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ELEMENTARY CHAINS OF INVARIANT SUBSPACES OF A BANACH SPACE

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ABSTRACT. We will generalize Ringrose's notion of a simple chain of closed invariant subspaces of a compact operator acting on a Banach space, to that of an elementary chain of invariant subspaces of a subalgebra of compact operators. With this we expand the notion of diagonal coefficients to that of diagonal representations and subsequently generalize Ringrose's theorem equating the spectrum of an operator to the collection of diagonal coefficients. This in turn, in conjunction with some results from the theory of Polynomial Identity algebras, allows us to generalize Murphy's theorem which states that a closed subalgebra $\mathcal{A}$ of compact operators is simultaneously triangularizable if and only if $\mathcal{A}/\operatorname{rad}(\mathcal{A})$ is commutative. Let $\mathcal{A}$ be an algebra of compact operators acting on a Banach space with a norm $\|\cdot\|_{\mathcal{A}}$ which dominates the operator norm, and under which $\mathcal{A}$ is complete. Then $\mathcal{A}$ has an elementary chain of invariant subspaces of bound n if and only if $\mathcal{A}/\operatorname{rad}(\mathcal{A})$ satisfies the standard polynomial $P_{2n}(x_1,x_2,\ldots,x_{2n}) = \sum_{\tau\in S_{2n}} \operatorname{sgn}(\tau)x_{\tau(1)}x_{\tau(2)}\cdots x_{\tau(2n)}$.

0. **Introduction.** We are interested in characterizing subalgebras $\mathcal{A}$ of the compact operators which have a collection of closed subspaces invariant under each of the operators in our collection. In particular, we will consider conditions under which certain types of chains of closed $\mathcal{A}$ invariant subspaces exist.

We denote by $\mathcal{K}(X)$ the set of compact operators acting on a Banach space X. Throughout this section, unless otherwise stated, we will assume that $\mathcal{A}$ is a norm-closed subalgebra of $\mathcal{K}(X)$.

1. Elementary chains and the existence of idempotents.

DEFINITION 1.1. A *complete chain*, $\mathcal{N}$, of subspaces of a Banach space X is a linearly ordered set of closed subspaces of X, such that:

i) $\{0\}, X \in \mathcal{N}$
ii) For every subfamily $\mathcal{N}_0$ of $\mathcal{N}$, the intersection and the closed linear span of the union of sets in $\mathcal{N}_0$ are in $\mathcal{N}$.

We denote the intersection, and the closed linear span of the union of elements of a subfamily $\mathcal{N}_0$ by $\bigwedge\{M : M \in \mathcal{N}_0\}$ and $\bigvee\{M : M \in \mathcal{N}_0\}$ respectively.

To each element $M \in \mathcal{N}$ we associate its *predecessor*

$$M_- = \bigvee\{L \in \mathcal{N} : L \subset M \text{ and } L \neq M\}.$$

A chain $\mathcal{N}$ will be called:

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i) *invariant* for a collection of operators $\mathcal{A}$, if for each $M \in \mathcal{N}$, and each $T \in \mathcal{A}$, $T(M) \subseteq M$;

ii) *simple* if it is a complete chain such that for each $M \in \mathcal{N}$, $\dim(M/M_-) \leq 1$.

iii) *elementary* if it is a complete chain such that for each $M \in \mathcal{N}$, $\dim(M/M_-) < \infty$.

An elementary chain is *bounded*, of *bound n*, if there exists a natural number n such that $\max\{\dim(M/M_-) : M \in \mathcal{N}\} = n$.

$\mathcal{N}$ will be said to have a *gap* at M whenever $0 < \dim(M/M_-)$.

We begin by showing that the existence of non-zero idempotent elements in our algebra imply the existence of gaps in our $\mathcal{A}$-invariant chain $\mathcal{N}$.

LEMMA 1.2. *Let $\mathcal{N}$ be a complete maximal $\mathcal{A}$-invariant chain of closed subspaces. If there exists an idempotent $G \in \mathcal{A} : G^2 = G \neq 0$ then there exists a corresponding $M = M_G \in \mathcal{N}$ such that* $\dim(M/M_-) > 0$.

PROOF. Note that $\dim(GN)$ is an increasing function of $N \in \mathcal{N}$, left continuous in the order topology on $\mathcal{N}$. It is also right continuous. For if $N_0 = \inf_{N>N_0} N$, then since G is finite rank, $k = \inf\{\dim(GN) : N > N_0\}$ is attained at some N_1. Hence $GN_1 = GN$ for all $N_0 < N \leq N_1$. Consequently GN_1 is contained in $\bigcap_{N>N_0} N = N_0$, so that $GN_1 = GN_0$ also has dimension k.

Now let M_G be the infimum of all elements $N \in \mathcal{N}$ such that $\dim(GN) \neq 0$. ∎

We denote the Jacobson radical of $\mathcal{A}$ by $\mathrm{rad}(\mathcal{A})$. An irreducible representation will refer to a topologically irreducible representation, while a strictly irreducible representation is understood to mean an algebraically irreducible representation.

Two subcollections of $\mathcal{A}$ will play a central role in the development.

DEFINITION 1.3. i) We call a non-zero element G a *minimal idempotent* in $\mathcal{A}$ if, whenever $E \in \mathcal{A}$ with $0 \neq E = E^2$ and $GE = EG = E$, it follows that $E = G$. We denote the set of minimal idempotents of $\mathcal{A}$ by $\mathcal{M}(\mathcal{A})$.

ii) The pullback, under the canonical quotient homomorphism, of the center of $\mathcal{A}/\,\mathrm{rad}(\mathcal{A})$ will be denoted by $\mathcal{Z}_r(\mathcal{A})$. We have

$$\mathcal{Z}_r(\mathcal{A}) = \{T \in \mathcal{A} : TS - ST \in \mathrm{rad}(\mathcal{A}), \text{ for all } S \in \mathcal{A}\}.$$

We will often use the notation of the Lie bracket, $[T, S]$, to denote $TS - ST$. We use $I \triangleleft J$, to indicate that I is a two-sided ideal in J, and $I \leq J$ to indicate that I is a subalgebra of J, or simply a subspace when no other algebraic structure is present.

LEMMA 1.4. *Let $\mathcal{A}$ be an arbitrary subalgebra of $\mathcal{B}(X)$ and let $\mathcal{N}$ be a complete maximal $\mathcal{A}$-invariant chain with a gap at M. Define*

$$\pi_M \colon \mathcal{A} \longrightarrow B(M/M_-) \quad by \quad \pi_M(T)(x + M_-) = Tx + M_-.$$

If $\dim(M/M_-) > 1$, *or if* $\pi_M \neq 0$ *then*

i) π_M is an irreducible representation, and

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ii) if $\mathcal{N}$ has a finite dimensional gap at M then π is strictly irreducible.

PROOF. Notice that we are not assuming that $\mathcal{A}$ is a subalgebra of $\mathcal{K}(X)$, nor that it is norm-closed.

For i), suppose there exists a non-zero subspace of M/M_- which is invariant under $\pi_M(\mathcal{A})$. Such a subspace is of the form $\{x + M_- : x \in L\}$ where $M_- < L \leq M$. If the norm closure of $\{x + M_- : x \in L\}$ is properly contained in M/M_- then it follows that $M_- < \bar{L} \leq M$. By maximality of $\mathcal{N}$, this implies that $\bar{L} = M$.

For ii), again suppose that there exists $\{x+M_- : x \in L\}$ as above. Then since M/M_- is finite dimensional, there exists $\{x_i\}_{i=1}^n \subset X$ such that $\{x+M_- : x \in L\} = \operatorname{span}\{x_i+M_-\}$. But then $M_- < L \leq M$ and $L = \operatorname{span}\{x_i\} \oplus M_-$. Hence by maximality of $\mathcal{N}$, $L = M$. ∎

THEOREM 1.5. *Let $\mathcal{N}$ be a complete maximal $\mathcal{A}$-invariant chain with a finite dimensional gap at M, and define $\pi_M: \mathcal{A} \longrightarrow B(M/M_-)$ as in Lemma 1.4. Then there exists a corresponding $G \in \mathcal{M}\big(\mathcal{Z}_r(\mathcal{A})\big)$ such that $\pi_M(G)$ is the identity in $B(M/M_-)$.*

PROOF. Define

$$\Pi_M: \mathcal{A}/\operatorname{rad}(\mathcal{A}) \longrightarrow B(M/M_-) \quad \text{by} \quad \Pi_M\big(T + \operatorname{rad}(\mathcal{A})\big)(x + M_-) = Tx + M_-.$$

Notice that $\operatorname{rad}(\mathcal{A}) \subseteq \ker(\pi_M)$, by Lemma 1.4, so that Π_M is well defined. Furthermore, again by Lemma 1.4, Π_M is surjective by Burnside's Theorem, since $\pi_M(\mathcal{A})$ and hence also $\Pi_M(\mathcal{A})$ are transitive. Now let $J = \ker(\Pi_M)$. Since $B(M/M_-)$ is a simple algebra with identity, it follows that J is a maximal modular ideal in $\mathcal{A}/\operatorname{rad}(\mathcal{A})$.

Denoting $\mathcal{A}/\operatorname{rad}(\mathcal{A})$ by $\tilde{\mathcal{A}}$, and the image of T in $\mathcal{A}/\operatorname{rad}(\mathcal{A})$ by $\tilde{T}$, note that since $T \in \mathcal{K}(X)$, $\operatorname{sp}_{\tilde{\mathcal{A}}}(\tilde{T})$ has no non-zero limit points. Furthermore, using Theorem 4.2 of [Bar2] we see that $\tilde{\mathcal{A}}$ is a modular annihilator algebra. From this, and Lemma 2.8.10 of [Ric], it follows that the left and right annihilators of J ($\operatorname{lan}(J)$ and $\operatorname{ran}(J)$ respectively) are identical and are not equal to $\{0\}$.

Now let $K = \operatorname{lan}(J) = \operatorname{ran}(J)$. Note that, since $(K \cap J)^2 = \{0\}$, $K \cap J = \{0\}$. Then, since J is maximal and K is finite dimensional, $J + K = \tilde{\mathcal{A}}$, and therefore, $J \oplus K = \tilde{\mathcal{A}}$.

Now $K \cong \tilde{\mathcal{A}}/J \cong B(M/M_-)$, so that there exists $F \in K \leq \tilde{\mathcal{A}}$ such that F is the identity in K and $F^2 = F \neq 0$. In particular, by Theorem 2.3.9 of [Ric] there exists $G \in \mathcal{A}$ such that $G^2 = G \neq 0$ and the image of G in $\tilde{\mathcal{A}}$ is equal to F. We claim finally that $G \in \mathcal{M}\big(\mathcal{Z}_r(\mathcal{A})\big)$.

That $G \in \mathcal{Z}_r(\mathcal{A})$ follows from the fact that $\tilde{G}$ is the identity in K, the annihilator of J.

G is a minimal idempotent in $\mathcal{Z}_r(\mathcal{A})$ since, given any $E \in \mathcal{Z}_r(\mathcal{A})$ such that $E^2 = E \neq 0$ and $EG = GE = E$, it follows that $\tilde{E} = \widetilde{EG} \in K$ since K is an ideal in $\tilde{\mathcal{A}}$. Furthermore, $\tilde{E}$ is a non-zero central idempotent in $K \cong \mathrm{M}_n(\mathbb{C})$, where $n = \dim(M/M_-)$. Hence $\tilde{E} = \tilde{G}$, the unique non-zero central idempotent in K. Since $G - E \in \operatorname{rad}(\mathcal{A})$ and $GE = EG = E$ implies that $(G - E)^2 = G - E$, $G = E$. ∎

We let $\mathcal{QN}(\mathcal{A})$ denote the set of topologically nilpotent or quasi-nilpotent elements of $\mathcal{A}$, and $\mathcal{QI}(\mathcal{A})$ denote the set of quasi-invertible elements of $\mathcal{A}$.

Denoting the bounded finite rank operators by $\mathcal{F}(X)$, we let $\mathcal{F}(\mathcal{A})$ denote the set $\mathcal{F}(X)\cap\mathcal{A}$. It is easy to see that given $T\in\mathcal{F}(\mathcal{A})$ with $\dim\big(\mathcal{R}(T)\big)=n$ it follows that $\dim(T\mathcal{A}T)\leq n^2$.

Recall that a Banach algebra $\mathcal{A}$ is semiprime if, whenever $J\lhd\mathcal{A}$ such that $J^2=0$, it follows that $J=0$.

LEMMA 1.6. *Let $\mathcal{N}$ be a complete maximal $\mathcal{A}$-invariant chain. If there exists $G\in\mathcal{M}\big(Z_r(\mathcal{A})\big)$, then there exists a corresponding $M\in\mathcal{N}$ such that* $\dim(M/M_-)=n<\infty$. *Furthermore, if $\pi_M\colon\mathcal{A}\longrightarrow B(M/M_-)$ is defined as in Lemma 1.4, then $\pi_M(G)$ is the identity in $B(M/M_-)$.*

PROOF. Let $M=M_G\in\mathcal{N}$ be as in Lemma 1.2, so that $0<\dim(M/M_-)$.

Notice that by Lemma 1.4, π_M is an irreducible representation. We claim that

i) $\mathcal{A}/\ker\pi_M$ is semiprime; and

ii) under the canonical quotient homomorphism $\mathcal{A}\longrightarrow\mathcal{A}/\ker\pi_M$, $\operatorname{rad}(\mathcal{A})$ maps into $\operatorname{rad}(\mathcal{A}/\ker\pi_M)$.

Proof of i) Given $J\lhd\mathcal{A}/\ker\pi_M$, such that $J^2=\{0\}$, and

$$\Pi_M\colon\mathcal{A}/\ker\pi_M\longrightarrow B(M/M_-)$$

the algebra homomorphism induced, as in Theorem 1.6 above, by π_M, then either $\Pi_M(J)(M/M_-)=\{0\}$ in which case $\Pi_M(J)=\{0\}$, which then forces $J=\{0\}$, or $\Pi_M(J)(M/M_-)$ is dense in (M/M_-), from which, since $\Pi_M(J^2)=\{0\}$, it follows that $J=\{0\}$.

Proof of ii) This follows from Proposition 24.16.iii) of [BD].

Notice that if $G\in\mathcal{M}\big(Z_r(\mathcal{A})\big)$, then $G\in\mathcal{F}(\mathcal{A})$. Using Proposition 32.5 of [BD], it follows that $\big(\operatorname{rad}(\mathcal{A})+\ker\pi_M\big)(G+\ker\pi_M)=0+\ker\pi_M$. That is, if $T\in\operatorname{rad}(\mathcal{A})$, then $TG(x+M_-)=0+M_-$ for all $x\in M$, so that $TG(M)\subseteq M_-$. Therefore $\operatorname{rad}(\mathcal{A})$ annihilates $G(M/M_-)=\{Gx+M_-:x\in M\}$.

Finally we claim that $\{0\}\neq G(M/M_-)$ is invariant under $\mathcal{A}$. Since $G\in Z_r(\mathcal{A})$, for all $T\in\mathcal{A}$, $TG-GT\in\operatorname{rad}(\mathcal{A})$; hence $(TG-GT)(Gx+M_-)=0$, so

$$T(Gx+M_-)=TG(Gx+M_-)=GT(Gx+M_-)=G(y+M_-),$$

where $y=TGx\in M$. Consequently, $G(M/M_-)=M/M_-$, by maximality, and hence $\dim(M/M_-)=n<\infty$. ∎

EXAMPLE 1.7. It is shown in Theorem 5 of [CD] that if X and Y are Banach spaces with Schauder bases, then a linear transformation T is in $\mathcal{K}(X,Y)$ if and only if $\lim_{n\to\infty}\|T_n\|=0$, where $T_n(x)$ is defined by the matrix

$$a_{ij}^{(n)}=\begin{cases}a_{ij}, & \text{if } i\geq n\\ 0, & \text{if } i<n\end{cases}$$

where (a_{ij}) is the matrix representation of T, relative to the given bases of X and Y.

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Fixing X a Banach space with a Schauder basis, consider $\mathcal{A}$ the algebra of all operators in $\mathcal{K}(X)$ of the following form:

$$\mathcal{A} = \left\{ \begin{pmatrix} A_1 & * & * & * & \cdots \\ (0) & A_2 & * & * & \cdots \\ (0) & (0) & A_3 & * & \cdots \\ \vdots & & & & \ddots \end{pmatrix} \in \mathcal{K}(X) : A_i \in M_{n_i}(\mathbb{C}) \right\}.$$

If $A \in \mathcal{A}$, then the "diagonal" of A is made up of blocks, each of which is itself a matrix. In fact, in each of these blocks of $\mathcal{A}$ we have the entire matrix algebra, $M_{n_i}(\mathbb{C})$. Consequently the (non-trivial) invariant subspaces of $\mathcal{A}$ are $M_k = \{x \in X : x_i = 0$ for $i > \sum_{j=1}^{k} n_j\}$. Hence the complete maximal $\mathcal{A}$-invariant chain of closed subspaces $\mathcal{N}$ consists of the chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_k \subseteq \cdots \subseteq X$, where $\dim(M_k/M_{k-1}) = n_k$.

Notice that $R \equiv \{A \in \mathcal{A} : A_i = 0 \text{ for all } i\} = \operatorname{rad}(\mathcal{A})$. Consequently,

$$\mathcal{A}/\operatorname{rad}(\mathcal{A}) \cong \left\{ \begin{pmatrix} A_1 & (0) & (0) & (0) & \cdots \\ (0) & A_2 & (0) & (0) & \cdots \\ (0) & (0) & A_3 & (0) & \cdots \\ \vdots & & & & \ddots \end{pmatrix} \in \mathcal{K}(X) : A_i \in M_{n_i}(\mathbb{C}) \right\}.$$

As in Theorem 1.5, to a gap that occurs at $M_i \in \mathcal{N}$ there corresponds a non-zero idempotent in $\mathcal{M}\big(\mathcal{Z}_r(\mathcal{A})\big)$. This is the element $E_i \in \mathcal{A}$ such that the i-th diagonal block of E_i consists of the identity matrix $I \in M_{n_i}(\mathbb{C})$, and all of the other of whose entries are zero. First, clearly E_i is a non-zero idempotent in $\mathcal{A}$. While given any $A \in \mathcal{A}$,

$$[E_i, A] = \begin{pmatrix} (0) & (0) & (0) & * & (0) & \cdots \\ \vdots & (0) & (0) & * & (0) & \cdots \\ \vdots & & & & & \cdots \\ \vdots & & & (0) & * & * \\ \vdots & & & & (0) & \cdots \end{pmatrix} \in \operatorname{rad}(\mathcal{A}).$$

Furthermore, $E_i \in \mathcal{M}\big(\mathcal{Z}_r(\mathcal{A})\big)$, that is E_i is a minimal idempotent in $\mathcal{Z}_r(\mathcal{A})$. For suppose that there exists a $G \in \mathcal{Z}_r(\mathcal{A})$ such that $G^2 = G \neq 0$ and $GE_i = E_iG = G$. Then G must be of the form:

$$G = \begin{pmatrix} (0) & (0) & (0) & \cdots & \cdots & \cdots \\ \vdots & (0) & (0) & \cdots & \cdots & \cdots \\ \vdots & & & \cdots & \cdots & \cdots \\ \vdots & & & G_i & (0) & \cdots \\ \vdots & & & & (0) & \cdots \end{pmatrix}.$$

Furthermore, since $G \in \mathcal{Z}_r(\mathcal{A})$, for all $T \in \mathcal{A}$ we have $[G, T] \in \operatorname{rad}(\mathcal{A})$. But this implies that $G_iT_i - T_iG_i = 0$ for all $T_i \in M_{n_i}(\mathbb{C})$, that is G_i is in the center of $M_{n_i}(\mathbb{C})$, and is an idempotent. It follows therefore, that $G = E_i$, and hence $E_i \in \mathcal{M}\big(\mathcal{Z}_r(\mathcal{A})\big)$.

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2. **Characterizations of $\mathcal{M}(\mathcal{A})$ and the diagonal representations.** Recall (Theorem 4.3.4 of [Rin]) that if T is a compact linear operator acting on a complex Banach space X, then there exists a simple chain, $\mathcal{N}$, of closed, T-invariant subspaces. To each $M \in \mathcal{N}$ is associated an α_M, the diagonal coefficient of T at M. For a given T, we wish to view $\alpha_M(T)$ as the operator induced by T on M/M_-, a subspace of dimension at most one.

In Theorem 4.3.10 of [Rin], Ringrose shows that under the assumptions above, $\lambda \neq 0$ is an eigenvalue of T if and only if it is a diagonal coefficient of T. Furthermore, if we consider the nest algebra in $\mathcal{K}(X)$ associated to $\mathcal{N}$, $\mathrm{Alg}(\mathcal{N})$, then for $\alpha_M(T)$ defined as above, $\alpha_M\colon \mathrm{Alg}(\mathcal{N}) \longrightarrow \mathbb{C}$ is a one-dimensional representation on $\mathrm{Alg}(\mathcal{N})$, for each $M \in \mathcal{N}$.

We establish the analogous notion in the case where $\mathcal{A}$ is a norm-closed subalgebra of $\mathcal{K}(X)$ with a complete maximal chain of $\mathcal{A}$-invariant subspaces, $\mathcal{N}$, which is elementary.

DEFINITION 2.1. Let $\mathcal{A}$ be a subalgebra of $\mathcal{K}(X)$ with a maximal chain of closed $\mathcal{A}$-invariant subspaces, $\mathcal{N}$, which is elementary. Denote by $\mathcal{N}^*$ the elements $M \in \mathcal{N}$ such that $\dim(M/M_-) \neq 0$. For $M \in \mathcal{N}^*$, define $\pi_M\colon \mathcal{A} \longrightarrow B(M/M_-)$ as in Lemma 1.4. For $M \in \mathcal{N}^*$, let $\alpha_M(T) = \mathrm{sp}_{\pi_M(\mathcal{A})}\,\pi_M(T)$, while if $M = M_-$, let $\alpha_M(T) = \{0\}$. We call $\alpha_M(T)$ the *spectral coefficients* of T at M. For $M \in \mathcal{N}^*$, we will call $\pi_M(T)$ the *representation* of T at M, and the collection $\{\pi_M(T)\colon M \in \mathcal{N}^*\}$ the *diagonal representations of* T.

It is well known that, if $\mathcal{B}$ is a Banach algebra, then for each $T \in \mathcal{B}$,

$$\{0\} \cup \mathrm{sp}_{\mathcal{B}}(T) = \bigcup\Big\{\mathrm{sp}_{\pi(\mathcal{B})}\big(\pi(T)\big) \,:\, \pi \text{ is a strictly irreducible representation}\Big\} \cup \{0\}.$$

We wish to establish that any strictly irreducible representation of $\mathcal{A}$ is equivalent to π_M for some $M \in \mathcal{N}^*$, hence that

$$\{0\} \cup \mathrm{sp}_{\mathcal{A}}(T) = \bigcup\Big\{\mathrm{sp}_{\pi_M(\mathcal{A})}\big(\pi_M(T)\big) \,:\, M \in \mathcal{N}^*\Big\} \cup \{0\}.$$

We begin with an alternative characterization of $\mathcal{M}(\mathcal{A})$ in the following proposition, the proof of which, since G is finite rank, is straightforward.

PROPOSITION 2.3. *$G \in \mathcal{M}(\mathcal{A})$ if and only if G is a non-zero idempotent and*

$$G\mathcal{A}G \subseteq \mathbb{C}G + \mathrm{rad}(\mathcal{A}).$$

With this characterization our definition of minimal idempotent agrees, in the case where $\mathcal{A}$ is semisimple, with the other more standard analytic definition (see *e.g.* Section 30 of [BD]). It will also allow us to exploit some unpublished work on minimal idempotents supplied by Dr. Bruce A. Barnes (see Lemma 2.6 and Theorem 2.8 below). As preliminaries, we need the following proposition whose proof is also straightforward.

PROPOSITION 2.4. *Let $\mathcal{A}$ be a subalgebra of $\mathcal{K}(X)$, and suppose that there exists an element $G \in \mathcal{A}$ such that $G^2 = G \neq 0$. Then*

i) there exists a non-zero $F \in \mathcal{M}(\mathcal{A})$ such that $FG = GF = F$ and *ii) there exist* $\{E_k\}_1^n \subseteq \mathcal{M}(\mathcal{A})$ *such that* $E_iE_j = 0$ *if* $i \neq j$, *and* $G = E_1 + E_2 + \cdots + E_n$.

COROLLARY 2.5. *For* $\mathcal{A}$ *a closed subalgebra of* $\mathcal{K}(X)$, *if* $\pi: \mathcal{A} \longrightarrow \mathcal{L}(Y)$ *is a strictly irreducible representation on a linear space* Y, *then there exists* $G \in \mathcal{M}(\mathcal{A})$ *such that* $\pi(G) \neq 0$.

PROOF. Note that if T is quasinilpotent, then so is $\pi(T)$. Since the image of $\mathcal{A}$ is semisimple, it follows that $\pi(T) = 0$. Choose $T \in \mathcal{A}$ such that $\pi(T) \neq 0$, and note that the spectral radius $\rho\big(\pi(T)\big) = \rho \neq 0$. From the spectral theory for compact operators there exists a spectral idempotent $P \in \mathcal{A}$ such that

$$P = \frac{1}{2\pi i}\int_\gamma (\lambda - T)^{-1} d\lambda \quad \text{with} \quad \pi(P) = \frac{1}{2\pi i}\int_\gamma \big(\lambda - \pi(T)\big)^{-1} d\lambda \neq 0.$$

The result now follows from the above. ∎

LEMMA 2.6. *Let* $\mathcal{A}$ *be a Banach algebra in which* $\mathcal{M}(\mathcal{A})$ *is not empty. Then*

i) If $\mathcal{L}$ *is a left ideal of* $\mathcal{A}$ *such that* $\mathcal{L}^2 \subseteq \mathrm{rad}(\mathcal{A})$, *then* $\mathcal{L} \subseteq \mathrm{rad}(\mathcal{A})$.

ii) If $\mathcal{L}$ *is a left ideal of* $\mathcal{A}$, *and* $E \in \mathcal{M}(\mathcal{A})$, *such that* $\mathcal{A}E \cap \mathrm{rad}(\mathcal{A}) \subseteq \mathcal{L} \subseteq \mathcal{A}E$, *then either* $\mathcal{L} = \mathcal{A}E \cap \mathrm{rad}(\mathcal{A})$, *or* $\mathcal{L} = \mathcal{A}E$.

iii) For $E \in \mathcal{M}(\mathcal{A})$, *the left regular representation of* $\mathcal{A}$ *on* $\mathcal{A}E/\mathcal{A}E \cap \mathrm{rad}(\mathcal{A})$ *is strictly irreducible.*

PROOF. i) Using Theorem 2.3.2 i) of [Ric], if $\mathcal{L}^2 \subseteq \mathrm{rad}(\mathcal{A})$, then $\mathcal{L}^2$ is contained in every primitive ideal. Now using Theorem 2.2.9iv) of [Ric], this implies that $\mathcal{L} \subseteq P$ for every primitive ideal P.

ii) Suppose that $\mathcal{L} \neq \mathcal{A}E \cap \mathrm{rad}(\mathcal{A})$. Then from i) it follows that $\mathcal{L}^2 \nsubseteq \mathrm{rad}(\mathcal{A})$. So we can choose TE and SE in $\mathcal{L}$ such that $TESE \notin \mathrm{rad}(\mathcal{A})$, hence $ESE \notin \mathrm{rad}(\mathcal{A})$. Furthermore, $ESE = \lambda E + R$, for some complex number $\lambda \neq 0$ and some $R \in \mathrm{rad}(\mathcal{A})$. But $ESE = \lambda E + RE$ from which it follows that $E = \lambda^{-1}(ESE - RE)$. But then $E \in \mathcal{L}$, and hence $\mathcal{L} = \mathcal{A}E$.

iii) With the action of $\mathcal{A}$ on $\mathcal{A}E/\mathcal{A}E \cap \mathrm{rad}(\mathcal{A})$ given by

$$S\big(T + \mathcal{A}E \cap \mathrm{rad}(\mathcal{A})\big) = ST + \mathcal{A}E \cap \mathrm{rad}(\mathcal{A}),$$

strict irreducibility is an immediate consequence of ii) above. ∎

Recall that if $\rho: \mathcal{A} \longrightarrow \mathcal{L}(X)$ and $\pi: \mathcal{A} \longrightarrow \mathcal{L}(Y)$ are algebra homomorphisms of $\mathcal{A}$ into the set of linear maps of X and Y respectively, then we say that ρ and π are *equivalent representations*, denoted $\rho \equiv \pi$, if there exists a linear bijection $W: X \longrightarrow Y$ such that, $W\rho(T) = \pi(T)W$ for all $T \in \mathcal{A}$.

Given $E \in \mathcal{M}(\mathcal{A})$, let $X_E = \mathcal{A}E/\mathcal{A}E \cap \mathrm{rad}(\mathcal{A})$. Denote by π_E the left regular representation of $\mathcal{A}$ on X_E.

THEOREM 2.7. *Let* π *be a strictly irreducible representation of* $\mathcal{A}$ *on a linear space* Y. *If* $\pi(E) \neq 0$ *for some* $E \in \mathcal{M}(\mathcal{A})$, *then* $\pi \equiv \pi_E$.

PROOF. Note first that the operator $\pi(E)$ is a non-zero projection. Choose a non-zero $y_0 \in Y$ such that $\pi(E)y_0 = y_0$. From the strict irreducibility of π, it follows that

$Y = \{\pi(T)y_0 : T \in \mathcal{A}\}$. Define the linear map $V: \mathcal{A}E \longrightarrow Y$ by

$$V(TE) = \pi(TE)y_0 = \pi(T)y_0.$$

Then $V(\mathcal{A}E) = Y$. Let $L = \{S \in \mathcal{A}E : V(S) = 0\}$, so that L is a left ideal in $\mathcal{A}$. Notice that if $R \in \mathrm{rad}(\mathcal{A})$, then $\pi(R) = 0$, as π is a strictly irreducible representation. Therefore $\mathcal{A}E \cap \mathrm{rad}(\mathcal{A}) \subseteq L \subseteq \mathcal{A}E$. As L is properly contained in $\mathcal{A}E$, it follows from Lemma 2.6 ii) above that $L = \mathcal{A}E \cap \mathrm{rad}(\mathcal{A})$.

Define $W: X_E \longrightarrow Y$ by

$$W\big(TE + \mathcal{A}E \cap \mathrm{rad}(\mathcal{A})\big) = V(TE).$$

With this definition it now follows easily that W is bijective, and that, for all $T \in \mathcal{A}$, $W\pi_E(T) = \pi(T)W$. ∎

We now apply these results in

THEOREM 2.8. *Let $\mathcal{A}$ be a closed subalgebra of $\mathcal{K}(X)$ and $\mathcal{N}$ a complete maximal $\mathcal{A}$-invariant chain which is elementary. If $\pi: \mathcal{A} \longrightarrow \mathcal{L}(Y)$ is a strictly irreducible representation of $\mathcal{A}$ on a linear space Y, then π is equivalent to a representation on M/M_- for some $M \in \mathcal{N}^*$.*

PROOF. By Corollary 2.5, there exists $G \in \mathcal{M}(\mathcal{A})$ such that $\pi(G) \neq 0$. By Lemma 1.2, there exists a corresponding $M = M_G \in \mathcal{N}^*$, such that $0 < \dim(M/M_-)$, and this dimension is finite, as $\mathcal{N}$ is elementary. Therefore if ρ is the left regular representation of $\mathcal{A}$ on M/M_-, ρ is strictly irreducible, and by the construction of $M = M_G$ in Lemma 1.2, the image of G under ρ is not 0. Hence by Theorem 2.7 it follows that $\pi \equiv \pi_G \equiv \rho$. ∎

If $\mathcal{N}$ is an elementary chain then every strictly irreducible representation π of $\mathcal{A}$ is (equivalent to) a representation on a finite dimensional vector space.

COROLLARY 2.9. *Let $\mathcal{A}$ and $\mathcal{N}$ be as above. For $M \in \mathcal{N}^*$ let π_M be the left regular representation of $\mathcal{A}$ on M/M_-. Then for all $T \in \mathcal{A}$,*

$$\{0\} \cup \mathrm{sp}_{\mathcal{A}}(T) = \bigcup \big\{\mathrm{sp}_{\pi_M(\mathcal{A})}\big(\pi_M(T)\big) : M \in \mathcal{N}^*\big\} \cup \{0\}.$$

As a particular case we have the following

COROLLARY 2.10 (RINGROSE, 1962). *Let $T \in \mathcal{K}(X)$ and let $\mathcal{N}$ be a complete simple T-invariant chain. Then the eigenvalues of T, with the possible exception of 0, are precisely the diagonal coefficients $\{\alpha_M(T) : M \in \mathcal{N}\}$.*

COROLLARY 2.11. *Let $\mathcal{A}$ and $\mathcal{N}$ be as in Theorem 2.8. Then the Jacobson radical and the Brown-McCoy radical coincide.*

3. **Elementary chains and polynomial identity algebras.** At this point we would like to explore the relationship between the type of subalgebras $\mathcal{A}$ of the compact operators acting on a Banach space that we have been considering, and the theory of Polynomial Identity Algebras. In particular, we would like to generalize a result due to Murphy, ([Mur], Theorem 1), which states that a closed subalgebra $\mathcal{A}$ of the compact operators $\mathcal{K}(X)$ has a complete maximal chain which is simple if and only if the algebra $\mathcal{A}/\operatorname{rad}(\mathcal{A})$ is commutative.

DEFINITION 3.1. We say that an n-tuple $(T_1, T_2, \ldots, T_n)$ satisfies a non-commuting polynomial P whenever it satisfies $P(T_1, T_2, \ldots, T_n) = 0$. Similarly, the algebra $\mathcal{A}$ itself is said to satisfy the polynomial identity P if any ordered n-tuple $(T_1, T_2, \ldots, T_n)$ of elements from $\mathcal{A}$ satisfies P. The algebra $\mathcal{A}$ is said to satisfy a polynomial identity of degree m if it satisfies some non-commuting polynomial of degree m. Finally, an algebra $\mathcal{A}$ is said to be a PI-algebra if $\mathcal{A}$ satisfies a multilinear polynomial identity all of whose coefficients are ± 1.

Throughout this section, S_n will denote the symmetric group on n elements, that is, the group of permutations of the set $\{1, 2, \ldots, n\}$. For each element $\tau \in S_n$, $\operatorname{sgn}(\tau) = \pm 1$, according as τ is an even or odd permutation.

The standard polynomial in n non-commuting variables is defined by

$$P_n(x_1, x_2, \ldots, x_n) = \sum_{\tau \in S_n} \operatorname{sgn}(\tau) x_{\tau(1)} x_{\tau(2)} \cdots x_{\tau(n)},$$

and will also be denoted $[x_1, x_2, \ldots, x_n]$, generalizing the Lie bracket notation.

Notice that $[x_1, x_2, \ldots, x_n]$ is an n-linear, alternating polynomial of degree n, so that an algebra $\mathcal{A}$ satisfies $[x_1, x_2, \ldots, x_n]$ if and only if every set of n distinct elements chosen from a basis for $\mathcal{A}$ is a root of the polynomial. Furthermore, if $\mathcal{A}$ satisfies P_n, then $\mathcal{A}$ satisfies P_{n+k}, for all natural numbers k.

We will need several results from the theory of PI-algebras, which we will state without proof. The first result is due to Shimon Amitsur and Jacob Levitzki (1950), and can be found in Theorems 1.4.1 and 1.4.5 of [Row].

THEOREM 3.2 (AMITSUR-LEVITZKI, 1950). *The algebra $M_n(\mathbb{C})$ satisfies the standard polynomial P_{2n} of degree $2n$ and does not satisfy any polynomial identity of degree less than $2n$.*

A fundamental result in the theory is due to Irving Kaplansky and is stated as Theorem X.5.1 of [Jac].

THEOREM 3.3 (KAPLANSKY, 1948). *Let $\mathcal{A}$ be a primitive algebra over a field k which satisfies a polynomial identity of degree n. Then the center of $\mathcal{A}$ is a field and $\mathcal{A}$ has dimension over its center less than or equal to $(\frac{n}{2})^2$.*

In the development we will need the following immediate corollary to this theorem, which follows from the Gelfand-Mazur theorem.

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COROLLARY 3.4. *If $\mathcal{A}$ is a primitive Banach algebra satisfying a polynomial identity of degree n, then $\mathcal{A}$ is finite dimensional.*

As the algebras that we are most concerned with are non-unital subalgebras of the compact operators on an infinite dimensional Banach space, we make a slight alteration of a definition found in Section 22 of [Krup].

DEFINITION 3.5. An algebra $\mathcal{A}$ over the complex field will be said to possess a sufficient family of n-dimensional representations if there exists a collection of non-zero algebra homomorphisms $\Phi = \{\varphi_\alpha\}_{\alpha\in\mathrm{A}}$,

$$\varphi_\alpha : \mathcal{A} \longrightarrow \mathrm{M}_k(\mathbb{C}), \quad k = k(\alpha) \leq n,$$

such that $S \in QI(\mathcal{A})$ if and only if $\det[I - \varphi_\alpha(S)] \neq 0$ for all $\alpha \in \mathrm{A}$.

Notice that $T \in \mathrm{Inv}(\mathcal{A}_u)$ if and only if $\det[\varphi_\alpha(T)] \neq 0$ for all $\alpha \in \mathrm{A}$ whenever $\Phi = \{\varphi_\alpha\}_{\alpha\in\mathrm{A}}$ is a sufficient family of n-dimensional representations on $\mathcal{A}$.

LEMMA 3.6. *If $\mathcal{A}$ is a Banach algebra which has a sufficient family of n-dimensional representations, then $\mathcal{A}/\operatorname{rad}(\mathcal{A})$ satisfies the standard polynomial of degree $2n$.*

PROOF. First, if φ is an algebra homomorphism on a non-unital algebra $\mathcal{A}$ into the matrix algebra $M_k(\mathbb{C})$, then we extend φ to the unitization of $\mathcal{A}$, $\mathcal{A}_u$, by defining $\varphi(\lambda 1 + T) = \lambda I + \varphi(T)$, where I is the identity in $M_k(\mathbb{C})$.

Now let $\Phi = \{\varphi_\alpha\}_{\alpha\in\mathrm{A}}$ be a sufficient family of n-dimensional representations on $\mathcal{A}$. For all $T \in \bigcap\{\ker(\varphi_\alpha) : \alpha \in \mathrm{A}\}$, if $\lambda \neq 0$, then $\det[\lambda I - \varphi_\alpha(T)] \neq 0$, for all $\alpha \in \mathrm{A}$, so that $\bigcap\{\ker(\varphi_\alpha) : \alpha \in \mathrm{A}\} \subseteq \operatorname{rad}(\mathcal{A})$. While by Amitsur-Levitzki, for all $\varphi_\alpha \in \Phi$, and for all ordered $2n$-tuples, $(T_1, T_2, \ldots, T_{2n})$ chosen from $\mathcal{A}$, $[T_1, T_2, \ldots, T_{2n}] \in \bigcap\{\ker(\varphi_\alpha) : \alpha \in \mathrm{A}\} \subseteq \operatorname{rad}(\mathcal{A})$. ∎

THEOREM 3.7. *Let $\mathcal{A}$ be a closed subalgebra of $\mathcal{K}(X)$, with $\mathcal{N}$ a complete maximal $\mathcal{A}$-invariant chain. Then $\mathcal{N}$ is a bounded elementary chain with bound n if and only if $\mathcal{A}/\operatorname{rad}(\mathcal{A})$ satisfies $[x_1, x_2, \ldots, x_{2n}]$.*

Before we begin the proof of the theorem, we return to material developed above. For $\mathcal{A}$ as in the theorem and $\mathcal{N}$ a complete maximal $\mathcal{A}$-invariant chain we define the subcollection $\mathcal{N}^* = \{M \in \mathcal{N} : \dim(M/M_-) > 0\}$. Recall that the associated maps π_M and Π_M defined in Lemma 1.4 and Theorem 1.5 are irreducible representations of $\mathcal{A}$.

LEMMA 3.8. *Let π_M be the representation defined above and $\mathcal{K} = \ker(\pi_M)$. Then $\mathcal{A}/\mathcal{K}$ is a i) prime, ii) semi-simple, iii) modular annihilator algebra.*

PROOF. For i), suppose that $\tilde{I}, \tilde{J} \lhd \mathcal{A}/\mathcal{K}$ such that $\tilde{I}\tilde{J} = 0 \in \mathcal{A}/\mathcal{K}$. Let $I = \{T \in \mathcal{A} : T + \mathcal{K} \in \tilde{I}\}$, the pullback of I under the canonical homomorphism $\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{K}$, and similarly for J. Then $I, J \lhd \mathcal{A}$, and $\pi_M(IJ) = \{0\}$. But if $\pi_M(J)(M/M_-) \neq \{0\}$, then since π_M is an irreducible representation and J is an ideal in $\mathcal{A}$, it follows that the norm closure $\overline{\pi_M(J)(M/M_-)} = M/M_-$. Hence since $\pi_M(I)(M/M_-) = \pi_M(I)\overline{\pi_M(J)(M/M_-)} = \{0\}$, we must have $\pi_M(I) = \{0\}$, *i.e.*, $I \subseteq \ker(\pi_M) = \mathcal{K}$. Consequently $\tilde{I} = \{0\}$.

For ii), note that $\mathcal{A}/\mathcal{K}$ is semiprime. Furthermore, $\mathcal{A}/\mathcal{K}$ is not a radical algebra. To see this, we may assume $\dim(M/M_-) > 1$. If $\mathcal{A}/\mathcal{K}$ is a radical algebra, using Corollary 3.5 of [LNRR], it follows that there exists a simple chain of closed $\mathcal{A}/\mathcal{K}$-invariant subspaces of M/M_-, contradicting the maximality of $\mathcal{N}$. Consequently, by Proposition 32.5 of [BD], the socle exists and is non-zero. So from i) above it follows that $\mathrm{rad}(\mathcal{A}/\mathcal{K}) = \{0\}$.

For iii), we use Theorem 4.2 of [Bar2] to see that $\mathcal{A}/\mathcal{K}$ is a modular annihilator algebra. ∎

As established in Definition 3.1 of [Bar1], given $E \in \mathcal{M}(\mathcal{A})$, $\mathcal{A}E$ is a minimal left ideal and $\mathcal{A}(1-E)$ is a maximal modular left ideal of $\mathcal{A}$. Consequently $P^E = \{T \in \mathcal{A} : T\mathcal{A} \subseteq \mathcal{A}(1-E)\}$ is a primitive ideal. Furthermore Barnes proves in Proposition 3.1 that $P^E = \mathrm{lan}(\mathcal{A}E) = \mathrm{ran}(E\mathcal{A})$.

LEMMA 3.9. *For $\mathcal{A}$ and $\mathcal{K}$ as above, $\mathcal{A}/\mathcal{K}$ is a primitive algebra.*

PROOF. We temporarily denote $\mathcal{A}/\mathcal{K}$ by $\tilde{\mathcal{A}}$. Then from above $\tilde{\mathcal{A}}$ is a modular annihilator algebra, which is prime. Furthermore, as in Section 3 of [Bar1], since $\mathrm{soc}(\tilde{\mathcal{A}})$ exists and is non-zero, there exists $E \in \mathcal{M}(\tilde{\mathcal{A}})$. Then from above, $P^E(\tilde{\mathcal{A}}E) = \{0\}$ implies that $P^E(\tilde{\mathcal{A}}E\tilde{\mathcal{A}}) = \{0\}$, hence $P^E = \{0\}$. ∎

PROOF OF THEOREM 3.7. Suppose that $\mathcal{N}$ is a bounded elementary chain with bound n. For each $M \in \mathcal{N}^*$, let π_M denote the associated representation as above. It is clear that $\Phi = \{\pi_M : M \in \mathcal{N}^*\}$ is a sufficient family of representations. Consequently by Lemma 3.6, it follows that $\mathcal{A}/\mathrm{rad}(\mathcal{A})$ satisfies $[x_1, x_2, \ldots, x_{2n}]$.

Conversely, suppose that $\mathcal{A}/\mathrm{rad}(\mathcal{A})$ satisfies $[x_1, x_2, \ldots, x_{2n}]$, so that for all $\{T_i\}_1^{2n} \subseteq \mathcal{A}$, $[T_1, T_2, \ldots, T_{2n}] \in \mathrm{rad}(\mathcal{A})$. To see that $\mathcal{N}$ is an elementary chain, assume that there is an $M \in \mathcal{N}$ such that $\dim(M/M_-) = \infty$. Using Corollary 1.5.28 of [Row], we see that

$$\mathrm{rad}(\mathcal{A})/\mathrm{rad}(\mathcal{K}) = \mathrm{rad}(\mathcal{A})/[\mathrm{rad}(\mathcal{A}) \cap \mathcal{K}] \cong [\mathrm{rad}(\mathcal{A}) + \mathcal{K}]/\mathcal{K} \subseteq \mathrm{rad}(\mathcal{A}/\mathcal{K}).$$

Since $\mathrm{rad}(\mathcal{A}/\mathcal{K}) = \{0\}$ by Lemma 3.8, $\mathrm{rad}(\mathcal{A}) = \mathrm{rad}(\mathcal{K})$ from which it follows that

$$[T_1 + \mathcal{K}, T_2 + \mathcal{K}, \ldots, T_{2n} + \mathcal{K}] = [T_1, T_2, \ldots, T_{2n}] + \mathcal{K} \in \mathrm{rad}(\mathcal{A}) + \mathcal{K} \subseteq \mathcal{K},$$

that is, $\mathcal{A}/\ker(\pi_M)$, satisfies $[x_1, x_2, \ldots, x_{2n}]$.

By the corollary to Kaplansky's theorem it follows that $\mathcal{A}/\ker(\pi_M)$ is finite dimensional, contradicting the assumption that $\dim(M/M_-) = \infty$. We conclude that $\mathcal{N}$ is an elementary chain.

Finally, that $\mathcal{N}$ is a bounded elementary chain with bound n, follows from Amitsur-Levitzki. ∎

Recall that an algebra $\mathcal{A}$ of operators on a Banach space X is simultaneously triangularizable if there exists a maximal chain of closed subspaces of X invariant under all elements of $\mathcal{A}$.

COROLLARY 3.10 (MURPHY). *Let $\mathcal{A}$ be a closed subalgebra of the compact operators acting on a Banach space X. Then $\mathcal{A}$ is simultaneously triangularizable if and only if $\mathcal{A}/\,\mathrm{rad}(\mathcal{A})$ is commutative.*

PROOF. Lemma 4.3.3 of [Rin]. ∎

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