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ON MODULI OF CONTINUITY FOR GAUSSIAN AND l^2-NORM SQUARED PROCESSES GENERATED BY ORNSTEIN-UHLENBECK PROCESSES

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1. Introduction. Let $\{Y(t), -\infty < t < \infty\} = \{X_k(t), -\infty < t < \infty\}_{k=1}^{\infty}$ be a sequence of independent Ornstein-Uhlenbeck processes with coefficients γ_k and λ_k, i.e., $X_k(\cdot)$ is a Gaussian process with $EX_k(t) = 0$ and

$$EX_k(s)X_k(t) = (\gamma_k/\lambda_k)\exp(-\lambda_k|t-s|), \quad (\gamma_k, \lambda_k > 0, k = 1, 2, \ldots).$$

The process $Y(\cdot)$ was first studied by Dawson (1972) as the stationary solution of the infinite array of stochastic differential equations

$$(1.1) \qquad dX_k(t) = -\lambda_k X_k(t)dt + (2\gamma_k)^{1/2}dW_k(t) \quad (k = 1, 2, \ldots),$$

where $\{W_k(t), -\infty < t < \infty\}$ are independent Wiener processes (cf. also **[6]**, **[19]**, and **[1]**). If we assume that the l^2-norm squared process

$$(2) \qquad \chi^2(t) = \|Y(t)\|^2 = \sum_{k=1}^{\infty} X_k^2(t)$$

has finite mean, i.e., if

$$(1.3) \qquad E\chi^2(t) = \sum_{k=1}^{\infty}(\gamma_k/\lambda_k) < \infty,$$

then $Y(t) \in l^2$ at fixed times. This does not guarantee that $Y(t) \in l^2$ for all t however (cf. **[16]**, for example). With $\gamma_k = 1$ $(k = 1, 2, \ldots)$ and assuming that for large j we have also $cj^{i+\delta} \leqq \lambda_j \leqq dj^{1+\delta}$ for some $c > 0$, $d > 0$ and $\delta > 0$, Dawson (1972) showed $Y(\cdot)$ in l^2 to be almost surely (a.s.) continuous. Since the coordinate Ornstein-Uhlenbeck processes $X_k(\cdot)$ are continuous, it follows from standard Hilbert space theory that to demonstrate l^2 continuity of $Y(\cdot)$ it is enough to show that the real valued process $\chi^2(\cdot) = \|Y(\cdot)\|^2$ is continuous. Iscoe and McDonald (1986), Schmuland (1988b) developed techniques for studying the latter process and showed that $\chi^2(\cdot)$, and hence also $Y(\cdot)$ in l^2, is continuous if, in addition to (1.3), we have also the condition

$$(1.4) \qquad \Gamma_2 = \sum_{k=1}^{\infty} \gamma_k^2/\lambda_k < \infty.$$

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This result is not sharp in that γ_k can be a lot larger and we will still have continuity. Iscoe, Marcus, McDonald, Talagrand and Zinn (1989) showed for example that if, in addition to (1.3), we have also

$$\max_{k\geqq 1}\gamma_k((\log\gamma_k)\vee 0)^r/(\lambda_k\vee 1)<\infty \quad \text{for some } r>1,$$

then $Y(\cdot)$ is a.s. l^2 continuous. In a somewhat more general context, Fernique (1989) gave a complete solutioin for the latter continuity problem. A special case of his Théorème reads as follows:

For each $x\in\mathbf{R}^+$, let

$$K(x)=\{k\in\mathbf{N}:\gamma_k>\lambda_k x\} \text{ and } \lambda(x)=\sup\{\lambda_k:k\in K(x)\}.$$

Then $Y(\cdot)\in l^2$ is a.s. *continuous if and only if we have* (1.3) *and*

$$\int((\log[\lambda(x)])\vee 0)dx<\infty$$

as well.

Consequently, (cf. Corollary 1 of [**7**]), for $Y(\cdot)$

$$\sum_{k=1}^{\infty}(\gamma_k/\lambda_k)(1+((\log\lambda_k)\vee 0))<\infty. \tag{1.5}$$

On the other hand, finiteness of Γ_2 of (1.4) gives more than just continuity of $Y(\cdot)$ in l^2. Using variations of the condition (1.4), Schmuland (1988a) established various orders of Hölder continuity for $Y(\cdot)$ in l^2 as well as for $\chi^2(\cdot)$.

Another real valued process which is also closely related to $Y(\cdot)\in l^2$ is the stationary mean zero Gaussian process $X(\cdot)$ defined by

$$\{X(t),-\infty<t<\infty\}=\left\{\sum_{k=1}^{\infty}X_k(t),-\infty<t<\infty\right\}, \tag{1.6}$$

where the $X_k(\cdot)$ are again the independent coordinate Ornstein-Uhlenbeck processes of $Y(\cdot)$. This process can of course be studied by well developed techniques for Gaussian processes. In particular $X(\cdot)$ is a.s. continuous if and only if it satisfies Fernique's necessary and sufficient condition for continuity of a stationary Gaussian process (cf. Corollary 2.5 of Section IV.2 in [**10**]), i.e., if and only if in this case

$$E|X(t)-X(s)|^2=\phi^2(|t-s|),$$

where $\phi(u)$ is an increasing function in $u>0$, we have that $\phi(u)/(u(\log(1/u))^{1/2})$ is integrable at zero. Using this condition one can also compare the processes

$Y(\cdot) \in l^2$ and $X(\cdot)$. For example, the condition (1.4) with $\gamma_k = 1$ ($k = 1, 2, \ldots$) reduced to (1.3), and hence it is sharp for the a.s. continuity of $Y(\cdot)$ in l^2. However, in this case Iscoe and McDonald (1986, Example 3 (due to D. A. Dawson)) show that with

$$\sum_{k=1}^{\infty} \lambda_k^{-1} < \infty$$

but

$$\sum_{k=1}^{\infty} \lambda_k^{-1} (\log \lambda_k)^{1/2} = \infty$$

(e.g. $\lambda_k = k(\log k)^{3/2}$)$Y(\cdot) \in l^2$ is a.s. continuous but $X(\cdot)$ does not satisfy the just mentioned Fernique condition. On the other hand we have

$$(1.7) \qquad E\|Y(t) - Y(s)\|^2 = E|X(t) - X(s)|^2,$$

and consequently, in general, checking Fernique's necessary and sufficient condition for the a.s. continuity of the real valued, stationary, mean zero Gaussian process $X(\cdot)$ should be also sufficient for that of the stationary, mean zero Gaussian process $Y(\cdot)$ in l^2. In particular, just like studying the process $\chi^2(\cdot)$ on its own, that of $X(\cdot)$ is also of interest. For example, when proposing mathematical models for neural response, one of the processes figuring in Walsh's work (1981) is $X(\cdot)$ of (1.6).

The main aim of this exposition is to establish exact moduli of continuity for the processes $X(\cdot)$ and $\chi^2(\cdot)$. There are many papers dealing with a.s. upper bounds for the moduli of continuity of various Gaussian processes (cf. for example [**2**], [**17**], [**11**], [**12**], [**14**], [**13**] and the references of these works). Our exact moduli of continuity results in (1.10) and (1.11) do not follow from those of the just mentioned papers. Rather, they are fashioned after the P. Lévy exact moduli of continuity for Brownian motion as presented and proved in [**4**, Theorem 1.1.1 and Remark 1.1.2]. The results of (1.12), (1.13) and (1.15), (1.16) constitute exact moduli of continuity for the non-Gaussian l^2-norm squared process $\chi^2(\cdot)$ under the condition (1.4). They also provide a contrast to those of (1.10) and (1.11) for $X(\cdot)$ under the condition

$$(1.8) \qquad \Gamma_1 = \sum_{k=1}^{\infty} \gamma_k < \infty.$$

We now state our moduli of continuity results for $X(\cdot)$ and $\chi^2(\cdot)$. First we note that we will assume throughout

$$(1.9) \qquad \Gamma_0 = E|X(t)|^2 = E\|Y(t)\|^2 = \sum_{k=1}^{\infty} (\gamma_k/\lambda_k) < \infty,$$

a condition which is shared by both of the processes $X(\cdot)$ and $\chi^2(\cdot)$ of our present interest, as well as by $Y(\cdot) \in l^2$ via $\chi^2(t) = \|Y(t)\|^2$.

The organization of this exposition is as follows. In this section we state and comment on our exact moduli of continuity results for the two processes $X(\cdot)$ and $\chi^2(\cdot)$. In Section 2 we state and prove our large deviation propositions which we need for proving in Section 3 the moduli of continuity results of Theorems 1 and 2, stated right below. The large deviation results of Section 2 may also be of interest on their own.

THEOREM 1. *Let $\Gamma_0 < \infty$, and assume that $T_h \uparrow \infty$ continuously as $h \longrightarrow 0$. Then, if $\Gamma_1 < \infty$, we have*

$$(1.10) \qquad \lim_{h\to 0} \sup_{|t|\leq T_h} \frac{|X(t+h) - X(t)|}{(2h\Gamma_1)^{1/2}(2\log(T_h/h))^{1/2}} = 1, \text{ a.s.,}$$

$$(1.11) \qquad \lim_{h\to 0} \sup_{|t|\leq T_h} \sup_{0\leq s\leq h} \frac{|X(t+s) - X(t)|}{(2h\Gamma_1)^{1/2}(2\log(T_h/h))^{1/2}} = 1, \text{ a.s.}$$

THEOREM 2. *Let $\Gamma_0 < \infty$, $M = \max_{j\geq 1}\gamma_j^2/\lambda_j$, and assume that $T_h \uparrow \infty$ continuously as $h \longrightarrow 0$. Then, if $\Gamma_2 < \infty$, we have*

$$(1.12) \qquad \limsup_{h\to 0} \sup_{|t|\leq T_h} \frac{|\chi^2(t+h) - \chi^2(t)|}{(8hM)^{1/2} 2\log(T_h/h)} \leqq 1 \text{ a.s.,}$$

$$(1.13) \qquad \limsup_{h\to 0} \sup_{|t|\leq T_h} \sup_{0\leq s\leq h} \frac{|\chi^2(t+s) - \chi^2(t)|}{(8hM)^{1/2} 2\log(T_h/h)} \leqq 1 \text{ a.s.}$$

If, in addition, the continuous nondecreasing function T_h satisfies also

$$(1.14) \qquad \log T_h/\log(1/h) \longrightarrow \infty \quad \textit{as } h \longrightarrow 0,$$

then, if $\Gamma_2 < \infty$, we have as well

$$(1.15) \qquad \lim_{h\to 0} \sup_{|t|\leq T_h} \frac{|\chi^2(t+h) - \chi^2(t)|}{(8hM)^{1/2} 2\log(T_h/h)} = 1 \text{ a.s.,}$$

$$(1.16) \qquad \lim_{h\to 0} \sup_{|t|\leq T_h} \sup_{0\leq s\leq h} \frac{|\chi^2(t+s) - \chi^2(t)|}{(8hM)^{1/2} 2\log(T_h/h)} = 1 \text{ a.s.}$$

Remark . Under condition (1.14) $\log(T_h/h)$ in (1.15) and (1.16) can be replaced by $\log T_h$. In this case we can, for example, take

$$T_k = \exp\{(\log 1/h) \log\log \cdots \log(1/h)\},$$

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where for small enough $h > 0$, $\log\log\cdots\log(1/h)$ stands for taking log any given finite number of times, resulting in the modulus $(8hM)^{1/2}2(\log(1/h))\log\log\cdots\log(1/h)$ for the $\chi(\cdot)$ process.

Taking $T_h = 1/h$ in (1.10) and (1.11), their norming function looks like that of the classical P. Lévy moduli of continuity for Brownian motion (cf., e.g., Theorem 1.1.1 and Remark 1.1.2 in [**4**]). Clearly, if $\lambda_k = 1$ $(k = 1, 2, \ldots)$ then the condition $\Gamma_1 < \infty$ is sharp for the a.s. continuity of the process $X(\cdot)$, as well as for (1.10) and (1.11), since then $\Gamma_0 = \Gamma_1$, i.e., the existence condition $\Gamma_0 < \infty$ for $X(t)$ to be a Gaussian random variable for each fixed t with mean zero and variance Γ_0, coincides with requiring the finiteness of Γ_1 for the continuity of $X(\cdot)$. If $\Gamma_0 < \infty$ and $\Gamma_0 \neq \Gamma_1$, then of course $\Gamma_1 < \infty$ is only a sufficient condition for the a.s. continuity of $X(\cdot)$, but then it is always sufficient also for the stronger than continuity statements of (1.10) and (1.11) as well. The coefficients λ_k of the Ornstein- Uhlenbeck processes $X_k(\cdot)$ of (1.1) measure the strength of their drift toward the origin, while their coefficients γ_k reflect their tendency to diffuse outward. Thus, in the light of (1.10) and (1.11), the condition that Γ_1 should be finite amounts to saying that if there is only a finite global "amount of tendency" in the system (1.1) to diffuse, then the infinite series of its solutions, namely the process $X(t)$, will behave like Brownian motion on $\mathbf{R}^1$, provided only that the latter is a Guassian random variable with variance $\Gamma_0 < \infty$ for each fixed t.

As to the results in (1.12), (1.13), (1.15) and (1.16), it is interesting to call attention to their much bigger norming function as compared to that of (1.10) and (1.11). Instead of $(2\log(T_h/h))^{1/2}$ in the latter, we now have the same function squared. Also, instead of the "expected" Γ_2 we have ended up with M. These changes can be easily understood, however, by having a look at the large deviation results (2.2) and (2.3) on which (1.12), (1.13), (1.15) and (1.16) are based. More interesting now is to note that if $\lambda_k = 1$ $(k = 1, 2, \ldots)$, then the condition $\Gamma_2 < \infty$ is not only sharp for the a.s. continuity of $Y(\cdot)$ in l^2 on account of $\Gamma_0 = \Gamma_2$, but then it gives also (1.12) and (1.13), as well as (1.15) and (1.16) when combined with (1.14). Also, if $\Gamma_0 < \infty$ and $\Gamma_0 \neq \Gamma_2$, then $\Gamma_2 < \infty$ is not only a sufficient condition for the a.s. continuity of $Y(\cdot) \in l^2$ and that of $\chi^2(\cdot)$, but it yields these exact moduli as well.

2. Large deviations.

LEMMA 2.1. *Assume $\Gamma_0 < \infty$. For any $\epsilon > 0$ there exist $h(\epsilon) > 0$ and $C_i = C_i(\epsilon) > 0$ $(i = 1, 2)$ such that for any $T > h(\epsilon)$, $h < h(\epsilon)$ we have*

$$(2.1) \qquad P\left\{\sup_{|t|\leqq T_h}\sup_{0\leqq s\leqq h}|X(t+s)-X(t)| \geqq v(2h\Gamma_1)^{1/2}\right\} \leqq (C_1T/h)\exp\left(-\frac{v^2}{2+\epsilon}\right)$$

for any $v > 0$, and if $\Gamma_2 < \infty$ as well, then we have also

$$(2.2) \qquad P\{|\chi^2(t+h)-\chi^2(t)| \geqq v(8hM)^{1/2}\} \geqq \frac{1}{7v}\exp\left(-\frac{v}{1-\epsilon}\right).$$

$$(2.3)\quad P\left\{\sup_{|t|\leqq T_h}\sup_{0\leqq s\leqq h}|\chi^2(t+s)-\chi^2(t)| \geqq v(8hM)^{1/2}\right\}\leqq (C_2T/h)\exp\left(-\frac{v}{1+\epsilon}\right)$$

for any $v \geqq (8/\epsilon^2)(\Gamma_2/M)^{1/2}$.

In addition to Lemma 2.1, the proof of Theorem will also utilize the next, well-known result.

LEMMA 2.2 ([**18**]). *Let $G(t)$ and $G^*(t)$ be Gaussian processes on $\mathbf{R}^+$, possessing continuous sample path functions, with $EG(t) = EG^*(t) = 0$, $EG^2(t) = EG^{*2}(t) = 1$, and let $\rho(s,t)$ and $\rho^*(s,t)$ be their respective covariance functions. Suppose that we have $\rho(s,t) \geqq \rho^*(s,t)$, $s,t \in \mathbf{R}^+$. Then*

$$(2.4)\quad P\left\{\sup_{0\leqq s\leqq h} G(t)\leqq u\right\}\geqq P\left\{\sup_{0\leqq s\leqq h} G^*(t)\leqq u\right\}.$$

Proof of Lemma 2.1. First we prove (2.1). Obviously, it is enough to consider the case of $\Gamma_1 < \infty$. Assuming $\Gamma_0 < \infty$, we have by definition

$$X(t+s)-X(t) \stackrel{\mathfrak{D}}{=} N\left(0, 2\sum_{j=1}^{\infty}(\gamma_j/\lambda_j)(1-\exp(-\lambda_j s))\right).$$

By using L'Hospital's rule and dominated convergence, we have

$$\sum_{j=1}^{\infty}(\gamma_j/\lambda_j)(1-\exp(-\lambda_j s))/(s\Gamma_1)\longrightarrow 1 \quad \text{as } s\longrightarrow 0.$$

Let $r = r(\epsilon)$ be a positive number to be specified later on. Putting $r_1 = h/2^r$ and $t_r = [t/r_1]r_1$, we have

$$\begin{aligned}|X(t+s)-X(t)| &\leqq |X((t+s)_r)-X(t_r)|\\ &= \sum_{j=0}^{\infty}|X((t+s)_{r+j+1})-X((t+s)_{r+j})|\\ &+\sum_{j=0}^{\infty}|X(t_{r+j+1})-X(t_{r+j})|,\end{aligned}$$

and, on choosing $r = r(\epsilon)$ large enough, we obtain

$$(2.5)\quad P\left\{\sup_{|t|\leqq T_h}\sup_{0\leqq s\leqq h}|X((t+s)_r)-X(t_r)|\geqq v(1-\epsilon/6)(2h\Gamma_1)^{1/2}\right\} \leqq \frac{4Th}{r_1^2}\exp\left(-\frac{v^2}{2+\epsilon}\right)\leqq\frac{CT}{h}\exp\left(-\frac{v^2}{2+\epsilon}\right),$$

$$(2.6)\qquad P\left\{\sup_{|t|\leqq T_h}\sup_{0\leqq s\leqq h}\sum_{j=1}^{\infty}|X((t+s)_{r+j+1})-X((t+s)_{r+j})|\right.$$
$$\left.\geqq\sum_{j=1}^{\infty}(2h\Gamma_1(v^2+6j)/2^{r+j+1})^{1/2}\right\}$$
$$\leqq\sum_{j=1}^{\infty}P\left\{\sup_{|t|\leqq T_h}\sup_{0\leqq s\leqq h}|X((t+s)_{r+j+1})-X((t+s)_{r+j})|\right.$$
$$\left.\geqq(2h\Gamma_1(v^2+6j)/2^{r+j+1})^{1/2}\right\}$$
$$\leqq\sum_{j=1}^{\infty}\frac{4T}{h}2^{2(r+j+1)}\exp\left(-\frac{v^2+6j}{2+\epsilon}\right)\leqq\frac{CT}{h}\exp\left(-\frac{v^2}{2+\epsilon}\right),$$

and, similarly,

$$(2.7)\qquad P\left\{\sup_{|t|\leqq T_h}\sup_{0\leqq s\leqq h}\sum_{j=1}^{\infty}|X(t_{r+j+1})-X(t_{r+j})|\right.$$
$$\left.\geqq\sum_{j=0}^{\infty}(2h\Gamma_1(v^2+6j)/2^{r+j+1})^{1/2}\right\}$$
$$\leqq\frac{CT}{h}\exp\left(-\frac{v^2}{2+\epsilon}\right).$$

We can assume without loss of generality that $v\geqq 1$. Then

$$\sum_{j=1}^{\infty}\left(\frac{v^2+6j}{2^{r+j+1}}\right)^{1/2}\leqq\frac{v}{2^{r/2}}\sum_{j=0}^{\infty}\frac{1}{2^{(j+1)/2}}+\frac{1}{2^{r/2}}\sum_{j=1}^{\infty}\left(\frac{3j}{2^j}\right)^{1/2}\leqq\frac{\epsilon}{12}v,$$

provided $r=r(\epsilon)$ is large enough.

Now the proof of (2.1) is completed by combining these inequalities.

Next we prove (2.2) and (2.3). Put

$$M_n=\max_{1\leqq j\leqq n}\gamma_j^2/\lambda_j,\quad \sigma_k^2=E(X_k(t+h)+X_k(t))^2\quad\text{and}$$
$$\sigma'2_k=E(X_k(t+h)-X_k(t))^2.$$

Then

$$(2.8)\qquad E(X_k^2(t+h)-X_k^2(t))^2=\sigma_k^2\sigma_k'^2=4(\gamma_k/\lambda_k)^2(1-\exp(-2\lambda_k h)).$$

At first we let

$$p_n(v)=P\left\{\left|\sum_{j=1}^{n}(X_j^2(t+h)-X_j^2(t))\right|\geqq v(8hM_n)^{1/2}\right\}$$

and prove that for large n we have

$$\frac{1}{7v}\exp\left(-\frac{v}{1-\epsilon}\right) \leqq p_n(v) \leqq 2\exp\left(-\frac{v}{1+\epsilon}\right), \tag{2.9}$$

provided $v \geqq (8/\epsilon^2)(\Gamma_2/M_n)^{1/2}$.

Let k_0 be an integer such that $\gamma_{k_0}^2/\lambda_{k_0} = M_n$. We put

$$Y = \sum_{j=1}^{n}(X_j^2(t+h) - X_j^2(t)) - (X_{k_0}^2(t+h) - X_{k_0}^2(t))$$

and note that Y is independent of $X_{k_0}^2(t+h) - X_{k_0}^2(t)$. Since

$$\sum_{j=1}^{n}(X_j^2(t+h) - X_j^2(t)) = \sum_{j=1}^{n}(X_j(t+h) + X_j(t))(X_j(t+h) - X_j(t))$$

is symmetric, we have

$$\begin{aligned} p_n(v) &= 2P\left\{\sum_{j=1}^{n}(X_j^2(t+h) - X_j^2(t)) \geqq v(8hM_n)^{1/2}\right\} \\ &\geqq 2P\{X_{k_0}^2(t+h) - X_{k_0}^2(t) \geqq v(8hM_n)^{1/2}, Y \geqq 0\} \\ &= 2P\{X_{k_0}^2(t+h) - X_{k_0}^2(t) \geqq v(8hM_n)^{1/2}\}P\{Y \geqq 0\} \\ &\geqq P\{X_{k_0}^2(t+h) - X_{k_0}^2(t) \geqq v(8hM_n)^{1/2}\}. \end{aligned} \tag{2.10}$$

Now we estimate $P\{X_k^2(t+h) - X_k^2(t) \geqq v\sigma_k\sigma_k'\}$. Let f_k denote the denisty function of $X_k^2(t+h) - X_k^2(t)$. By independence of $X_k(t+h) + X_k(t)$ and $X_k(t+h) - X_k(t)$ we have

$$f_k(x) = \frac{1}{\pi\sigma_k\sigma_k'}\int_0^\infty \frac{1}{y}\exp\left\{-\frac{x^2}{2\sigma_k^2y^2} - \frac{y^2}{2\sigma_k'^2}\right\}dy,$$

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and on using tail probability estimates of a normal variable we obtain

$$
\begin{aligned}
(2.11)\quad & P\{X_k^2(t+h)-X_k^2(t)\geqq v\sigma_k'\}\\
&=\frac{1}{\pi\sigma_k\sigma_k'}\int_0^\infty\frac{1}{y}\left(\int_{v\sigma_k\sigma_k'}^\infty\exp\left\{-\frac{x^2}{2\sigma_k^2y^2}\right\}dx\right)\exp\left\{-\frac{y^2}{2\sigma_k'^2}\right\}dy\\
&\geqq\frac{1}{\pi\sigma_k'^2v}\int_0^\infty y\left(1-\frac{y^2}{v^2\sigma_k'^2}\right)\exp\left\{-\frac{v^2\sigma'^2}{2y^2}-\frac{y^2}{2\sigma_k'^2}\right\}dy\\
&\geqq\frac{1}{\pi v}\int_{v^{1/2}}^{v^{3/4}}y\left(1-\frac{y^2}{v^2}\right)\exp\left\{-\frac{v^2}{2y^2}-\frac{y^2}{2}\right\}dy\\
&\geqq\frac{e^v}{\pi v^{1/2}}\int_{2v^{1/2}}^{v^{3/4}+v^{1/4}}\exp\{-t^2/2\}dt\\
&\geqq\frac{e^v}{\pi v^{1/2}}\left\{\left(\frac{1}{2v^{1/2}}-\frac{1}{8v^{3/2}}\right)\exp\{-2v\}\right.\\
&\left.-\frac{1}{v^{3/4}+v^{1/4}}\exp\{-(v^{3/4}+v^{1/4})^2/2\}\right\},\\
&\geqq\frac{1}{7v}\exp(-v),
\end{aligned}
$$

provided that v is large enough, where above we used the change of variable $t=y+v/y$. Since

$$\sigma_{k_0}^2\sigma_{k_0}'^2/(8hM_n)\longrightarrow 1\quad\text{as } h\longrightarrow 0,$$

by (2.10) and (2.11) we get the left hand side inequality of (2.9).

Now we proved the right hand side inequality of (2.9). For $0\leqq t\leqq 1/(\sigma_j\sigma_j')$ we have

$$
\begin{aligned}
&E\exp\{t(X_j^2(t+h)-X_j^2(t))\}\\
&\quad=E\{E[\exp\{t(X_j(t+h)+X_j(t))(X_j(t+h)-X_j(t))\}\mid\\
&\quad(X_j(t+h)+X_j(t))]\}\\
&\quad=E\exp\left\{\tfrac{1}{2}t^2(X_j(t+h)+X_j(t))^2\sigma_j'^2\right\}\\
&\quad=(1-t^2\sigma_j'^2\sigma_j^2)^{-1/2}.
\end{aligned}
$$

Consequently, for $0\leqq t\leqq 1/(\sigma_{k_0}\sigma_{k_0}')$, we have

$$(2.12)\quad p_n(v)\geqq 2\exp\{-tv(8hM_n)^{1/2}\}\prod_{j=1}^n(1-t^2\sigma_j^2\sigma_j'^2)^{-1/2}.$$

Let $t=(1-\epsilon/2)/(\sigma_{k_0}\sigma_{k_0}')$. Then

$$t^2\sigma_{k_0}^2\sigma_{k_0}'^2\leqq(1-\epsilon/2)^2\leqq 1-3\epsilon/4,$$

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and by using the inequality $1-x \geqq e^{-x/\epsilon}$ for $0 \leqq x \leqq 1-\epsilon$, $0<\epsilon<1$, we get

$$\prod_{j=1}^{n}(1-t^2\sigma_j^2\sigma_j'^2)^{-1/2} \leqq \exp\left\{\left(\frac{1}{2}\right)\left(\frac{4}{3\epsilon}\right)t^2\sum_{j=1}^{n}\sigma_j^2\sigma_j'^2\right\}$$
$$\leqq \exp\left\{\frac{1}{\epsilon}\left(\sum_{j=1}^{n}\sigma_j^2\sigma_j'^2\right)/(\sigma_{k_0}^2\sigma_{k_0}'^2)\right\}.$$

Hence, on assuming that $v \geqq (8/\epsilon^2)(\Gamma_2/M_n)^{1/2}$, by (2.12) we get

$$p_n(v) \leqq 2\exp\left\{-(1-\epsilon/2)v(8hM_n)^{1/2}/(\sigma_{k_0}\sigma_{k_0}')\right.$$
$$\left.=\frac{1}{\epsilon}\left(\sum_{j=1}^{n}\sigma_j^2\sigma_{k_0}'^2\right)/(\sigma_{k_0}^2\sigma_{k_0}'^2)\right\}$$
$$\leqq 2\exp\{-(1-2\epsilon/3)v\} \leqq 2\exp\left\{-\frac{v}{1+\epsilon}\right\},$$

provided that h is small enough. This also completes the proof of (2.9).

By assuming $\Gamma_2<\infty$, $M_n = M$ for all large n. For all such n (2.9) remains true when M_n is replaced by M in the definition of $p_n(v)$. Consequently (2.9) yields

$$\text{(2.13)} \quad \frac{1}{7v}\exp\left\{-\frac{v}{1-\epsilon}\right\} \leqq P\{|\chi^2(t+h)-\chi^2(t)| \geqq v(8hM)^{1/2}\}$$
$$\leqq 2\exp\left\{-\frac{v}{1+\epsilon}\right\}$$

if $v \geqq (8/\epsilon^2)(\Gamma_2/M)^{1/2}$. The left hand side inequality of (2.13) is (2.2), while (2.3) can be proved along the lines of the proof of (2.1) with the help of the right hand side inequality of (2.13). Hence we omit these details.

3. Proofs of theorems 1 and 2.

Proof of Theorem 1. For any given $\epsilon>0$, let h_n be such that

$$\text{(3.1)} \quad \sum_{n=1}^{\infty}(h_{n-1}/T_{h_n})^{\epsilon} < \infty$$

and, as $n \longrightarrow \infty$,

$$\text{(3.2)} \quad (h_{n-1}/T_{h_n})/(h_n/T_{h_{n+1}}) \longrightarrow 1.$$

By Lemma 2.1 we have

$$P\left\{\sup_{|t|\leqq T_{h_n}}\sup_{0\leqq s\leqq h_{n-1}}|X(t+s)-X(t)|\right.$$
$$\left.\geqq(1+\epsilon)((2h_{n-1}\Gamma_1)2\log(T_{h_n}/h_{n-1}))^{1/2}\right\}$$
$$\leqq(C_1T_{h_n}/h_{n-1})\exp\left\{-\frac{2(1+\epsilon)^2}{2+\epsilon}\log(T_{h_n}/h_{n-1})\right\}$$
$$\leqq C(h_{n-1}/T_{h_n})^{\epsilon}.$$

The latter combined with (3.1) yields

$$\limsup_{n\to\infty}\sup_{|t|\leqq T_{h_n}}\sup_{0\leqq s\leqq h_{n-1}}\frac{|X(t+s)-X(t)|}{((2h_{n-1}\Gamma_1)2\log(T_{h_n}/h_{n-1}))}\leqq 1+\epsilon \text{ a.s.},$$

which, by condition (3.2), results in

$$\text{(3.3)}\qquad \limsup_{h\to0}\sup_{|t|\leqq T_h}\sup_{0\leqq s\leqq h}\frac{|X(t+s)-X(t)|}{((2h\Gamma_1)2\log(T_h/h))^{1/2}}\leqq 1+\epsilon \text{ a.s.}$$

Next we prove

$$\text{(3.4)}\qquad \liminf_{h\to0}\sup_{|t|\leqq T_h}\frac{|X(t+h)-X(t)|}{((2h\Gamma_1)2\log(T_h/h))^{1/2}}\geqq 1-\epsilon \text{ a.s.}$$

To this end, define h_n by

$$h_n=\sup\left\{h:\frac{T_{h_{n-1}}}{h}\geqq(\log n)^{3/\epsilon}\text{ and } h<h_{n-1}\right\}.$$

Then $h_n\to0$ and $h_n/h_{n+1}\to1$ as $n\to\infty$. For $i<j$ we have

$$\text{(3.5)}\qquad E(X((i+1)h_n)-X(ih_n))(X((j+1)h_n)-X(jh_n))$$
$$=\sum_{l=1}^{\infty}(\gamma_l/\lambda_l)e^{-\lambda_l jh_n}(2e^{\lambda_l ih_n}-e^{\lambda_l(i-1)h_n}-e^{\lambda_l(i+1)h_n})<0,$$

and hence, by using Lemma 2.2, we obtain

$$P\left\{\max_{|k|\leqq[T_{h_{n-1}}/h_n]}(X((k+1)-X(kh_n))\right.$$
$$\left.\leqq(1-\epsilon)((2h_n\Gamma_1)2\log(T_{h_{n-1}}/h_n))^{1/2}\right\}$$
$$\leqq(P\{(X(h_n)-X(0))\leqq(1-\epsilon)((2h_n\Gamma_1)2\log(T_{h_{n-1}}/h_n))^{1/2}\})^{2[T_{h_{n-1}}/h_n]+1}$$
$$\leqq\left(1-\frac{1}{6(\log(T_{h_{n-1}}/h_n))^{1/2}}\exp\{-(1-\epsilon/2)\log(T_{h_{n-1}}/h_n)\}\right)^{2T_{h_{n-1}}/h_n}$$
$$\leqq\exp\{-2(T_{h_{n-1}}/h_n)^{\epsilon/3}\}\leqq 1/n^2.$$

Hence,

$$\liminf_{n\to\infty} \max_{|k|\leqq [T_{h_{n-1}}/h_n]} \frac{(X((k+1)h_n) - X(kh_n))}{((2h_n\Gamma_1)2\log(T_{h_{n-1}}/h_n))^{1/2}} \geqq 1-\epsilon \text{ a.s.},$$

which, in turn, implies

$$\liminf_{n\to\infty} \sup_{|t|\leqq T_{h_{n-1}}} \frac{|X(t+h_n) - X(t)|}{((2h_n\Gamma_1)2\log(T_{h_{n-1}}/h_n))^{1/2}} \geqq 1-\epsilon \text{ a.s.} \tag{3.6}$$

As $n \longrightarrow \infty$, for $h_n \leqq h < h_{n-1}$ we have

$$h_n \log(T_{h_{n-1}}/h_n)/(h\log(T_h/h)) \longrightarrow 1. \tag{3.7}$$

Furthermore,

$$\begin{aligned} &\sup_{h_n\leqq h<h_{n-1}} \sup_{|t|\leqq T_{h_{n-1}}} \frac{|X(t+h) - X(t+h_n)|}{((2h_n\Gamma_1)2\log(T_{h_{n-1}}/h_n))^{1/2}} \\ &\leqq \sup_{0\leqq s\leqq h_{n-1}-h_n} \sup_{|t|\leqq T_{h_{n-1}}+h_n} \frac{|X(t+s) - X(t)|}{((2h_n\Gamma_1)2\log(T_{h_{n-1}}/h_n))^{1/2}}. \end{aligned} \tag{3.8}$$

By definition of h_n, as $n \longrightarrow \infty$, $(h_{n-1} - h_n)/h_n \longrightarrow 0$ and we have also

$$\frac{(h_{n-1} - h_n)\log((T_{h_{n-1}} + h_n)/(h_{n-1} - h_n))}{(h_n \log(T_{h_{n-1}}/h_n))} \longrightarrow 0. \tag{3.9}$$

Moreover, by (3.3),

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{|t|\leqq T_{h_{n-1}}+h_n} \sup_{0\leqq s\leqq h_{n-1}-h_n} \\ &\frac{|X(t+s) - X(t)|}{(2(h_{n-1} - h_n)\Gamma_1 2\log((T_{h_{n-1}} + h_n)/(h_{n-1} - h_n)))^{1/2}}. \\ &\leqq 1+\epsilon \text{ a.s.} \end{aligned}$$

Combining the latter with (3.8) and (3.9), we get

$$\limsup_{n\to\infty} \sup_{h_n\leqq h<h_{n-1}} \sup_{|t|\leqq T_{h_{n-1}}} \frac{|X(t+h) - X(t+h_n)|}{((2h_n\Gamma_1)2\log(T_{h_{n-1}}/h_n))^{1/2}} = 0 \text{ a.s.}, \tag{3.10}$$

and now (3.6), (3.7) and (3.10) yield (3.4). Consequently, by (3.3) and (3.4) we obtain (1.10) and (1.11). This also completes the proof of Theorem 1.

Proof of Theorem 2. Given any $\epsilon > 0$, we can prove

$$\limsup_{h\to 0} \sup_{|t|\leqq T_h} \sup_{0\leqq s\leqq h} \frac{|\chi^2(t+h) - \chi^2(t)|}{(8hM)^{1/2} 2\log(T_h/h)} \leqq 1+\epsilon \text{ a.s.} \tag{3.11}$$

along the lines of the proof of (3.3) by using (2.3) instead of (2.1). Hence we omit these details, and conclude that (1.13), and hence also (1.12) are true.

In order to prove (1.15) and (1.16) it is enough to show that under the condition (1.14) we have

$$\liminf_{h\to 0} \sup_{|t|\leqq T_h} \frac{|\chi^2(t+h)-\chi^2(t)|}{(8hM)^{1/2} 2\log(T_h/h)} \geqq 1-\epsilon \text{ a.s.} \tag{3.12}$$

The proof of this statement becomes somewhat involved due to the fact that we cannot apply Slepian's lemma to study the increments of the non-Gaussian $\chi^2(\cdot)$ process.

From the condition $\Gamma_2 < \infty$, and similarly to (3.11), it is easy to see that

$$\limsup_{h\to 0} \sup_{|t|\leqq T_h} \frac{\left|\sum_{k=K}^{\infty}(X_k^2(t+h)-X_k^2(t))\right|}{(8hM)^{1/2} 2\log(T_h/h)} \leqq \epsilon \text{ a.s.,} \tag{3.13}$$

provided that $K = K(\epsilon)$ is large enough. Fixing the value of K, by (3.13) the statement of (3.12) becomes equivalent to

$$\liminf_{h\to 0} \sup_{|t|\leqq T_h} \frac{\left|\sum_{k=1}^{K}(X_k^2(t+h)-X_k^2(t))\right|}{(8hM)^{1/2} 2\log(T_h/h)} \geqq 1-\epsilon \text{ a.s.} \tag{3.14}$$

Define h_n such that $T_{h_{n-1}}/h_n = n$ and put

$$\xi_l^k = X_k((l+1)h_n) - X_k(lh_n), \quad \eta_l^k = X_k((l+1)h_n) + X_k(lh_n).$$

Then

$$\sigma_{ll}^k := E(\xi_l^k)^2 = \frac{2\gamma_k}{\lambda_k}(1-e^{-\lambda_k h_n}),$$

$$\alpha_{ll}^k := E(\eta_l^k)^2 = \frac{2\gamma_k}{\lambda_k}(1+e^{-\lambda_k h_n}),$$

and for $l < r$,

$$\sigma_{lr}^k := E\xi_l^k\xi_r^k = \frac{\gamma_k}{\lambda_k} e^{-\lambda_k(r-l)h_n}(2-e^{-\lambda_k h_n}-e^{\lambda_k h_n}),$$

$$\alpha_{lr}^k := E\eta_l^k\eta_r^k = \frac{\gamma_k}{\lambda_k} e^{-\lambda_k(r-l)h_n}(2+e^{-\lambda_k h_n}-e^{\lambda_k h_n}),$$

$$\tau_{lr}^k := E\xi_l^k\eta_r^k = \frac{\gamma_k}{\lambda_k} e^{-\lambda_k(r-l)h_n}(e^{\lambda_k h_n}-e^{-\lambda_k h_n}),$$

and, clearly, $\tau_{rl}^{l} = -\tau_{lr}^{k}$. Let

$$\xi_{1,l}^{k} = \xi_{l}^{k} - \frac{\sigma_{1l}^{k}}{\sigma_{11}^{k}}\,\xi_{1}^{k} - \frac{\tau_{l1}^{k}}{\alpha_{11}^{k}}\,\eta_{1}^{k}, \quad \eta_{1,l}^{k} = \eta_{l}^{k} - \frac{\tau_{1l}^{k}}{\sigma_{11}^{k}}\,\xi_{1}^{k} - \frac{\alpha_{1l}^{k}}{\alpha_{11}^{k}}\,\eta_{1}^{k}.$$

It is easy to see that (ξ_1^k, η_1^k) is independent of $(\xi_{1,l}^k, \eta_{1,l}^k)$. We write

$$\begin{aligned}\sum_{k=1}^{K} \xi_l^k \eta_l^k &= \sum_{k=1}^{K} \Big\{ \xi_{1,l}^k \eta_{1,l}^k + \frac{\sigma_{1l}^k}{\sigma_{11}^k}\,\xi_1^k\eta_l^k - \frac{\sigma_{1l}^k\tau_{1l}^k}{(\sigma_{11}^k)^2}\,(\xi_1^k)^2 - \frac{\sigma_{1l}^k\alpha_{1l}^k}{\sigma_{11}^k\alpha_{11}^k}\,\xi_1^k\eta_1^k \\ &\quad + \frac{\tau_{l1}^k}{\alpha_{11}^k}\,\eta_1^k\eta_l^k - \frac{\tau_{1l}^k\tau_{l1}^k}{\sigma_{11}^k\alpha_{11}^k}\,\xi_1^k\eta_1^k - \frac{\alpha_{1l}^k\tau_{l1}^k}{(\alpha_{11}^k)^2}\,(\eta_1^k)^2 \\ &\quad + \frac{\tau_{1l}^k}{\sigma_{11}^k}\,\xi_1^k\xi_l^k + \frac{\alpha_{1l}^k}{\alpha_{11}^k}\,\xi_l^k\eta_1^k \Big\} \\ &:= \sum_{k=1}^{K} \xi_{1,l}^k\eta_{1,l}^k + H_l.\end{aligned}$$

We put

$$\lambda^m = \min\{\lambda_k, k \leqq K\} > 0, \quad L = [(\lambda^m h_n)^{-1}\log(T_{h_{n-1}}/h_n)],$$
$$A_n = (1-\epsilon)(8h_nM)^{1/2}\log(T_{h_{n-1}}/h_n).$$

We have

$$\begin{aligned}(3.15)\quad & P\left\{\max_{|l|\leqq[T_{h_{n-1}}/h_n]}\left|\sum_{k=1}^{K}(X_k^2((l+1)h_n) - X_k^2(lh_n))\right| \leqq A_n\right\} \\ &\leqq P\left\{\max_{1\leqq j\leqq T_{h_{n-1}}/(Lh_n)}\left|\sum_{k=1}^{K}(X_k^2((jL+1)h_n) - X_k^2(jLh_n))\right| \leqq A_n\right\} \\ &\leqq P\left\{\left|\sum_{k=1}^{K}(X_k^2((L+1)h_n) - X_k^2(Lh_n))\right| \leqq A_n\right\} \\ &\quad \times P\left\{\max_{2\leqq j\leqq T_{h_{n-1}}/(Lh_n)}\left|\sum_{k=1}^{K}\xi_{1,jL}^k\eta_{1,jL}^k\right| \leqq A_n(1+(T_{h_{n-1}}/h_n)^{-1})\right\} \\ &\quad + P\left\{\max_{2\leqq j\leqq T_{h_{n-1}}/(Lh_n)}|H_{jL}| > A_n(T_{h_{n-1}}/h_n)^{-1}\right\}.\end{aligned}$$

At first we wish to estimate the last probability. A typical enough term in H_{jL} is

$$\sum_{k=1}^{K}(\sigma_{1,jL}^k/\sigma_{11}^k)\xi_1^k\eta_{jL}^k,$$

which we now proceed to estimate in probability. We put

$$\bar{\eta}^k_{jL} = \eta^k_{jL} - (\tau^k_{1,jL}/\sigma^k_{11})\xi^k_1$$

and note that the latter is independent of ξ^k_1. We have

$$E(\bar{\eta}^k_{jL})^2 = \alpha^k_{jL,jL} - (\tau^k_{1,jL})^2/\sigma^k_{11}.$$

It suffices to estimate only

$$\sum_{k=1}^{K}(\sigma^k_{1,jL}/\sigma^k_{11})\xi^k_1\bar{\eta}^k_{jL}.$$

An inequality like that of (2.12) with $n = 1$ and $t = 1/(2\sigma_{k_0}\sigma'_{k_0})$ here gives

$$\text{(3.16)}\quad P\left\{\left|\sum_{k=1}^{K}(\sigma^k_{1,jL}/\sigma^k_{11})\xi^k_1\bar{\eta}^k_{jL}\right| > \frac{1}{16}\,A_n(T_{h_{n-1}}/h_n)^{-1}\right\}$$
$$\leqq 3K\exp\left\{-\frac{1}{16K}\,A_n(T_{h_{n-1}}/h_n)^{-1}/\right.$$
$$\left.(4(\sigma^{k_0}_{1,jL})^2(\sigma^{k_0}_{11})^{-1}(\alpha^{k_0}_{jL,jL} - (\tau^{k_0}_{1,jL})^2/\sigma^{k_0}_{11}))^{1/2}\right\},$$

where

$$\begin{aligned}(\sigma^k_{1,jL})^2(\sigma^k_{11})^{-1} &\leqq (\sigma^k_{1L})^2(\sigma^k_{11})^{-1}\\ &= O((\gamma_k\lambda_k h_n^2(T_{h_{n-1}}/h_n)^{-1})^2/(\gamma_k/\lambda_k))\\ &= O(\gamma_k\lambda_k^3 h_n^6 T_{h_{n-1}}^{-2})\end{aligned}$$

and

$$\alpha^k_{jL,jL} - (\tau^k_{1,jL})^2/\sigma^k_{11} \sim 2\gamma_k/\lambda_k.$$

Inserting these into (3.16) yields

$$P\left\{\left|\sum_{k=1}^{K}(\sigma^k_{1,jL}/\sigma^k_{11})\xi^k_1\bar{\eta}^k_{jL}\right| \geqq \frac{1}{16}\,A_n(T_{h_{n-1}}/h_n)^{-1}\right\}$$
$$\leqq 3K\exp\{-ch_n^{-3/2}\log(T_{h_{n-1}}/h_n)\} \leqq (T_{h_{n-1}}/h_n)^{-4},$$

provided that n is large enough, where c here, and also later on, stands for a positive constant which does not depend on n, but may take different values on the occasions when it occurs. Consequently we have also

$$P\left\{\max_{2\leqq j\leqq T_{h_{n-1}}/(Lh_n)}\left|\sum_{k=1}^{K}(\sigma^k_{1,jL}/\sigma^k_{11})\xi^k_1\bar{\eta}^k_{jL}\right| \geqq \frac{1}{16}\,A_n(T_{h_{n-1}}/h_n)^{-1}\right\}$$
$$\leqq L^{-1}(T_{h_{n-1}}/h_n)^{-3}.$$

For the other terms of H_{jL} we have similar estimations, and thus we arrive at

$$P\left\{\max_{2\leqq j\leqq T_{h_{n-1}}/(Lh_n)}|H_{jL}|\geqq A_n(T_{h_{n-1}}/h_n)^{-1}\right\}\leqq cL^{-1}(T_{h_{n-1}}/h_n)^{-3}.$$

Using a similar procedure for estimating the second probability on the right hand side of the inequality of (3.15), we obtain

$$\text{(3.17)}\quad P\left\{\max_{2\leqq j\leqq T_{h_{n-1}}/(Lh_n)}\left|\sum_{k=1}^{K}\xi_{1,jL}^{k}\eta_{1,jL}^{k}\right|\leqq A_n(1+(T_{h_{n-1}}/h_n)^{-1})\right\}$$
$$\leqq\left\{\max_{2\leqq j\leqq T_{h_{n-1}}/(Lh_n)}\left|\sum_{k=1}^{K}\xi_{jL}^{k}\eta_{jL}^{k}\right|\leqq A_n(1+2(T_{h_{n-1}}/h_n)^{-1}\right\}$$
$$+cL^{-1}(T_{h_{n-1}}/h_n)^{-3}.$$

Inserting the latter upper estimate into (3.15) and then repeating the same procedure for estimating the probabilities on the right hand side of the inequality of (3.17), we continue this procedure until we obtain

$$\text{(3.18)}\quad P\left\{\max_{|l|\leqq[T_{h_{n-1}}/h_n]}\left|\sum_{k=1}^{K}(X_k^2((l+1)h_n)-X_k^2(lh_n))\right|\leqq A_n\right\}$$
$$\leqq\prod_{j=1}^{[T_{h_{n-1}}/(Lh_n)]}P\left\{\left|\sum_{k=1}^{K}(X_k^2((jL+1)h_n)-X_k^2(jLh_n))\right|\right.$$
$$\left.\leqq A_n(1+j(T_{h_{n-1}}/h_n)^{-1})\right\}$$
$$+c(LT_{h_{n-1}}/h_n)^{-2}$$
$$\leqq\left(P\left\{\left|\sum_{k=1}^{K}(X_k^2(h_n)-X_k^2(0))\right|\leqq\left(1+\frac{\epsilon}{3}\right)A_n\right\}\right)^{[T_{h_{n-1}}/(Lh_n)]}$$
$$+c(LT_{h_{n-1}}/h_n)^{-2}.$$

Having taken the value of $K=K(\epsilon)$ large enough, and using now (2.2) and our condition (1.14), the first term on the right hand side of the last inequality of (3.18) does not exceed

$$\left(P\left\{|\chi^2(h_n)-\chi^2(0)|\right.\right.$$
$$\left.\left.\leqq\left(1-\frac{3\epsilon}{3}\right)(8h_nM)^{1/2}\log(T_{h_{n-1}}/h_n)\right\}\right)^{[T_{h_{n-1}}/(Lh_n)]}$$
$$\leqq\left(1-\frac{1}{8\log(T_{h_{n-1}}/h_n)}\exp\left\{-\left(1-\frac{\epsilon}{2}\right)\log(T_{h_{n-1}}/h_n)\right\}\right)^{[T_{h_{n-1}}/(Lh_n)]}$$
$$\leqq\exp\{-(T_{h_{n-1}}/h_n)^{\epsilon/3}L^{-1}\}\leqq\exp\{-T_{h_{n-1}}^{\epsilon/4}\}\leqq n^{-2},$$

provided that n is large enough. The rest of this proof of (3.12), which also completes those of (1.15) and (1.16), is similar to that of (3.4), and hence we omit these further details.

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