lambda \in d(0, r^-) \setminus \{0\}$. Let $f(x) = x$ and $g(x) = x + \lambda$. Since $|g(x) - f(x)| < r$ for all $x \in D$, we check that $g(x)$ lies in D for every $x \in D$, and hence $g(D) \subset D = f(D)$. Conversely, given $x \in D$, we see that $x - \lambda$ lies in D and satisfies $g(x - \lambda) = x$. Hence, $g(D) = D$. □

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