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ON DEGENERATE SIGMA-FUNCTIONS IN GENUS 2

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Abstract. We obtain explicit expressions for genus 2 degenerate sigma-function in terms of genus 1 sigma-function and elementary functions as solutions of a system of linear partial differential equations satisfied by the sigma-function. By way of application, we derive a solution for a class of generalized Jacobi inversion problems on elliptic curves, a family of Schrödinger-type operators on a line with common spectrum consisting of a point and two segments, explicit construction of a field of three-periodic meromorphic functions. Generators of rank 3 lattice in $\mathbb{C}^2$ are given explicitly.

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1. Introduction. The concept of a sigma-function in higher genus was introduced by Klein [**1**] in 1886 as an extensive generalization of elliptic Weierstrass sigma-function [**2**]. The importance of a sigma-function lies in the fact that it is a convenient generator of Abelian functions in g complex variables, i.e., meromorphic multiply periodic functions that possess the maximal number $2g$ of periods. From this viewpoint, sigma-function in genus 2 was studied since the time of Klein, and the classical results were well documented in Baker monograph [**3**].

Theory of sigma-functions progresses in several ways. The first approach considers a generalization called in [**4**] Kleinian sigma-function $\sigma(u) = \exp(\frac{1}{2}u^t\eta\omega^{-1}u)\theta(\omega^{-1}u; \omega, \omega')$, which is a modular invariant representative of the class of theta functions[1] with $u \in \mathbb{C}^g$. Dealing with this expression for the multivariate sigma-function the authors have examined fields of Abelian functions associated with hyperelliptic curves, described Jacobi and Kummer varieties as algebraic varieties, developed contemporary applications of Abelian functions to completely integrable equations of theoretical and mathematical physics. Further, theoretical developments in study of sigma, its relation to tau function, and some algebraic expressions are achieved in [**5**, **6**] and further papers. Modular definition of sigma-function for generic algebraic curves is proposed in [**7**], whilst the case of (n, s)-curves and, in particular, of hyperelliptic curves is studied in this context in [**8**].

Another approach, aimed to construct series expansions for multivariate sigma-functions by means of the operator algebras anihilating these functions. This theory

[1]Here, Riemann θ-function is used. As usual, ω and η denote first and second kind integrals along $\mathfrak{a}$-cycles, and ω' denotes first kind integrals along $\mathfrak{b}$-cycles.

is developed in the sequential papers [**9**, **10**, **11**, **12**], and gives many byproducts. For example, the canonical basis of second kind differentials associated to the first kind differentials are constructed without introducing Kleinian bi-differential unlike the first approach; in their turn the second kind differentials can be used for constructing the bi-differential. Algebraic identities and addition laws in the field of Abelian functions are easily obtained from a sigma-series expansion, as well as Hirota bilinear equations for integrable systems associated with a curve. The theory is applicable for both hyper- and non-hyperellitic plane curves.

The equivariant approach is based on covariance with respect to transformations of an algebraic curve, see, for example, [**13**]. For hyperelliptic curves, the transformations are induced by SL(2,$\mathbb{C}$) action. Identities for multivariate Abelian functions arise as finite-dimensional irreducible representations of the corresponding algebra $\mathfrak{sl}(2,\mathbb{C})$; and the covariant form of Kleinian bi-differential is constructed.

There are also computational approach, where series expansions for multivariate sigma-functions in higher genera and algebraic identities between the corresponding Abelian functions are constructed by numerical methods independently of the second approach, but partly inspired by it, see, for example, [**14**]. Series for some sigma-functions in genera 3, 4, 6 were obtained as well as identities for Abelian functions $\wp_{[k]}(u) = -\partial^{[k]}\log\sigma(u)$.

The present paper is essentially based on the theory of operator algebras anihilating multivariative sigma-functions, mostly on the paper [**11**]. The theory originates from Weierstrass's definition [**2**] of sigma-function as the entire function depending on three variables $(u; g_2, g_3) \in \mathbb{C}\times\mathbb{C}^2$ and satisfying the set of differential equations

$$Q_0(\sigma) = 0, \qquad Q_2(\sigma) = 0, \tag{1}$$
$$Q_0(\sigma) = -u\sigma_u + 4g_2\sigma_{g_2} + 6g_3\sigma_{g_3} + \sigma,$$
$$Q_2(\sigma) = -\tfrac{1}{2}\sigma_{u,u} - \tfrac{1}{24}g_2u^2\sigma + 6g_3\sigma_{g_2} + \tfrac{1}{3}g_2^2\sigma_{g_3},$$

with initial condition $\sigma(u; 0; 0) = u$; here g_2 and g_3 are parameters of Weierstrass elliptic curve $y^2 = 4x^3 - g_2x - g_3$. A multivariative sigma-function in genus g is an entire function $\sigma(u;\lambda)$ of $3g - m$ complex variables $(u;\lambda) \in \mathbb{C}^g \times \mathbb{C}^{2g-m}$, where m is modality (in the hyperelliptic case, $m = 0$, for more details see [**12**]). The multivariate sigma-function is defined by a set of heat equations [**11**] similar to (1) with an initial condition in the form of so called Schur–Weierstrass polynomial [**9**].

In this paper, we study cases when sigma-function in genus 2 can be represented as an aggregate of sigma-function in genus 1 and elementary functions. For $g = 2$, sigma-function depends on six variables $(u;\lambda) \in \mathbb{C}^2 \times \mathbb{C}^4$, where λ is the set of parameters of a genus 2 curve

$$y^2 = x^5 + \lambda_4x^3 + \lambda_6x^2 + \lambda_8x + \lambda_{10}. \tag{2}$$

In fact, curve (2) has *actual genus* 2 only if a certain constraint is imposed on λ. We say that the curve has *virtual genus* 2 and focus on the cases when its actual genus is lower. That is the cases when genus 2 sigma-function can be expressed in terms of genus 1 sigma-function and elementary functions, we call it a degenerate sigma-function. Strata of the space of parameters λ corresponding to fixed actual genera of (2) are analyzed in Sections 3 and 4. Then, we proceed to our main question by carrying out an analysis of the system of linear partial differential equations, so called heat equations in a non-holonomic frame [**11**], that are satisfied by sigma-function in Section 5, and derive our main result in Section 6.

In Section 7, we apply the result to a few selected problems: solution of a generalized Jacobi inversion problem, construction of a Schrödinger-type operator with spectrum composed of two segments and a point, description of the structure of a field of three periodic functions in two complex variables. In the course of our research, we also obtain a stratification of the space of parameters of genus 2 curves (2) with respect to the rank of a period lattice corresponding to the curve.

We emphasize, that our approach is based on the aforementioned operator algebras, and does not appeal to theta-functions. The paper aims to show that one can stay within the Weierstrass construction dealing with higher genera hyperelliptic (and also non-hyperelliptic) curves, which is the subject of our further investigations.

2. Notation. Below in the paper, we consider the space $\mathcal{C}$ of genus 2 curves of the form

$$x^5 - y^2 + \lambda_4 x^3 + \lambda_6 x^2 + \lambda_8 x + \lambda_{10} = 0. \tag{3}$$

The parameters $\lambda = (\lambda_4, \lambda_6, \lambda_8, \lambda_{10})$ run over $\mathbb{C}^4$. Genus 2 sigma-function is denoted by $\boldsymbol{\sigma}(u; \lambda)$, where $u = (u_3, u_1)$. We assign Satō weights to the variables by the rule $\deg \lambda_i = i$, $\deg u_k = -k$. Accordingly, with $\deg x = 2$ and $\deg y = 5$, the left-hand side of (3) is homogeneous of weight 10. It is important, that most of functions and operators appearing below respect Satō weights, in particular, $\deg \boldsymbol{\sigma}(u; \lambda) = -3$.

In what follows, we also deal with a family of genus 1 curves

$$X^3 - Y^2 + \gamma_4 X + \gamma_6 = 0, \tag{4}$$

here, $\deg X = 2$ and $\deg Y = 3$. To avoid confusion, we denote the corresponding genus 1 sigma-function by $\sigma(u_1)$, which stands for standard Weierstrass sigma-function with invariants $(g_2, g_3) = (-4\gamma_4, -4\gamma_6)$.

For brevity, we use the notation ∂_x in the place of $\partial/\partial x$.

3. Stratification of the space of parameters. The space of parameters Λ is naturally stratified into three strata: Λ_2, Λ_1 and Λ_0 that correspond to curves in genus $g = 2$, 1 and 0, respectively.

Proposition 1. *The space Λ is a disjoint union $\Lambda_2 \cup \Lambda_1 \cup \Lambda_0$ and*

$$\begin{aligned}
\Lambda_2 &= \{\lambda \in \mathbb{C}^4 \mid \Delta(\lambda) \neq 0\},\\
\Lambda_1 &= \{\lambda \in \mathbb{C}^4 \mid \Delta(\lambda) = 0,\ \Gamma(\lambda) \neq 0\},\\
\Lambda_0 &= \{\lambda \in \mathbb{C}^4 \mid \Gamma(\lambda) = 0\},
\end{aligned}$$

where

$$\begin{aligned}
\Delta(\lambda) = {} & 3125\lambda_{10}^4 - 3750\lambda_{10}^3\lambda_6\lambda_4 + 2000\lambda_{10}^2\lambda_8^2\lambda_4 + 2250\lambda_{10}^2\lambda_8\lambda_6^2\\
& - 1600\lambda_{10}\lambda_8^3\lambda_6 + 256\lambda_8^5 - 900\lambda_{10}^2\lambda_8\lambda_4^3 + 825\lambda_{10}^2\lambda_6^2\lambda_4^2 + 560\lambda_{10}\lambda_8^2\lambda_6\lambda_4^2\\
& - 630\lambda_{10}\lambda_8\lambda_6^3\lambda_4 + 108\lambda_{10}\lambda_6^5 - 128\lambda_8^4\lambda_4^2 + 144\lambda_8^3\lambda_6^2\lambda_4 - 27\lambda_8^2\lambda_6^4\\
& + \big(108\lambda_{10}^2\lambda_4^5 - 72\lambda_{10}\lambda_8\lambda_6\lambda_4^4 + 16\lambda_{10}\lambda_6^3\lambda_4^3 + 16\lambda_8^3\lambda_4^4 - 4\lambda_8^2\lambda_6^2\lambda_4^3,
\end{aligned} \tag{5}$$

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and

$$\Gamma(\lambda) = \begin{pmatrix} 50\lambda_{10}\lambda_6 - 80\lambda_8^2 + 36\lambda_8\lambda_4^2 - 27\lambda_6^2\lambda_4 - 4\lambda_4^4 \\ 200\lambda_{10}\lambda_8 - 40\lambda_{10}\lambda_4^2 - 36\lambda_8\lambda_6\lambda_4 + 27\lambda_6^3 + 4\lambda_6\lambda_4^3 \\ 625\lambda_{10}^2 - 720\lambda_8^2\lambda_4 + 135\lambda_8\lambda_6^2 + 308\lambda_8\lambda_4^3 - 216\lambda_6^2\lambda_4^2 - 32\lambda_4^5 \\ 1600\lambda_8^3 - 1040\lambda_8^2\lambda_4^2 + 360\lambda_8\lambda_6^2\lambda_4 + 135\lambda_6^4 + 224\lambda_8\lambda_4^4 - 88\lambda_6^2\lambda_4^3 - 16\lambda_4^6 \end{pmatrix}.$$

Proof. Consider a curve (3) with at least one double point at $(x, y) = (a_2, 0)$. It has the form

$$-y^2 + (x - a_2)^2\big(x^3 + 2a_2x^2 + \mu_4x + \mu_6\big) = 0. \tag{6}$$

By subtracting (6) from (3) and collecting coefficients at the power of x, we find the following polynomials in $(\lambda_{10}, \lambda_8, \lambda_6, \lambda_4; \mu_6, \mu_4, a_2)$:

$$\Upsilon(\lambda; \mu, a_2) = \begin{pmatrix} \lambda_4 - \big(\mu_4 - 3a_2^2\big) \\ \lambda_6 - \big(\mu_6 - 2a_2\mu_4 + 2a_2^3\big) \\ \lambda_8 - \big(-2a_2\mu_6 + a_2^2\mu_4\big) \\ \lambda_{10} - a_2^2\mu_6 \end{pmatrix}. \tag{7}$$

The polynomials vanish whenever a curve (3) has the form (6), that is the curve has genus not greater than 1, equivalently $\lambda \in \Lambda_1 \cup \Lambda_0$. The polynomials $\Upsilon(\lambda; \mu, a_2)$ generate an ideal $I_\Upsilon \subset \mathbb{C}[\lambda; \mu, a_2]$. Gröbner basis of $I_\Upsilon \cap \mathbb{C}[\lambda]$ is $\Delta(\lambda)$.

If $\delta(\mu, a_2) = 4\big(\mu_4 - \frac{4}{3}a_2^2\big)^3 + 27\big(\mu_6 - \frac{2}{3}a_2\mu_4 + \frac{16}{27}a_2^3\big)^2$ vanishes then the polynomial $x^3 + 2a_2x^2 + \mu_4x + \mu_6$, cf. (6), has a double root. This means the curve (6) has two double points and its genus is 0, equivalently $\lambda \in \Lambda_0$. The polynomials $\Upsilon(\lambda; \mu, a_2)$ and $\delta(\mu, a_2)$ generate an ideal $I_{(\Upsilon,\delta)} \subset \mathbb{C}[\lambda; \mu, a_2]$. Gröbner basis of $I_{(\Upsilon,\delta)} \cap \mathbb{C}[\lambda]$ is $\Gamma(\lambda)$.

To calculate Gröbner bases, we use Buchberger's method with lexicographic monomial order. □

REMARK 1. The polynomial $\Delta(\lambda)$ is in fact the discriminant of $x^5 + \lambda_4x^3 + \lambda_6x^2 + \lambda_8x + \lambda_{10}$, cf. (3), while the polynomial $\delta(\mu, a_2)$ is the discriminant of $x^3 + 2a_2x^2 + \mu_4x + \mu_6$, cf. (6).

Introduce variables γ_4, γ_6 by the formulas $\gamma_4 = \mu_4 - \frac{4}{3}a_2^2$ and $\gamma_6 = \mu_6 - \frac{2}{3}a_2\mu_4 + \frac{16}{27}a_2^3$. Then, the above polynomial $\delta(\mu, a_2)$ takes the form $\delta(\gamma) = 4\gamma_4^3 + 27\gamma_6^2$. In what follows, we shall need the following expressions

$$\mu_4 = \gamma_4 + \tfrac{4}{3}a_2^2, \qquad \mu_6 = \gamma_6 + \tfrac{2}{3}a_2\gamma_4 + \tfrac{8}{27}a_2^3. \tag{8}$$

Equations $\Upsilon(\lambda; \gamma, a_2) = 0$ with respect to (γ, a_2), here μ in (7) are replaced by γ according to (8), have no solution when $\lambda \in \Lambda_2$, a unique solution for (γ, a_2) when $\lambda \in \Lambda_1$, and two solutions when $\lambda \in \Lambda_0$. Indeed, if $\Delta(\lambda) \neq 0$ the equations are incompatible. Let $\Delta(\lambda) = 0$, suppose there exist two distinct points (γ, a_2) and (β, b_2) corresponding to the same point λ. Subtracting $\Upsilon(\lambda; \beta, b_2) = 0$ from $\Upsilon(\lambda; \gamma, a_2) = 0$ then eliminating $\gamma_4 - \beta_4$ and $\gamma_6 - \beta_6$, we come to a pair of algebraic equations of order five and four with respect to $t = a_2 - b_2$. These equations have a single common root $t=0$ iff $\delta(\gamma) \neq 0$, thus the points (γ, a_2) and (β, b_2) coincide. Now suppose, both $\Delta(\lambda)$ and $\delta(\gamma)$ vanish, then $(\gamma_6, \gamma_4) = (2t^3, -3t^2)$ for some value of $t \in \mathbb{C}$. The system $\Upsilon(\lambda; (2t^3, -3t^2), a_2) - \Upsilon(\lambda; (2s^3, -3s^2), b_2) = 0$ is satisfied by two solutions: $(s, b_2) = (t, a_2)$ and $(s, b_2) = \left(\frac{2}{3}t + \frac{5}{9}a_2,\ t - \frac{2}{3}a_2\right)$.

4. Frames in strata. To define a frame in the stratum Λ_2, we use a theorem due to Zakalyukin **[24]**, see also **[21]**, which puts into correspondence a vector field L tangent to hypersurface $\Delta(\lambda) = 0$ and a polynomial $p(x, y)$, namely,

$$Lf(x, y) = p(x, y)f(x, y) \mod (\partial_x f,\ \partial_y f),$$

where $f(x, y) = 0$ is a curve equation. In our case,

$$\mathbb{C}[x, y]/(\partial_x f,\ \partial_y f) = \operatorname{span}_{\mathbb{C}}\left(1,\ x,\ x^2,\ x^3\right),$$

and four vector fields $\{\ell_0,\ \ell_2,\ \ell_4,\ \ell_6\}$ correspondent to the polynomials

$$\begin{aligned} p_0(x, y) &= 10, & p_2(x, y) &= 10x,\\ p_4(x, y) &= 10x^2 + 6\lambda_4, & p_6(x, y) &= 10x^3 + 6\lambda_4 x + 4\lambda_6, \end{aligned}$$

provide a basis in Λ_2. Explicitly,

$$\begin{pmatrix} \ell_0 \\ \ell_2 \\ \ell_4 \\ \ell_6 \end{pmatrix} = V(\lambda) \begin{pmatrix} \partial_{\lambda_4} \\ \partial_{\lambda_6} \\ \partial_{\lambda_8} \\ \partial_{\lambda_{10}} \end{pmatrix}, \tag{9}$$

where

$$V(\lambda) = \begin{pmatrix} 4\lambda_4 & 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} \\ 6\lambda_6 & 8\lambda_8 - \frac{12}{5}\lambda_4^2 & 10\lambda_{10} - \frac{8}{5}\lambda_6\lambda_4 & -\frac{4}{5}\lambda_8\lambda_4 \\ 8\lambda_8 & 10\lambda_{10} - \frac{8}{5}\lambda_6\lambda_4 & 4\lambda_8\lambda_4 - \frac{12}{5}\lambda_6^2 & 6\lambda_{10}\lambda_4 - \frac{6}{5}\lambda_8\lambda_6 \\ 10\lambda_{10} & -\frac{4}{5}\lambda_8\lambda_4 & 6\lambda_{10}\lambda_4 - \frac{6}{5}\lambda_8\lambda_6 & 4\lambda_{10}\lambda_6 - \frac{8}{5}\lambda_8^2 \end{pmatrix}. \tag{10}$$

Vector fields $\ell = (\ell_0,\ \ell_2,\ \ell_4,\ \ell_6)$ are tangent to discriminant variety $\{\lambda \mid \Delta(\lambda) = 0\} \cong \Lambda_1 \cup \Lambda_0$, in fact,

$$\ell_k \Delta(\lambda) = \phi_k \Delta(\lambda), \qquad \phi_k \in \mathbb{C}[\lambda], \quad k = 0,\ 2,\ 4,\ 6; \tag{11}$$
$$\phi = (40,\ 0,\ 12\lambda_4,\ 4\lambda_6).$$

Vector fields ℓ are tangent to the variety $\{\lambda \mid \Gamma(\lambda)=0\} \cong \Lambda_0$, namely

$$\ell_k \Gamma(\lambda) = \psi_k \Gamma(\lambda), \qquad \psi_k \in \mathrm{Mat}(4; \mathbb{C}[\lambda]), \quad k = 0,\ 2,\ 4,\ 6; \tag{12}$$

$$\psi_0 = \mathrm{diag}(16,\ 18,\ 20,\ 24), \qquad \psi_2 = \begin{pmatrix} 0 & -6 & 0 & 0 \\ -\frac{116}{5}\lambda_4 & 0 & \frac{16}{5} & 0 \\ 27\lambda_6 & -77\lambda_4 & 0 & 0 \\ 72\lambda_6\lambda_4 & 240\lambda_8 - 56\lambda_4^2 & 0 & 0 \end{pmatrix},$$

$$\psi_4 = \begin{pmatrix} -\frac{32}{5}\lambda_4 & 0 & \frac{4}{5} & 0 \\ \frac{33}{5}\lambda_6 & 5\lambda_4 & 0 & 0 \\ 24\lambda_8 - \frac{432}{5}\lambda_4^2 & 0 & 12\lambda_4 & -\frac{12}{5} \\ 144\lambda_8\lambda_4 + 108\lambda_6^2 - \frac{176}{5}\lambda_4^3 & 0 & 0 & \frac{44}{5}\lambda_4 \end{pmatrix},$$

$$\psi_6 = \begin{pmatrix} -\frac{7}{5}\lambda_6 & -\frac{7}{5}\lambda_4 & 0 & 0 \\ 4\lambda_8 - \frac{128}{25}\lambda_4^2 & 0 & \frac{16}{25}\lambda_4 & 0 \\ 100\lambda_{10} - \frac{81}{5}\lambda_6\lambda_4 & -6\lambda_8 - \frac{81}{5}\lambda_4^2 & 0 & 0 \\ 72\lambda_8\lambda_6 - \frac{48}{5}\lambda_6\lambda_4^2 & 40\lambda_8\lambda_4 - \frac{48}{5}\lambda_4^3 & 0 & 0 \end{pmatrix}.$$

It follows from $\det V(\lambda) = \frac{16}{5}\Delta(\lambda)$ that ℓ defines a frame in the stratum Λ_2; next (11) and (12) imply that restrictions of ℓ to the strata Λ_1 and Λ_0 provide frames on the both strata. To analyze the restrictions in more detail, we need parameterization of Λ_1 and Λ_0. By combining (6) with (8) and comparing with (3), we observe that the subset of curves (3) with one double point is parameterized as follows:

$$\begin{aligned} \lambda_4 &= \gamma_4 - \tfrac{5}{3}a_2^2, \\ \lambda_6 &= \gamma_6 - \tfrac{4}{3}a_2\gamma_4 - \tfrac{10}{27}a_2^3, \\ \lambda_8 &= -2a_2\gamma_6 - \tfrac{1}{3}a_2^2\gamma_4 + \tfrac{20}{27}a_2^4, \\ \lambda_{10} &= a_2^2\gamma_6 + \tfrac{2}{3}a_2^3\gamma_4 + \tfrac{8}{27}a_2^5. \\ 4\gamma_4^3 &+ 27\gamma_6^2 \neq 0. \end{aligned} \tag{13}$$

LEMMA 1. *The restricted vector fields* $(\widetilde{\ell}_0,\ \widetilde{\ell}_2,\ \widetilde{\ell}_4) = (\ell_0,\ \ell_2,\ \ell_4)|_{\Lambda_1}$ *form a frame on the stratum* Λ_1. *In terms of parameterization* (13) *they are expressed as follows*

$$\widetilde{\ell}_0 = 2a_2\partial_{a_2} + 4\gamma_4\partial_{\gamma_4} + 6\gamma_6\partial_{\gamma_6},$$

$$\widetilde{\ell}_2 = \tfrac{2}{15}\big(6\gamma_4 + 5a_2^2\big)\partial_{a_2} + \tfrac{2}{3}\big(9\gamma_6 - 8a_2\gamma_4\big)\partial_{\gamma_4} - \tfrac{4}{3}\big(\gamma_4^2 + 6a_2\gamma_6\big)\partial_{\gamma_6},$$

$$\widetilde{\ell}_4 = \tfrac{2}{45}\big(27\gamma_6 + 9a_2\gamma_4 - 40a_2^3\big)\partial_{a_2} - \tfrac{4}{3}a_2\big(9\gamma_6 + a_2\gamma_4\big)\partial_{\gamma_4} - \tfrac{2}{3}a_2\big(3a_2\gamma_6 - 4\gamma_4^2\big)\partial_{\gamma_6}.$$

On Λ_1 *the vector field* ℓ_6 *is decomposed into*

$$\ell_6|_{\Lambda_1} = -a_2^3\widetilde{\ell}_0 - a_2^2\widetilde{\ell}_2 - a_2\widetilde{\ell}_4. \tag{14}$$

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Proof. The proof is straightforward. □

REMARK 2. The vector fields $(\widetilde{\ell}_0, \widetilde{\ell}_2, \widetilde{\ell}_4)$ on a curve (3) with a double point at $(a_2, 0)$ can be expressed in terms of the three vector fields: ∂_{a_2}, $L_0 = 4\gamma_4\partial_{\gamma_4} + 6\gamma_6\partial_{\gamma_6}$, and $L_2 = 6\gamma_6\partial_{\gamma_4} - \frac{4}{3}\gamma_4^2\partial_{\gamma_6}$ as follows:

$$\begin{aligned}\widetilde{\ell}_0 &= 2a_2\partial_{a_2} + L_0,\\ \widetilde{\ell}_2 &= \tfrac{2}{15}\big(6\gamma_4 + 5a_2^2\big)\partial_{a_2} - \tfrac{4}{3}a_2L_0 + L_2,\\ \widetilde{\ell}_4 &= \tfrac{2}{45}\big(27\gamma_6 + 9a_2\gamma_4 - 40a_2^3\big)\partial_{a_2} - \tfrac{1}{3}a_2^2L_0 - 2a_2L_2.\end{aligned} \tag{15}$$

The fields L_0, L_2 are tangent to the variety $\{\gamma \mid \delta(\gamma) = 0\}$.

In a similar way, from the generic form of a curve (3) with two double points at $(a_2, 0)$ and $(b_2, 0)$

$$-y^2 + (x - a_2)^2(x - b_2)^2(x + 2a_2 + 2b_2) = 0, \tag{16}$$

we obtain a parameterization of Λ_0

$$\begin{aligned}\lambda_4 &= -3a_2^2 - 4a_2b_2 - 3b_2^2,\\ \lambda_6 &= 2(a_2 + b_2)\big(a_2^2 + 3a_2b_2 + b_2^2\big),\\ \lambda_8 &= -a_2b_2\big(4a_2^2 + 7b_2a_2 + 4b_2^2\big),\\ \lambda_{10} &= 2a_2^2b_2^2(a_2 + b_2).\end{aligned} \tag{17}$$

LEMMA 2. *The restricted vector fields* $(\widetilde{\ell}_0, \widetilde{\ell}_2) = (\ell_0, \ell_2)|_{\Lambda_0}$ *form a frame on the stratum* Λ_0. *In terms of parameterization* (17) *they are expressed as follows*

$$\begin{aligned}\widetilde{\ell}_0 &= 2a_2\partial_{a_2} + 2b_2\partial_{b_2},\\ \widetilde{\ell}_2 &= -\tfrac{2}{5}\big(a_2^2 + 8a_2b_2 + 6b_2^2\big)\partial_{a_2} - \tfrac{2}{5}\big(6a_2^2 + 8a_2b_2 + b_2^2\big)\partial_{b_2}.\end{aligned}$$

On Λ_0 *the vector fields* ℓ_4 *and* ℓ_6 *are decomposed into*

$$\ell_4|_{\Lambda_0} = -(a_2^2 + a_2b_2 + b_2^2)\widetilde{\ell}_0 - (a_2 + b_2)\widetilde{\ell}_2, \tag{18a}$$

$$\ell_6|_{\Lambda_0} = a_2b_2(a_2 + b_2)\widetilde{\ell}_0 + a_2b_2\widetilde{\ell}_2. \tag{18b}$$

Proof. The proof is straightforward. □

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5. Annihilators of sigma-function. Following [**11**], we write down the operators producing heat equations in a non-holonomic frame in the case of genus 2 curve (3):

$$
\begin{aligned}
q_0 &= -\,u_1\partial_{u_1} - 3u_3\partial_{u_3} + 3 + \ell_0,\\
q_2 &= -\,\tfrac{1}{2}\partial_{u_1u_1} + \tfrac{4}{5}\lambda_4u_3\partial_{u_1} - u_1\partial_{u_3} + \tfrac{3}{10}\lambda_4u_1^2 - \tfrac{1}{10}\big(15\lambda_8 - 4\lambda_4^2\big)u_3^2 + \ell_2,\\
q_4 &= -\,\partial_{u_1u_3} + \tfrac{6}{5}\lambda_6u_3\partial_{u_1} - \lambda_4u_3\partial_{u_3} + \tfrac{1}{5}\lambda_6u_1^2 - \lambda_8u_1u_3\\
&\quad - \tfrac{1}{10}\big(30\lambda_{10} - 6\lambda_6\lambda_4\big)u_3^2 + \lambda_4 + \ell_4,\\
q_6 &= -\,\tfrac{1}{2}\partial_{u_3u_3} + \tfrac{3}{5}\lambda_8u_3\partial_{u_1} + \tfrac{1}{10}\lambda_8u_1^2 - 2\lambda_{10}u_1u_3 + \tfrac{3}{10}\lambda_8\lambda_4u_3^2\\
&\quad + \tfrac{1}{2}\lambda_6 + \ell_6.
\end{aligned}
$$

We define sigma-function $\boldsymbol{\sigma}(u_3,\, u_1;\, \lambda)$ on genus 2 curve (3) as a solution of the equations

$$
q_k\boldsymbol{\sigma}(u_3,\, u_1;\, \lambda) = 0, \qquad k = 0,\, 2,\, 4,\, 6
$$

with the initial condition $\boldsymbol{\sigma}(u_3,\, 0;\, 0) = u_3$. Since the solution is unique [**11**], this completely defines the sigma-function.

According to relation (14) from Lemma 1, the operator $Q_6 = -2(q_6 + a_2q_4 + a_2^2q_2 + a_2^3q_0)|_{\Lambda_1}$ does not include derivatives over γ and a_2, namely:

$$
Q_6 = \Big(\partial_{u_3} + a_2\partial_{u_1} + a_2^2u_1 + (\gamma_4 + \tfrac{4}{3}a_2^2)a_2u_3\Big)^2 - \big(\gamma_6 + \tfrac{5}{3}a_2\gamma_4 + \tfrac{125}{27}a_2^3\big). \tag{20}
$$

Introduce a new variable U_1 by the formula $u_1 = U_1 + a_2u_3$, then (20) becomes an ordinary differential operator

$$
Q_6 = D^2 - d(a_2,\gamma)^2, \tag{21}
$$

with the operator D and the function $d(a_2,\gamma)$ given by

$$
D = \partial_{u_3} + a_2^2U_1 + (\gamma_4 + \tfrac{7}{3}a_2^2)a_2u_3 \qquad d(a_2,\gamma)^2 = \gamma_6 + \tfrac{5}{3}a_2\gamma_4 + (\tfrac{5}{3}a_2)^3.
$$

The operator $Q_4 = -(q_4 + 2a_2q_2 + 3a_2^2q_0)|_{\Lambda_1}$ has the form

$$
\begin{aligned}
Q_4 = \Big(\partial_{U_1} + 2a_2U_1 + (\gamma_4 + \tfrac{28}{3}a_2^2)u_3\Big)D - \tfrac{6}{5}d(a_2,\gamma)\partial_{a_2}\big(d(a_2,\gamma)\,\cdot\,\big)&\\
-\tfrac{1}{5}\big(U_1^2 + 12a_2U_1u_3 + 3(\gamma_4 + 7a_2^2)u_3^2\big)d(a_2,\gamma)^2.&
\end{aligned} \tag{22}
$$

Then, $Q_0 = q_0|_{\Lambda_1}$ and $Q_2 = q_2 + \tfrac{4}{3}a_2q_0|_{\Lambda_1}$ take the form

$$
Q_0 = -\,U_1\partial_{U_1} - 3u_3\partial_{u_3} + 2a_2\partial_{a_2} + L_0 + 3, \tag{23a}
$$

$$
\begin{aligned}
Q_2 = -\,\tfrac{1}{2}\partial_{U_1U_1} - \tfrac{1}{3}a_2\big(U_1 + 3a_2u_3\big)\partial_{U_1} - \big(U_1 + 5a_2u_3\big)\partial_{u_3}&\\
+ \tfrac{2}{15}\big(6\gamma_4 + 25a_2^2\big)\partial_{a_2} + L_2 + \tfrac{1}{10}\big(3\gamma_4 - 5a_2^2\big)\big(U_1 + 2a_2u_3\big)U_1&\\
+ \tfrac{1}{30}\big(90a_2\gamma_6 + 12\gamma_4^2 - 16a_2^2\gamma_4 - 15a_2^4\big)u_3^2 + 4a_2.&
\end{aligned} \tag{23b}
$$

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A solution $\mathcal{Z}(u_3, U_1, a_2, \gamma)$ of the system

$$Q_k\mathcal{Z} = 0, \quad k = 0,\ 2,\ 4,\ 6, \qquad \mathcal{Z}(u_3, 0, 0, 0) = u_3$$

at $U_1 = u_1 - a_2 u_3$ is a degenerate sigma-function and coincides with $\boldsymbol{\sigma}(u_3, u_1; \lambda)$ restricted to Λ_1. We construct this solution explicitly in the next section.

6. Degenerate sigma-function.

THEOREM 1. *Suppose $\lambda \in \Lambda_1$. Sigma-function associated with a curve* (3) *has the form*

$$\begin{aligned}\boldsymbol{\sigma}(u_3, u_1, \lambda)|_{\Lambda_1} = \; & \frac{\mathrm{e}^{-\frac{3}{5}\wp(\alpha)\left(\left(\frac{1}{2}\gamma_4 + \frac{3}{25}\wp(\alpha)^2\right)u_3^2 + \frac{2}{5}\wp(\alpha)u_1u_3 + \frac{1}{6}u_1^2\right)}}{\wp'(\alpha)\sigma(\alpha)} \\ & \times \left(\sigma\left(\alpha + u_1 - \tfrac{3}{5}\wp(\alpha)u_3\right)\mathrm{e}^{\frac{1}{2}\wp'(\alpha)u_3 - \zeta(\alpha)\left(u_1 - \frac{3}{5}\wp(\alpha)u_3\right)}\right. \\ & \left. - \sigma\left(\alpha - u_1 + \tfrac{3}{5}\wp(\alpha)u_3\right)\mathrm{e}^{-\frac{1}{2}\wp'(\alpha)u_3 + \zeta(\alpha)\left(u_1 - \frac{3}{5}\wp(\alpha)u_3\right)}\right),\end{aligned} \tag{24}$$

where σ, ζ, $\wp$ are Weierstrass functions associated with the curve (4)*, and α is defined by $\wp(\alpha) = \frac{5}{3}a_2$.*

Proof. First, we consider the equation

$$Q_6\mathcal{Z}(u_3, U_1, a_2, \gamma) = 0,$$

where Q_6 is defined by (21). The gauge transformation

$$\mathcal{Z}(u_3, U_1, a_2, \gamma) = \exp\left\{-\tfrac{1}{2}a_2\left(\gamma_4 + \tfrac{7}{3}a_2^2\right)u_3^2 - a_2^2 U_1 u_3\right\}\rho(u_3, U_1, a_2, \gamma) \tag{25}$$

leads to a simpler equation

$$\partial_{u_3u_3}\rho(u_3, U_1, a_2, \gamma) - d(a_2, \gamma)^2\rho(u_3, U_1, a_2, \gamma) = 0.$$

As fundamental solutions of the equation, we choose $c_\epsilon(U_1, a_2, \gamma)\exp(\epsilon d(a_2, \gamma)u_3)$, where ϵ is unary operator: $\epsilon = \pm$. Then,

$$\rho(u_3, U_1, a_2, \gamma) = c_+(U_1, a_2, \gamma)\mathrm{e}^{u_3 d(a_2,\gamma)} + c_-(U_1, a_2, \gamma)\mathrm{e}^{-u_3 d(a_2,\gamma)}. \tag{26}$$

Next, consider the equation

$$Q_4\mathcal{Z}(u_3, U_1, a_2, \gamma) = 0.$$

Taking into account (25) and (26), we obtain the following equations for c_ϵ

$$\epsilon\partial_{U_1}c_\epsilon - \tfrac{6}{5}\partial_{a_2}\left(d(a_2, \gamma)c_\epsilon\right) = \left(-2\epsilon a_2 U_1 + \tfrac{1}{5}d(a_2, \gamma)U_1^2\right)c_\epsilon.$$

The substitution

$$c_\epsilon(U_1, a_2, \gamma) = \exp\left\{\varphi_\epsilon(U_1, a_2, \gamma)\right\}/d(a_2, \gamma)$$

leads to a linear non-homogeneous partial differential equation

$$\big(\epsilon\partial_{U_1} - \tfrac{6}{5}d(a_2,\gamma)\partial_{a_2}\big)\varphi_\epsilon = -2\epsilon a_2 U_1 + \tfrac{1}{5}d(a_2,\gamma)U_1^2. \tag{27}$$

We solve the associated homogeneous equation by the method of characteristics:

$$-\epsilon \mathrm{d}U_1 = \frac{5}{6}\frac{\mathrm{d}a_2}{d(a_2,\gamma)} = \frac{\mathrm{d}\big(\frac{5}{3}a_2\big)}{-2\sqrt{\big(\frac{5}{3}a_2\big)^3 + \frac{5}{3}a_2\gamma_4 + \gamma_6}}.$$

The characteristics is defined by the equation

$$\alpha(a_2,\gamma) + \epsilon U_1 = \text{const},$$

where

$$\alpha(a_2,\gamma) = \int_\infty^{\frac{5}{3}a_2} \frac{\mathrm{d}X}{-2\sqrt{X^3+\gamma_4 X+\gamma_6}}, \qquad \deg\alpha = 1.$$

We write down a general solution of the homogeneous equation as

$$\varphi_\epsilon^{(h)} = \log s_\epsilon\big(\alpha(a_2,\gamma)+\epsilon U_1,\gamma\big).$$

In what follows, we need elliptic functions σ, ζ, $\wp$, $\wp'$ associated with the curve (4). Here,

$$\wp(\alpha) = \tfrac{5}{3}a_2 \qquad \wp'(\alpha) = 2d(a_2,\gamma) = -2\sqrt{\big(\tfrac{5}{3}a_2\big)^3 + \tfrac{5}{3}a_2\gamma_4 + \gamma_6}.$$

The functions $\wp$, $\wp'$ satisfy the equation $(\wp')^2 = 4\wp^3 + 4\lambda_4\wp + 4\lambda_6$, thus, they are standard Weierstrass functions with the invariants $(g_2, g_3) = (-4\lambda_4,\ -4\lambda_6)$, see **[15]**.

Next, we construct a particular solution of non-homogeneous equation (27) in the form

$$\varphi_\epsilon^{(\mathrm{nh})} = C_2(a_2,\gamma)U_1^2 + C_1(a_2,\gamma)U_1 + C_0(a_2,\gamma).$$

By substituting the ansatz and collecting coefficients at the powers of U_1, we obtain a system of equations for C_2, C_1 and C_0:

$$\partial_{a_2}C_2 = -\tfrac{1}{6}, \quad \tfrac{6}{5}d(a_2,\gamma)\partial_{a_2}C_1 = 2\epsilon\big(C_2+a_2\big), \quad \tfrac{6}{5}d(a_2,\gamma)\partial_{a_2}C_0 = \epsilon C_1.$$

Observe that $\partial_\alpha = \frac{6}{5}d(a_2,\gamma)\partial_{a_2}$. Whence

$$\begin{aligned} &C_2(a_2,\gamma) = -\tfrac{1}{6}a_2, && \\ &\partial_\alpha C_1 = \epsilon\tfrac{5}{3}a_2 = \epsilon\wp(\alpha) &\Rightarrow\qquad & C_1(a_2,\gamma) = -\epsilon\zeta(\alpha), \\ &\partial_\alpha C_0 = -\zeta(\alpha) &\Rightarrow\qquad & C_0(a_2,\gamma) = -\log\sigma(\alpha), \end{aligned}$$

above we have used the standard relations:

$$\zeta(u) = -\int_\infty^u \wp(v)\mathrm{d}v, \qquad \log\sigma(u) = \int_\infty^u \zeta(v)\mathrm{d}v.$$

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Summing up, the general solution $\varphi_\epsilon^{(h)} + \varphi_\epsilon^{(nh)}$ of (27) has the form

$$\varphi_\epsilon(U_1, a_2, \gamma) = \log s_\epsilon\big(\alpha(a_2, \gamma) + \epsilon U_1, \gamma\big) - \tfrac{1}{6} a_2 U_1^2 - \epsilon\zeta(\alpha) U_1 - \log\sigma(\alpha).$$

Therefore, we come to the following expression for c_ϵ:

$$c_\epsilon(U_1, a_2, \gamma) = \frac{s_\epsilon\big(\alpha(a_2, \gamma) + \epsilon U_1, \gamma\big)}{\wp'(\alpha(a_2, \gamma))\sigma(\alpha(a_2, \gamma))} \mathrm{e}^{-\frac{1}{6} a_2 U_1^2 - \epsilon\zeta(\alpha(a_2,\gamma)) U_1}. \tag{28}$$

Taking into account the form (28) of dependence of c_ϵ on a_2, for the next step, we change the variables on Λ_1 from $(a_2,\ \gamma_4,\ \gamma_6)$ to $(\alpha,\ \gamma_4,\ \gamma_6)$.

$$c_\epsilon(U_1, \alpha, \gamma) = \frac{s_\epsilon\big(\alpha + \epsilon U_1, \gamma\big)}{\wp'(\alpha)\sigma(\alpha)} \mathrm{e}^{-\frac{1}{10}\wp(\alpha) U_1^2 - \epsilon\zeta(\alpha) U_1}.$$

Under the change of variables the operators Q_2, Q_0 map to new operators $\widetilde{Q}_2$, $\widetilde{Q}_0$, where the map is defined by the following formula (cf. Remark 2)

$$\big(\partial_{a_2},\ L_2,\ L_0\big) \mapsto \left(\frac{5}{3\wp'(\alpha)}\partial_\alpha,\ L_2 - \frac{L_2\big(\wp(\alpha)\big)}{\wp'(\alpha)}\partial_\alpha,\ L_0 - \frac{L_0\big(\wp(\alpha)\big)}{\wp'(\alpha)}\partial_\alpha\right).$$

Applying the operator $\widetilde{Q}_2$ to $\mathcal{Z}(u_3, U_1, \frac{3}{5}\wp(\alpha), \gamma)$ with ansatz (28) and using the relations

$$\begin{aligned}
L_2\sigma(\alpha) &= \sigma(\alpha)\big(-\tfrac{1}{6}\gamma_4\alpha^2 + \tfrac{1}{2}\zeta(\alpha)^2 - \tfrac{1}{2}\wp(\alpha)\big),\\
L_2\zeta(\alpha) &= -\tfrac{1}{3}\gamma_4\alpha - \zeta(\alpha)\wp(\alpha) - \tfrac{1}{2}\wp'(\alpha),\\
L_2\wp(\alpha) &= \tfrac{4}{3}\gamma_4 + 2\wp(\alpha)^2 + \zeta(\alpha)\wp'(\alpha),\\
L_2\wp'(\alpha) &= \zeta(\alpha)\big(6\wp(\alpha)^2 + 2\gamma_4\big) + 3\wp(\alpha)\wp'(\alpha).
\end{aligned}$$

we come to the equation

$$\Big(-\tfrac{1}{2}\partial_{U_1U_1} + \tfrac{1}{6}\gamma_4\big(\alpha + \epsilon U_1\big)^2 + L_2\Big)s_\epsilon(\alpha + \epsilon U_1, \gamma) = 0. \tag{29}$$

Similarly, the operator $\widetilde{Q}_0$ leads to the equation

$$\Big(-(\alpha + \epsilon U_1)\partial_{U_1} + L_0 + 1\Big)s_\epsilon(\alpha + \epsilon U_1, \gamma) = 0. \tag{30}$$

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Further, consider the power series expansion for $\mathcal{Z}(u_3, 0, \frac{3}{5}\wp(\alpha), \gamma)$ in u_3 near zero. We obtain

$$\mathcal{Z}(u_3, 0, \tfrac{3}{5}\wp(\alpha), \gamma) = \frac{s_+(\alpha, \gamma) + s_-(\alpha, \gamma)}{\sigma(\alpha)\wp'(\alpha)} + \frac{s_+(\alpha, \gamma) - s_-(\alpha, \gamma)}{2\sigma(\alpha)} u_3 + O(u_3^2).$$

Comparing the expansion with the initial condition $\mathcal{Z}(u_3, 0, 0, 0) = u_3$ for entire function $\mathcal{Z}$ and taking into account that at $\gamma = (0, 0)$ the value of $\alpha(a_2, \gamma)$ tends to infinity as $a_2 \to 0$, we find

$$s_+\big(\alpha, 0\big) = -s_-\big(\alpha, 0\big),$$
$$s_+\big(\alpha, 0\big) = \sigma(\alpha)|_{\gamma=0} = \alpha.$$

Therefore, $s_\epsilon\big(\alpha, 0\big) = \epsilon\alpha$. Thus, the initial condition singles out a unique solution of equations (29) and (30) that is

$$s_\epsilon(\alpha + \epsilon U_1, \gamma) = \epsilon\sigma(\alpha + \epsilon U_1).$$

Combining all of the above results, we write down the final expression for $\mathcal{Z}$. □

Remark 3. Note that the genus 2 degenerate sigma-function (24) can be represented with the help of elliptic Baker function Φ

$$\Phi(u, \alpha) = \frac{\sigma(\alpha - u)}{\sigma(\alpha)\sigma(u)}\, \mathrm{e}^{\zeta(\alpha)u}. \tag{31}$$

Indeed, we have

$$\begin{aligned}
&\boldsymbol{\sigma}(u_3, u_1, \lambda)|_{\Lambda_1} \\
&\quad = -\mathrm{e}^{-\frac{3}{5}\wp(\alpha)\left(\left(\frac{1}{2}\gamma_4+\frac{3}{25}\wp(\alpha)^2\right)u_3^2+\frac{2}{5}\wp(\alpha)u_1u_3+\frac{1}{6}u_1^2\right)} \frac{\sigma\big(u_1 - \frac{3}{5}\wp(\alpha)u_3\big)}{\wp'(\alpha)} \\
&\qquad \times \left(\Phi\big(-u_1 + \tfrac{3}{5}\wp(\alpha)u_3, \alpha\big)\mathrm{e}^{\frac{1}{2}\wp'(\alpha)u_3} + \Phi\big(u_1 - \tfrac{3}{5}\wp(\alpha)u_3, \alpha\big)\mathrm{e}^{-\frac{1}{2}\wp'(\alpha)u_3}\right).
\end{aligned}$$

Remark 4. Visibly the right-hand side of (24) is singular when $\sigma(\alpha) = 0$ or $\wp'(\alpha) = 0$. The first case corresponds to $a_2 = \infty$, which does not belong to Λ_1, otherwise the equation (3) would not include the term x^5. In the second case, 2α is a period of the Weierstrass elliptic function, that is $\wp(\alpha)$ becomes a branch point e_i of (4). Then, by Lôpital rule,

$$\boldsymbol{\sigma}(u_3, u_1, \lambda)|_{\Lambda_1, \wp'(\alpha)=0} = \frac{\mathrm{e}^{-\frac{3}{5}\wp(\alpha)\left(\left(\frac{1}{2}\gamma_4+\frac{3}{25}\wp(\alpha)^2\right)u_3^2+\frac{2}{5}\wp(\alpha)u_1u_3+\frac{1}{6}u_1^2\right)}}{2\sigma(\alpha)}$$

$$\times\left(\left(u_3+\wp(\alpha)\frac{u_1-\frac{3}{5}\wp(\alpha)u_3}{\gamma_4+3\wp(\alpha)^2}\right)\left(\sigma\left(\alpha+u_1-\tfrac{3}{5}\wp(\alpha)u_3\right)\mathrm{e}^{-\zeta(\alpha)\left(u_1-\frac{3}{5}\wp(\alpha)u_3\right)}\right.\right.$$

$$\left.+\,\sigma\left(\alpha-u_1+\tfrac{3}{5}\wp(\alpha)u_3\right)\mathrm{e}^{\zeta(\alpha)\left(u_1-\frac{3}{5}\wp(\alpha)u_3\right)}\right)$$

$$+\,\frac{1}{\gamma_4+3\wp(\alpha)^2}\left(\sigma'\left(\alpha+u_1-\tfrac{3}{5}\wp(\alpha)u_3\right)\mathrm{e}^{-\zeta(\alpha)\left(u_1-\frac{3}{5}\wp(\alpha)u_3\right)}\right.$$

$$\left.\left.-\,\sigma'\left(\alpha-u_1+\tfrac{3}{5}\wp(\alpha)u_3\right)\mathrm{e}^{\zeta(\alpha)\left(u_1-\frac{3}{5}\wp(\alpha)u_3\right)}\right)\right)$$

$$=\mathrm{e}^{-\frac{3}{5}e_i\left(\left(\frac{1}{2}\gamma_4+\frac{3}{25}e_i^2\right)u_3^2+\frac{2}{5}e_iu_1u_3+\frac{1}{6}u_1^2\right)}\times$$

$$\times\left(\left(u_3+\frac{e_i\left(u_1-\frac{3}{5}e_iu_3\right)}{\gamma_4+3e_i^2}\right)\sigma_i\left(u_1-\tfrac{3}{5}e_iu_3\right)+\frac{\sigma_i'\left(u_1-\frac{3}{5}e_iu_3\right)}{\gamma_4+3e_i^2}\right), \tag{32}$$

where $\alpha=\frac{1}{2}\omega_i$, $e_i=\wp(\frac{1}{2}\omega_i)$, $\eta_i=\zeta(\frac{1}{2}\omega_i)$, and $\sigma_i(u)=\exp(-u\eta_i)\sigma(u+\frac{1}{2}\omega_i)/\sigma(\frac{1}{2}\omega_i)$ denotes a sigma-function with characteristic **[15]**, p. 348 equation (22). We use the notation $\omega_1=\omega$, $\omega_2=-\omega-\omega'$, $\omega_3=\omega'$ for periods of the Weierstrass elliptic function, and take into account the following properties of sigma-functions with characteristic

$$\sigma_i(-z)=\sigma_i(z),$$

$$\frac{\sigma'(u+\frac{1}{2}\omega_i)}{\sigma(\frac{1}{2}\omega_i)}\,\mathrm{e}^{-\eta_iu}-\frac{\sigma'(-u+\frac{1}{2}\omega_i)}{\sigma(\frac{1}{2}\omega_i)}\,\mathrm{e}^{\eta_iu}=2\sigma_i'(z).$$

THEOREM 2. *Suppose* $\lambda\in\Lambda_0$. *Sigma-function associated with a curve* (3) *has the form*

$$\boldsymbol{\sigma}(u_3,u_1,\lambda)|_{\Lambda_0}=\frac{\mathrm{e}^{\frac{1}{2}\left(3a_2b_2(a_2+b_2)u_3^2+2a_2b_2u_1u_3-(a_2+b_2)u_1^2\right)}}{(a_2-b_2)}$$

$$\times\left(\cosh\left(\sqrt{2a_2+3b_2}(u_1-a_2u_3)\right)\frac{\sinh\left(\sqrt{3a_2+2b_2}(u_1-b_2u_3)\right)}{\sqrt{3a_2+2b_2}}\right.$$

$$\left.-\,\cosh\left(\sqrt{3a_2+2b_2}(u_1-b_2u_3)\right)\frac{\sinh\left(\sqrt{2a_2+3b_2}(u_1-a_2u_3)\right)}{\sqrt{2a_2+3b_2}}\right). \tag{33}$$

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The theorem is proven by an argument similar to the proof of Theorem 1.

REMARK 5. Evidently, when $b_2 = a_2$ an uncertainty arises in (33), which has the following limit

$$\boldsymbol{\sigma}(u_3, u_1, \lambda)|_{\Lambda_0, b_2=a_2} = \mathrm{e}^{a_2\left(3a_2^2u_3^2 + a_2u_1u_3 - u_1^2\right)} \times\left(u_3 + \frac{1}{10a_2}(u_1 - a_2u_3) - \frac{1}{4(5a_2)^{3/2}}\sinh\left(2\sqrt{5a_2}(u_1 - a_2u_3)\right)\right). \tag{34}$$

At $a_2 \to 0$, we come to the curve $y^2 = x^5$, and σ-function turns into the Schur–Weierstrass polynomial, introduced in [9]

$$\boldsymbol{\sigma}(u_3, u_1, \lambda)|_{\Lambda_0,\lambda=0} = u_3 - \frac{u_1^3}{3}. \tag{35}$$

7. Applications.

7.1. A generalized Jacobi inversion problem. Let (X_1, Y_1) and (X_2, Y_2) be a pair of finite points on the elliptic curve (4). Consider an inversion problem for integrals

$$\begin{aligned}
&\int_\infty^{(X_1,Y_1)} \frac{\mathrm{d}X}{-2Y} + \int_\infty^{(X_2,Y_2)} \frac{\mathrm{d}X}{-2Y} = U_1,\\
&\int_\infty^{(X_1,Y_1)} \frac{\mathrm{d}X}{-2Y(X-A)} + \int_\infty^{(X_2,Y_2)} \frac{\mathrm{d}X}{-2Y(X-A)} = U_3.
\end{aligned} \tag{36}$$

Denote

$$\mathcal{Z} = \boldsymbol{\sigma}\left(U_3, U_1 + \tfrac{3}{5}AU_3, \lambda(A,\gamma)\right),$$

and $\lambda(A, \gamma)$ is defined by

$$\begin{aligned}
\lambda_4 &= \gamma_4 - \tfrac{3}{5}A^2,\\
\lambda_6 &= \gamma_6 - \tfrac{4}{5}A\gamma_4 - \tfrac{2}{25}A^3,\\
\lambda_8 &= -\tfrac{6}{5}A\gamma_6 - \tfrac{3}{25}A^2\gamma_4 + \tfrac{12}{125}A^4,\\
\lambda_{10} &= \tfrac{9}{25}A^2\gamma_6 + \tfrac{18}{125}A^3\gamma_4 + \tfrac{72}{3125}A^5.
\end{aligned} \tag{37}$$

Further, let

$$\mathcal{P}_{ij} = -\partial_{U_iU_j}\log\mathcal{Z} \quad\text{and}\quad \mathcal{P}_{ijk} = -\partial_{U_iU_jU_k}\log\mathcal{Z}.$$

COROLLARY 1. *The solution of a generalized Jacobi inversion problem* (36) *is given by the formulas*

$$\begin{aligned}
X_1 + X_2 &= \mathcal{P}_{11} + \tfrac{4}{5}A,\\
X_1X_2 &= -\mathcal{P}_{13} + A\mathcal{P}_{11} + \tfrac{4}{25}A^2,\\
Y_k &= -\frac{1}{2}\mathcal{P}_{111} - \frac{\mathcal{P}_{113}}{2(X_k - A)}, \quad k = 1,\ 2.
\end{aligned} \tag{38}$$

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Proof. Consider the Jacobi inversion problem on a genus 2 curve of the form (3)

$$\begin{aligned} \int_{P_0}^{(x_1,y_1)} \frac{\mathrm{d}x}{-2y} + \int_{P_0}^{(x_2,y_2)} \frac{\mathrm{d}x}{-2y} &= u_3, \\ \int_{P_0}^{(x_1,y_1)} \frac{x\,\mathrm{d}x}{-2y} + \int_{P_0}^{(x_2,y_2)} \frac{x\,\mathrm{d}x}{-2y} &= u_1. \end{aligned} \tag{39}$$

The pair of points (x_1, y_1) and (x_2, y_2) on the curve is defined by formulas

$$\begin{aligned} x_1 + x_2 = \wp_{11}, \qquad x_1 x_2 = -\wp_{13}, \\ y_k = -\tfrac{1}{2}\big(x_k \wp_{111} + \wp_{113}\big), \quad k = 1,\ 2, \end{aligned} \tag{40}$$

where $\wp_{ij} = -\partial_{u_i u_j} \log \boldsymbol{\sigma}(u_3, u_1, \lambda)$ and $\wp_{ijk} = -\partial_{u_i u_j u_k} \log \boldsymbol{\sigma}(u_3, u_1, \lambda)$. For more details, see [**3**].

Indeed, relations (40) hold for all values of u and λ, where sigma-function does not vanish. Consider (39) with parameters λ as in (37). The substitution

$$\begin{aligned} x = X - \tfrac{2}{5}A, \qquad y = Y(X - A), \\ u_3 = U_3, \qquad u_1 = U_1 + \tfrac{3}{5} A U_3, \end{aligned} \tag{41}$$

transforms the problem (39) to the problem (36). Consequently, (40) transforms to (38). □

Introducing the following notation

$$\mathcal{P} = \frac{\sigma\big(\alpha + U_1\big)}{\sigma\big(\alpha - U_1\big)}\, \mathrm{e}^{\wp'(\alpha) U_3 - 2\zeta(\alpha) U_1}, \tag{42a}$$

$$\mathcal{S} = \frac{1}{2\big(\wp(U_1) - \wp(\alpha)\big)} \left(\wp'(U_1) - \wp'(\alpha) \frac{\mathcal{P}+1}{\mathcal{P}-1}\right), \tag{42b}$$

where $\wp(\alpha) = A$, we present explicit expressions for (38)

$$X_1 + X_2 = \mathcal{S}^2 - \wp(U_1), \tag{43a}$$

$$X_1 X_2 = \wp(U_1)\mathcal{S}^2 - \wp'(U_1)\mathcal{S} - \wp(\alpha)\big(\wp(U_1) + \wp(\alpha)\big) + \frac{\wp'(U_1)^2 - \wp'(\alpha)^2}{4\big(\wp(U_1) - \wp(\alpha)\big)}, \tag{43b}$$

and from $Y_k = -\frac{1}{2}\Big(X_k \partial_{U_1}(X_1 + X_2) - \partial_{U_1}(X_1 X_2)\Big) / \big(X_k - \wp(\alpha)\big)$

$$\begin{aligned} Y_k = &- \frac{X_k - \wp(U_1)}{X_k - \wp(\alpha)} \mathcal{S}^3 + \frac{\wp'(U_1)}{X_k - \wp(\alpha)} \mathcal{S}^2 \\ &+ \left(2\wp(U_1) + \wp(\alpha) + \frac{\wp'(U_1)^2 - \wp'(\alpha)^2}{4\big(X_k - \wp(\alpha)\big)\big(\wp(U_1) - \wp(\alpha)\big)}\right)\mathcal{S} + \frac{1}{2}\wp'(U_1). \end{aligned} \tag{43c}$$

Example 1. In the case when A is a branch point, say $e_1 = \wp(\omega/2)$, of the curve (4) the function $\boldsymbol{\sigma}$ is simplified dramatically, cf. (32). However, formula (43c) fails for one of the roots. The explicit solution has the form

$$(X_1,\ Y_1) = (e_1, 0), \quad (X_2,\ Y_2) = \big(\wp(U_1 + \omega/2), -\tfrac{1}{2}\wp'(U_1 + \omega/2)\big). \tag{44}$$

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Introducing variables ξ_k by the equalities $\wp(\xi_k) = X_k$, $k = 1, 2$, we rewrite the problem (36) in the form

$$\begin{gathered}\xi_1 + \xi_2 = U_1,\\ \int_0^{\xi_1} \frac{\mathrm{d}\xi}{\wp(\xi) - \wp(\alpha)} + \int_0^{\xi_2} \frac{\mathrm{d}\xi}{\wp(\xi) - \wp(\alpha)} = U_3.\end{gathered} \tag{45}$$

With the help of

$$\frac{\wp'(\alpha)}{\wp(\alpha) - \wp(\xi)} = 2\zeta(\alpha) - \zeta(\alpha - \xi) - \zeta(\alpha + \xi),$$

we explicitly integrate and reduce (45) to the following system

$$\xi_1 + \xi_2 = U_1, \qquad \frac{\sigma(\alpha - \xi_1)\sigma(\alpha - \xi_2)}{\sigma(\alpha + \xi_1)\sigma(\alpha + \xi_2)} = \mathrm{e}^{-2\zeta(\alpha)U_1 + \wp'(\alpha)U_3}. \tag{46}$$

Example 2. For the rational limit $(\gamma_4, \gamma_6) = 0$

$$\sigma(\xi) = \xi, \quad \zeta(\xi) = \xi^{-1}, \quad \wp(\xi) = \xi^{-2}, \quad \wp'(\xi) = -2\xi^{-3}$$

the problem (36) with usage of (46) is solved explicitly:

$$\xi_1 + \xi_2 = U_1, \qquad \xi_1\xi_2 = -\alpha^2 + \alpha U_1\left[\tanh\left(\frac{U_3}{\alpha^3} + \frac{U_1}{\alpha}\right)\right]^{-1}.$$

In the rational limit, the same relations are obtained from (43) with $X_k = \xi_k^{-2}$.

Consider the equation with respect to ξ

$$e^{\varkappa} = \frac{\sigma(\xi - \alpha)\sigma(\xi + \beta)}{\sigma(\xi + \alpha)\sigma(\xi - 2\alpha + \beta)}. \tag{47}$$

Similar equations appear in the theory of Bethe ansatz, see **[20]** and many other publications. By combining the substitutions

$$\beta = \alpha - U_1, \qquad \varkappa = -2\zeta(\alpha)U_1 + \wp'(\alpha)U_3.$$

the equation (47) is reduced to (46) and has two solutions ξ_1, ξ_2 defined by (43).

REMARK 6. The ratio $\dfrac{\sigma(\xi - \alpha)\sigma(\xi + \beta)}{\sigma(\xi + \alpha)\sigma(\xi - 2\alpha + \beta)}$ can be represented as a rational function in $\wp(\xi)$ and $\wp'(\xi)$. Then, the equation (47) is transformed to

$$e^{\varkappa} = \frac{\sigma(\beta)}{\sigma(2\alpha - \beta)} \cdot \frac{\wp(\alpha) - \wp(2\alpha - \beta)}{\wp(\alpha) - \wp(\beta)} \cdot \frac{\begin{vmatrix} \wp'(\xi) & \wp(\xi) & 1\\ \wp'(\alpha) & \wp(\alpha) & 1\\ -\wp'(\beta) & \wp(\beta) & 1\end{vmatrix}}{\begin{vmatrix} \wp'(\xi) & \wp(\xi) & 1\\ -\wp'(\alpha) & \wp(\alpha) & 1\\ \wp'(2\alpha - \beta) & \wp(2\alpha - \beta) & 1\end{vmatrix}}.$$

This is equivalent to an equation of the form $A\wp'(\xi) + B\wp(\xi) + C = 0$, which apparently has three roots. Two of the roots are functions in $\varkappa$ and provide a solution of (47) and in fact are the same as defined by (43). The extra root $\big(\wp(\xi), \wp'(\xi)\big) = \big(\wp(\alpha-\beta), -\wp'(\alpha-\beta)\big)$ is independent of $\varkappa$.

7.2. Schrödinger equation with periodic potential. Introduce the function

$$\Phi\big((u_3, u_1), (\beta_3, \beta_1)\big) = \frac{\boldsymbol{\sigma}(\beta_3 - u_3, \beta_1 - u_1, \lambda)}{\boldsymbol{\sigma}(u_3, u_1, \lambda)} \times \exp\left(-u_3 \int_\infty^{(b,y(b))} \frac{(3x^3 + \lambda_4 x)\,\mathrm{d}x}{-2y} - u_1 \int_\infty^{(b,y(b))} \frac{x^2\,\mathrm{d}x}{-2y}\right), \quad (48)$$

where (β_1, β_3) is the image of the point $(b, y(b))$ on the genus 2 curve (3) under the Abel map

$$\beta_3 = \int_\infty^{(b,y(b))} \frac{\mathrm{d}x}{-2y}, \qquad \beta_1 = \int_\infty^{(b,y(b))} \frac{x\,\mathrm{d}x}{-2y}.$$

The 1-forms $\frac{3x^3+\lambda_4 x}{-2y}\mathrm{d}x$ and $\frac{x^2}{-2y}\mathrm{d}x$ are second kind differentials associated to the first kind differentials $\frac{1}{-2y}\mathrm{d}x$ and $\frac{x}{-2y}\mathrm{d}x$. The function $\Phi\big((u_3, u_1), (\beta_3, \beta_1)\big)$ is a genus 2 analog of the elliptic Baker function (31).

Next, we exploit the fact that $\Phi\big((u_3, u_1), (\beta_3, \beta_1)\big)$ satisfies the equation

$$\big(\partial_{u_1 u_1} - 2\wp_{11}\big)\Phi = b\,\Phi, \quad (49)$$

which is similar to a Schrödinger-type equation

$$\big(\partial_{zz} - \mathcal{U}(z)\big)\psi(z) = \mathcal{E}\psi(z), \quad (50)$$

where $\mathcal{E}$ is a value of energy.

COROLLARY 2. *Suppose* $\lambda(\wp(\alpha), \gamma)) \in \Lambda_1$ *is defined by* (37) *with* $\wp(\alpha) = A$. *Then for all* $U_3 \in \mathbb{C}$ *the function*

$$\psi(U_1) = \frac{\boldsymbol{\sigma}\big(B_3 - U_3, B_1 - U_1 + \frac{3}{5}\wp(\alpha)U_3; \lambda(\wp(\alpha), \gamma)\big)}{\boldsymbol{\sigma}\big(U_3, U_1 + \frac{3}{5}\wp(\alpha)U_3; \lambda(\wp(\alpha), \gamma)\big)}\,\mathrm{e}^{U_1\varpi}, \quad (51)$$

where B_1 *is an arbitrary complex number,*

$$B_3 = \frac{1}{\wp'(\alpha)}\left(2\zeta(\alpha)B_1 + \log\frac{\sigma(\alpha - B_1)}{\sigma(\alpha + B_1)}\right),$$

$$\varpi = -\zeta(B_1) + \frac{1}{5}\wp(\alpha)\left(1 + \frac{18\zeta(\alpha)\wp(\alpha)}{5\wp'(\alpha)}\right)B_1 + \frac{9\wp(\alpha)^2}{25\wp'(\alpha)}\log\frac{\sigma(B_1 - \alpha)}{\sigma(B_1 + \alpha)}.$$

satisfies the Schrödinger equation (50) *with the potential and energy*

$$\mathcal{U}(U_1) = 2\mathcal{S}^2 - 2\wp(U_1) - 2\wp(\alpha), \qquad \mathcal{E} = \wp(B_1), \quad (52)$$

where the notation (42b) *is used.*

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Proof. Consider the equation (49) with respect to the variable $U_1 = u_1 - \frac{3}{5}\wp(\alpha)u_3$, and use $\mathcal{P}_{11}$ from Corollary 1 instead of $\wp_{11}$. The equation acquires the form

$$\big(\partial_{U_1U_1} - 2\mathcal{P}_{11}\big)\Psi = b\Psi,$$

where Ψ is obtained from Φ by applying the substitution (41)

$$\Psi\big((U_3, U_1), (B_3, B_1)\big) = \frac{\sigma\big(B_3 - U_3, B_1 - U_1 + \frac{3}{5}\wp(\alpha)U_3; \lambda(\wp(\alpha), \gamma)\big)}{\sigma\big(U_3, U_1 + \frac{3}{5}\wp(\alpha)U_3; \lambda(\wp(\alpha), \gamma)\big)} \times \exp\bigg(-U_3\int_\infty^{(b,y(b))} \mathrm{d}R_3 - U_1\int_\infty^{(b,y(b))} \mathrm{d}R_1\bigg), \qquad (53)$$

where $B_3 = \beta_3$, $B_1 = \beta_1 - \frac{3}{5}\wp(\alpha)\beta_3$, the set of parameters $\lambda(\wp(\alpha), \gamma)$ is defined by (37) with $A = \wp(\alpha)$. Under the substitution (41), we get

$$B_3 = \int_\infty^{b+\frac{2}{5}\wp(\alpha)} \frac{\mathrm{d}X}{-2(X - \wp(\alpha))Y(X)} = \int_0^{B_1} \frac{\mathrm{d}\xi}{\wp(\xi) - \wp(\alpha)}.$$

The factor $\exp\big(-U_3\int_\infty^{(b,y(b))}\mathrm{d}R_3\big)$ is inessential so can be safely omitted. Next, we compute

$$\mathrm{d}R_1 = \left(\frac{\frac{1}{5}\wp(\alpha)}{-2Y} + \frac{X}{-2Y} + \frac{\frac{9}{25}\wp(\alpha)^2}{-2Y(X - \wp(\alpha))}\right)\mathrm{d}X,$$

and obtain $\int_\infty^{(b,y(b))}\mathrm{d}R_1 = \varpi$. Finally, using (41), we find $b = \wp(B_1) - \frac{2}{5}\wp(\alpha)$. □

REMARK 7. The function $\mathcal{U}$ defined by (52) satisfies the KdV equation

$$4\partial_{U_3}\mathcal{U} = \partial_{U_1}^3\mathcal{U} - 6\mathcal{U}\partial_{U_1}\mathcal{U},$$

and is a stationary solution for higher equations of KdV hierarchy.

Suppose, the roots $\{e_j\}_{j=1}^5$, $\sum_j e_j = 0$, of polynomial $f(x, 0) = x^5 + \lambda_4x^3 + \lambda_6x^2 + \lambda_8x + \lambda_{10}$, that is branch points of the curve (3), are real numbers, and $e_1 \geqslant e_2 \geqslant e_3 \geqslant e_4 \geqslant e_5$. Then, the spectrum of operator in (49) is the union of three segments: $[e_5, e_4] \cup [e_3, e_2] \cup [e_1, \infty]$. When $\lambda \in \Lambda_1$ one of the segments, say $[e_5, e_4]$, contracts to produce a double point A. Under the conditions, we can interpret the results of Corollary 2 in the following way.

COROLLARY 3. *Let* (ω, ω') *be periods of Weierstrass functions and assume* $\operatorname{Im}\omega = 0$, $\operatorname{Re}\omega' = 0$. *Then, provided* $\wp(\alpha) \in \mathbb{R}$, *formula* (52) *defines one parametric families, with parameter* $\varphi \in [-\frac{1}{2}, \frac{1}{2}]$, *of real-valued potentials in variable x on real line*

$$\mathcal{V}_1(x) = \frac{1}{\omega^2}\mathcal{U}(\omega x), \qquad \text{with} \quad U_3 = \frac{2\pi\imath}{\wp'(\alpha)}\varphi;$$

$$\mathcal{V}_2(x) = \frac{1}{\omega^2}\mathcal{U}(\omega x + \tfrac{1}{2}\omega'), \qquad \text{with} \quad U_3 = \frac{2\pi\imath}{\wp'(\alpha)}\varphi + \frac{1}{\wp'(\alpha)}\big(\zeta(\alpha)\omega' - \alpha\eta'\big).$$

The operators $\partial_{xx} - \mathcal{V}_1(x)$ *and* $\partial_{xx} - \mathcal{V}_2(x)$ *share a common spectrum*

$$\{\wp(\alpha)\} \cup \big[\wp(\tfrac{1}{2}\omega'), \wp(\tfrac{1}{2}\omega + \tfrac{1}{2}\omega')\big] \cup \big[\wp(\tfrac{1}{2}\omega), \infty\big].$$

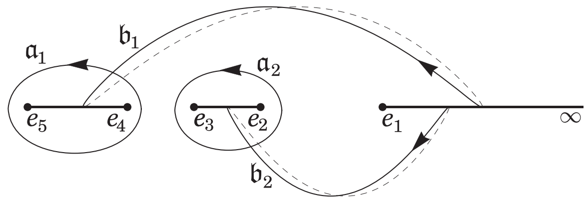

Figure 1. Sketch of branch points and basis homology cycles.

Proof. Under the assumptions $\wp(z)$ is real when z runs from the origin along the boundary of rectangle with sides $\frac{1}{2}\omega$ and $\frac{1}{2}\omega'$. Further, both $(\wp(x),\ \wp'(x))$ and $(\wp(x+\frac{1}{2}\omega'),\ \wp'(x+\frac{1}{2}\omega'))$ are real for $x \in \mathbb{R}$. Let $\alpha \in (0, \frac{1}{2}\omega)$, the functions $\mathcal{P}$ and $\mathcal{S}$ defined by (42) are real valued. At $\alpha \in (\frac{1}{2}\omega + \frac{1}{2}\omega', \frac{1}{2}\omega')$ value of $\wp'(\alpha)$ is real, and $\mathcal{P}/\mathcal{P}^* = 1$, as a result $\mathcal{S}$ is real. At $\alpha \in (\frac{1}{2}\omega', 0) \cup (\frac{1}{2}\omega, \frac{1}{2}\omega + \frac{1}{2}\omega')$ values of $\wp'(\alpha)$ are imaginary, and $\mathcal{P}\mathcal{P}^* = 1$ so $\mathcal{S}$ is imaginary. □

REMARK 8. The above potentials are unbounded except for $\mathcal{V}_2(x)$ with $\varphi \in (-\frac{1}{2}, \frac{1}{2})$ in three cases: (1) $\operatorname{Re}\alpha = 0$, (2) $\operatorname{Re}\alpha = \omega$, (3) $\operatorname{Im}\alpha = 0$.

7.3. Rank 3 lattices. Consider the space $\mathcal{C}$ of curves with a puncture at the common branch point at infinity. Choose the following basis of holomorphic differentials

$$h(x,y) = \big(1,\ x,\ -x^2,\ -(3x^3+\lambda_4 x)\big)^t \frac{\mathrm{d}x}{-2y}, \tag{54}$$

Denote by $\mathfrak{C} = \big(\mathfrak{a}_1,\ \mathfrak{a}_2,\ \mathfrak{b}_2,\ \mathfrak{b}_1\big)$ a basis of homology cycles such that $\mathfrak{a}_i \circ \mathfrak{b}_j = \delta_{ij}$, see Figure 1. Denote by Ω a matrix of integrals of $h(x,y)$ over $\mathfrak{C}$, that is $\Omega = \int_{\mathfrak{C}} h(x,y)$.

If $\lambda \in \Lambda_2$, then $\operatorname{rank}\Omega = 4$ and Ω satisfies Legendre identity

$$\Omega^t J \Omega = 2\pi\imath J, \tag{55}$$

for the symplectic matrix $J = \operatorname{codiag}(1,\ 1,\ -1,\ -1)$. First two rows of Ω generate a rank 4 lattice in $\mathbb{C}^2$, and thus define a two-dimensional complex torus as the quotient of $\mathbb{C}^2$ over the lattice. Meromorphic functions on the torus, that is *four-periodic functions on* $\mathbb{C}^2$, can be derived from sigma-function by taking logarithmic derivatives of order greater than 1. If $\lambda \in \Lambda_1 \cup \Lambda_0$, then $\operatorname{rank}\Omega < 4$.

Introduce the notation

$$\mathcal{F}_k = \{\lambda \in \mathbb{C}^4 \mid \operatorname{rank}\Omega = k\}, \qquad k = 0,\ 1,\ 2,\ 3,\ 4.$$

Evidently, the space $\Lambda \cong \mathbb{C}^4$ is a disjoint union $\Lambda = \cup_{k=0}^4 \mathcal{F}_k$ (cf. Proposition 1), where $\mathcal{F}_4 = \Lambda_2$. Next,

LEMMA 3. *$\mathcal{F}_3$ is the set of simple roots of the discriminant $\Delta(\lambda)$ of* (3)

$$\mathcal{F}_3 = \{\lambda \mid \Delta(\lambda) = 0,\ \partial_\lambda\Delta(\lambda) \neq 0\}.$$

Proof. Evidently, $\mathcal{F}_3 \subset \Lambda_1$. In the case of $\lambda \in \Lambda_1$, we use the transformations (41) to obtain elliptic parametrization $(x,y) = \big(\wp(\xi) - \frac{2}{5}\wp(\alpha),\ -\frac{1}{2}\wp'(\xi)\big(\wp(\xi) - \wp(\alpha)\big)\big)$ with

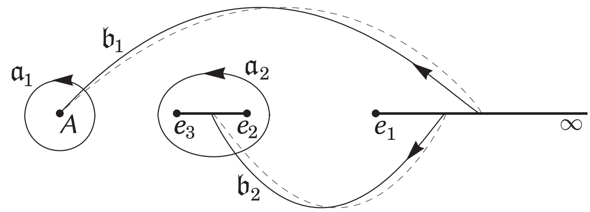

Figure 2. Sketch of branch points and basis homology cycles when two branch points contract.

the uniformizing parameter $\xi \in \mathbb{C}$. Compute the integrals $I(x, y) = \int_\infty^{(x,y)} h(x, y)$ as functions in ξ

$$\begin{aligned}
I_1(\xi) &= \frac{2\zeta(\alpha)}{\wp'(\alpha)}\xi + \frac{1}{\wp'(\alpha)} \log \frac{\sigma(\alpha - \xi)}{\sigma(\alpha + \xi)},\\
I_2(\xi) &= \xi + \tfrac{3}{5}\wp(\alpha) I_1(\xi),\\
I_3(\xi) &= \zeta(\xi) - \tfrac{6}{25}\wp(\alpha)^2 I_1(\xi) - \tfrac{1}{5}\wp(\alpha) I_2(\xi),\\
I_4(\xi) &= -\tfrac{1}{2}\wp'(\xi) - \tfrac{3}{5}\wp(\alpha)\big(\gamma_4 + \tfrac{12}{25}\wp(\alpha)^2\big) I_1(\xi) - \tfrac{9}{25}\wp(\alpha)^2 I_2(\xi) - \tfrac{3}{5}\wp(\alpha) I_3(\xi).
\end{aligned} \tag{56}$$

Now, we calculate the periods. Let $\Omega = \left(\begin{smallmatrix} T_1 & T_2 & T_3 & T_4 \\ H_1 & H_2 & H_3 & H_4 \end{smallmatrix}\right)$, where T_k and H_k are two-dimensional vectors. By taking expansion of $I(\xi)$ near $\xi = \alpha$, we find that

$$\begin{pmatrix} T_1 \\ H_1 \end{pmatrix} = 2\pi\imath \operatorname{Res}_{t=0} I(\alpha + t), \qquad \begin{pmatrix} T_4 \\ H_4 \end{pmatrix} = \infty.$$

For this computations, Figure 2 is instrumental.

On the other hand,

$$\begin{pmatrix} T_2 \\ H_2 \end{pmatrix} = I(\xi + \omega) - I(\xi), \qquad \begin{pmatrix} T_3 \\ H_3 \end{pmatrix} = I(\xi + \omega') - I(\xi).$$

Explicitly, for finite periods, we have

$$(T_1, T_2, T_3) = K_1 \begin{pmatrix} 0 & \omega & \omega' \\ -\frac{\imath\pi}{\alpha} & \eta & \eta' \end{pmatrix}, \tag{57}$$

$$(H_1, H_2, H_3) = K_2 (T_1, T_2, T_3) + K_3 \begin{pmatrix} 0 & \omega & \omega' \\ 0 & \eta & \eta' \end{pmatrix}, \tag{58}$$

where

$$K_1 = \begin{pmatrix} \frac{2}{\wp'(\alpha)}\zeta(\alpha) & -\frac{2}{\wp'(\alpha)}\alpha \\ 1 + \frac{6}{5}\frac{\wp(\alpha)}{\wp'(\alpha)}\zeta(\alpha) & -\frac{6}{5}\frac{\wp(\alpha)}{\wp'(\alpha)}\alpha \end{pmatrix}, \qquad K_2 = \begin{pmatrix} -\frac{9}{25}\wp(\alpha)^2 & 0 \\ 0 & -\big(\gamma_4 + \frac{12}{25}\wp(\alpha)^2\big) \end{pmatrix},$$

$$K_3 = \begin{pmatrix} -\frac{1}{5}\wp(\alpha) & 1 \\ \gamma_4 + \frac{6}{25}\wp(\alpha)^2 & -\frac{3}{5}\wp(\alpha) \end{pmatrix}.$$

and ω, ω' are periods of the Weierstrass function $\wp$, and $\eta = 2\zeta(\omega/2)$, $\eta' = 2\zeta(\omega'/2)$. Thus, when $\lambda \in \Lambda_1$ rank $\Omega = 3$ if and only if $\wp'(\alpha) \neq 0$.

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Table 1. Classification of strata of parameters *Lambda*.

$\deg(f)_0$	(1, 1, 1, 1, 1)	(2, 1, 1, 1)	(3, 1, 1)	(2, 2, 1)	(3, 2)	(4, 1)	(5)
genus g	2	1	1	0	0	0	0
$\#\deg(f)_0 - 1$	4	3	2	2	1	1	0
rank Ω	4	3	2	2	1	1	0

To complete the proof, it remains to notice that on Λ_1 the gradient of the discriminant (5) vanishes together with $\wp'(\alpha)$. Indeed,

$$\partial_\lambda \Delta(\lambda)|_{\Lambda_1} = \frac{1}{5}(4\gamma_4^3 + 27\gamma_6^2)\left(\wp'(\alpha)\right)^6\left(\left(\tfrac{3}{5}\wp(\alpha)\right)^3, \left(\tfrac{3}{5}\wp(\alpha)\right)^2, \tfrac{3}{5}\wp(\alpha), 1\right), \tag{59}$$

here ∂_λ stands for $(\partial_{\lambda_4}, \partial_{\lambda_6}, \partial_{\lambda_8}, \partial_{\lambda_{10}})$. By definition, $4\gamma_4^3 + 27\gamma_6^2$ does not vanish on Λ_1. □

Similarly, $\mathcal{F}_2$ is the set of double zeros of discriminant $\Delta(\lambda)$, and $\mathcal{F}_1$ is the set of triple zeros of discriminant $\Delta(\lambda)$:

$$\mathcal{F}_2 = \{\lambda \mid \partial_\lambda \Delta(\lambda) = 0,\ \partial_\lambda^2 \Delta(\lambda) \neq 0\},$$
$$\mathcal{F}_1 = \{\lambda \mid \partial_\lambda^2 \Delta(\lambda) = 0,\ \partial_\lambda^3 \Delta(\lambda) \neq 0\}.$$

Further, $\mathcal{F}_0$ is the set of 4-tuple zeros of discriminant $\Delta(\lambda)$, which is a singe point $\lambda = 0$.

On the other hand, let $f(x) = x^5 + \sum_{k=0}^{3} \lambda_{10-2k} x^k$. Divisor of zeros $(f)_0$ is a formal product $p_1^{d_1} p_2^{d_2} \dots p_5^{d_5}$, p_i are distinct points, integers d_i are non-negative and $\sum d_i = 5$, while $\sum d_i p_i = 0$. Assume $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_5$ and denote $\deg(f)_0 = (d_1, d_2, \dots)$, for non-zero d_i. Clearly, $\deg(f)_0$ takes values in partitions of number 5. Denote the dimension of corresponding subset of Λ by m, we have $m = \#\deg(f)_0 - 1$. In fact, m equals the dimension of a component of $\mathcal{F}_m$. Table 1 gives summary of all possible cases.

This completes description of the stratification of Λ by the rank of corresponding lattice.

REMARK 9. Since $\mathcal{F}_2$ has non-empty intersections with both Λ_1 and Λ_0, cf. Table 1, the two-periodic functions on the associated 'torus' can be of different nature: those that are essentially a combination of rational and elliptic functions, see Remark 4, and those that are combinations of exponential functions, see Theorem 2. The strata $\mathcal{F}_1$ and $\mathcal{F}_0$ are associated with exponential and rational functions, respectively.

7.4. Three-periodic functions. On the stratum $\mathcal{F}_3$ in the place of identity (55), we have

$$\begin{pmatrix} T_1 & T_2 & T_3 \\ H_1 & H_2 & H_3 \end{pmatrix}^t J \begin{pmatrix} T_1 & T_2 & T_3 \\ H_1 & H_2 & H_3 \end{pmatrix} = 2\pi\imath \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

COROLLARY 4. *For all $u = (u_3,\ u_1) \in \mathbb{C}^2$ sigma-function $\boldsymbol{\sigma}(u;\lambda)$ obeys the periodicity property*

$$\left.\frac{\boldsymbol{\sigma}(u \pm T_k;\lambda)}{\boldsymbol{\sigma}(u;\lambda)}\right|_{\lambda\in\mathcal{F}_3} = -\exp\left\{\pm H_k^t \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(u \pm \tfrac{1}{2}T_k\right)\right\}, \qquad k = 1,\ 2,\ 3.$$

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Proof follows directly from the periodicity property of genus 2 sigma-function.

REMARK 10. The function $\Phi(u,\beta)$ defined by (48) has Bloch property on $\mathcal{F}_4=\Lambda_2$ and keeps the property when restricted to $\mathcal{F}_3$

$$\Phi(u+T_i,\beta)=\Phi(u,\beta)\mathrm{e}^{M_iT_i},\qquad i=1,2,3.$$

The 'quasi-momenta' M_i are given by rather cumbersome expressions, which, however, can be readily deduced from (56), (57) and (58) in a condensed form. Let $\beta^t=(I_1(\alpha),I_2(\alpha))$ and $\rho=(I_4(\alpha),I_3(\alpha))$, where α is the image of double point A, that is $\wp(\alpha)=A$. Then, we have

$$M_1=\rho+\beta^tK_2,\qquad M_2=M_3=\rho+\beta^t(K_2+K_3K_1^{-1}).$$

Now, return to discussing three-periodic functions. Over $\mathcal{F}_3$ any order greater than 1 logarithmic derivative of sigma-function is a three-periodic function.

Introduce the function

$$\mathcal{P}(u_3,u_1)=\frac{\sigma\big(\alpha+u_1-\frac{3}{5}\wp(\alpha)u_3\big)}{\sigma\big(\alpha-u_1+\frac{3}{5}\wp(\alpha)u_3\big)}\,\mathrm{e}^{\big(\wp'(\alpha)+\frac{6}{5}\wp(\alpha)\zeta(\alpha)\big)u_3-2\zeta(\alpha)u_1},\tag{60}$$

with $\wp'(\alpha)\neq 0$. It is straightforward to verify that T_1,T_2,T_3 are periods of the function $\mathcal{P}(u_3,u_1)$.

COROLLARY 5. *Any meromorphic three-periodic function in two variables* (u_3,u_1) *with the periods* T_1,T_2,T_3 *is a rational function of* $\boldsymbol{\mathcal{P}}_{\text{basis}}=\big(\mathcal{P}(u_3,u_1),\ \wp(u_1-\frac{3}{5}\wp(\alpha)u_3),\ \wp'(u_1-\frac{3}{5}\wp(\alpha)u_3),\ \wp(\alpha),\ \wp'(\alpha)\big)$.

Proof. Any genus 2 Abelian function, that is a meromorphic four-periodic function of (u_3,u_1), has a unique representation as the rational function of $\boldsymbol{\wp}_{\text{basis}}=(\wp_{11},\wp_{13},\wp_{111},\wp_{113},\wp_{1111},\wp_{1113})$, in particular

$$\begin{aligned}
\lambda_4&=\tfrac{1}{2}\wp_{1111}-3\wp_{11}^2-2\wp_{13},\\
\lambda_6&=\tfrac{1}{2}\wp_{1113}-\tfrac{1}{2}\wp_{1111}\wp_{11}+\tfrac{1}{4}\wp_{111}^2+2\wp_{11}^3-2\wp_{13}\wp_{11},\\
\lambda_8&=-\tfrac{1}{2}\wp_{1113}\wp_{11}-\tfrac{1}{2}\wp_{1111}\wp_{13}+\tfrac{1}{2}\wp_{113}\wp_{111}+\wp_{13}^2+4\wp_{11}^2\wp_{13},\\
\lambda_{10}&=-\tfrac{1}{2}\wp_{1113}\wp_{13}+\tfrac{1}{4}\wp_{113}^2+2\wp_{13}^2\wp_{11}.
\end{aligned}\tag{61}$$

The composition of $\Delta(\lambda)$, see (5), with (61) defines a polynomial $\Delta\big(\lambda(\boldsymbol{\wp}_{\text{basis}})\big)$. When $\lambda\in\Lambda_1$, the polynomial $\Delta\big(\lambda(\boldsymbol{\wp}_{\text{basis}})\big)$ should vanish, while $\delta(\gamma)=4\gamma_4^3+27\gamma_6^2$ should be non-zero. Taking into account

$$\begin{aligned}
\gamma_4&=\frac{\wp'(U_1)^2-4\wp(U_1)^3-\big(\wp'(\alpha)^2-4\wp(\alpha)^3\big)}{4\big(\wp(U_1)-\wp(\alpha)\big)},\\
\gamma_6&=-\frac{\wp(\alpha)\big(\wp'(U_1)^2-4\wp(U_1)^3\big)-\wp(U_1)\big(\wp'(\alpha)^2-4\wp(\alpha)^3\big)}{4\big(\wp(U_1)-\wp(\alpha)\big)},
\end{aligned}$$

where $U_1=u_1-\frac{3}{5}\wp(\alpha)u_3$, we see the condition $4\gamma_4^3+27\gamma_6^2\neq 0$ turns into a condition on a polynomial in $\wp(U_1)$, $\wp'(U_1)$, $\wp(\alpha)$ and $\wp'(\alpha)$. By (40)–(43) we can express $\wp_{11}$,

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$\wp_{13}, \wp_{111}, \wp_{113}$ as rational functions of $\boldsymbol{\mathcal{P}}_{\text{basis}}$. Differentiating expressions for $\wp_{111}, \wp_{113}$ with respect to u_1 we get the rational functions

$$\begin{aligned}
\wp_{1111} &= 6\mathcal{S}^4 - 8\big(2\wp(U_1) + \wp(\alpha)\big)\mathcal{S}^2 + 4\wp'(U_1)\mathcal{S} \\
&\quad + 4\wp(U_1)^2 + 10\wp(\alpha)\wp(U_1) + 4\wp(\alpha)^2 - \frac{\wp'(U_1)^2 - \wp'(\alpha)^2}{2\big(\wp(U_1) - \wp(\alpha)\big)}, \\
\wp_{1113} &= -6\big(\wp(U_1) - \tfrac{2}{5}\wp(\alpha)\big)\mathcal{S}^4 + 6\wp'(U_1)\mathcal{S}^3 \\
&\quad + \left(4\wp(U_1)^2 + \tfrac{38}{5}\wp(\alpha)\wp(U_1) + \tfrac{14}{5}\wp(\alpha)^2 - \frac{3\big(\wp'(U_1)^2 - \wp'(\alpha)^2\big)}{2\big(\wp(U_1) - \wp(\alpha)\big)}\right)\mathcal{S}^2 \\
&\quad - \tfrac{2}{5}\wp'(U_1)\Big(10\wp(U_1) + 11\wp(\alpha)\Big)\mathcal{S} + 6\wp(\alpha)\big(\tfrac{2}{5}\wp(U_1)^2 + \wp(\alpha)\wp(U_1) + \tfrac{2}{5}\wp(\alpha)^2\big) \\
&\quad - \tfrac{4}{5}\big(\wp(U_1)^2 + \wp(\alpha)^2\big) + \frac{9\wp(U_1)\big(\wp'(U_1)^2 + \wp'(\alpha)^2\big)}{5\big(\wp(U_1) - \wp(\alpha)\big)}.
\end{aligned}$$

These rational expressions for $\boldsymbol{\wp}_{\text{basis}}$ substituted in $\Delta\big(\lambda(\boldsymbol{\wp}_{\text{basis}})\big)$ make it vanish identically. Furthermore, we can re-express a rational function of $\boldsymbol{\wp}_{\text{basis}}$ as a rational function of $\boldsymbol{\mathcal{P}}_{\text{basis}}$. $\square$

The parametrization of $\boldsymbol{\wp}_{\text{basis}}$ by rational functions of $\boldsymbol{\mathcal{P}}_{\text{basis}}$ is analogous to the parametrization of λ in terms of a_2 and γ, cf. (13). In fact, the former parametrization is induced by the latter, which is clearly seen if we follow the connection between sigma-function $\boldsymbol{\sigma}$ and generators $\boldsymbol{\wp}_{\text{basis}}$ of the field of fiber-wise Abelian functions on the universal space of genus 2 Jacobi varieties.

Remark 11. Note that the function $f(z_1, z_2) = \mathcal{P}(z_1/\wp'(\alpha), cz_2 + \frac{3}{5}\wp(\alpha)z_1/\wp'(\alpha))$, where $c \neq 0$ is an arbitrary number, is a solution of the following system of functional equations

$$f(z_1, z_2)f(z_1, -z_2) = \exp(z_1), \qquad f(z_1, z_2) = -f(-z_1, -z_2).$$

Proposition 2. *A field of three-periodic functions is a transcendental extension of the field of elliptic functions with transcendence degree 1.*

Proof follows from Corollary 5, the function $\mathcal{P}(u_3, u_1)$ serves as the transcendental element.

8. Concluding remarks. For all values of parameters λ sigma-function $\boldsymbol{\sigma}(u; \lambda)$ is essentially a function of the same nature, that is remains holomorphic and entire in all its arguments for singular curves as well. It possesses a periodicity property, in particular, for the case of actual genus 1 and $\partial_\lambda \Delta \neq 0$ it is given in Corollary 4, and the functions $-\partial_{u_i}\partial_{u_j} \log \boldsymbol{\sigma}(u; \lambda)$ are three-periodic. The whole classification of degeneration strata is given in Table 1. 'Degenerate' expressions (24) and (33), at special values of parameters λ, are useful for solving generalized Jacobi inversion problem, and Schrödinger equation with periodic potential.

The technique we use above can be extended almost literally to higher genera hyperelliptic sigma-functions. Generalization to non-hyperelliptic sigma-functions is a challenging problem. Based on (24) and well-known formula for degenerate Weierstrass

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sigma-function [15], namely when $(g_2, g_3) \mapsto (12a^2, -8a^3)$

$$\sigma(u) \mapsto \tfrac{1}{2}\sqrt{3a}\,\mathrm{e}^{-\frac{1}{2}au^2}\big(\mathrm{e}^{\sqrt{3a}u} - \mathrm{e}^{-\sqrt{3a}u}\big),$$

we conjecture that evaluation of a genus g hyperelliptic sigma-function at a stratum of parameters Λ_{g-1}, where genus of the underlying curve falls by 1, has similar structure

$$\boldsymbol{\sigma}(u) \mapsto C\mathrm{e}^{-u^t \mathcal{Q} u}\Big(\sigma(\mathcal{A}+u)\mathrm{e}^{\mathcal{M}^t u} - \sigma(\mathcal{A}-u)\mathrm{e}^{-\mathcal{M}^t u}\Big).$$

Here, sigma-function on the left-hand side is in genus g, while sigma-function on the right-hand side is in genus $g-1$, scalar C, $g \times g$ matrix $\mathcal{Q}$ and vectors $\mathcal{A}$ and $\mathcal{M}$ are expressed with the hep of first and second kind Abelian integral as functions of the coordinates of a double point and the parameters of genus $g-1$ curve corresponding to a point in Λ_{g-1}. From this viewpoint, sigma-function in genus 0 is a constant function, say, 1. We can regard the result of degeneration as an action of an operator $\mathcal{T}$, which is in essence an evaluation operator. Then, properly tuned operators $\mathcal{T}(a)$ and $\mathcal{T}(b)$ associated with double points at a and b commute with respect to composition, which opens a possibility to study further degeneration of sigma-function in a more abstract setting.

For the generalized Jacobi inversion problem considered in Section 7.1 a solution is known within the framework of the generalized Theta-function theory, which was originated by Clebsch and Gordan [**17**], whereas the algebraic description of generalized Jacobians arose in [**23**]. It was further developed by Previato [**22**], Fedorov [**19**], Braden and Yu. Fedorov [**16**] and others. A connection between the degenerate sigma-function (24) and the generalized Theta-function can be traced through the relation between sigma- and theta-functions in genus 1, see [**15**].

The subject of Section 7.2 may be viewed as the simplest examples of a potential of mixed solitonic and finite-gap nature. It is of considerable interest to explicitly construct potentials that possess arbitrary collection of points and segments in the place of spectra.

In general, lattices of odd ranks satisfying Riemann and Schottky conditions lead to generalized Jacobi varieties, see [**18**, **22**]. The rank three lattice from Section 7.3 is an example of that. The corresponding generalized Jacobi variety is a product of a cylinder and a torus. At the same time, we conjecture that a field of $2g+1$-periodic functions can be effectively constructed as a transcendental extension of the field of hyperelliptic Abelian functions in genus g with help of a single transcendental element of a form similar to (60), namely

$$\mathcal{P}(u_{g+1}, u) = \frac{\sigma(\alpha+u)}{\sigma(\alpha-u)} \exp\big\{c(\alpha)u_{g+1} + d(\alpha)^t u\big\},$$

where σ denotes genus g sigma-function, $u, \alpha \in \mathbb{C}^g$, $u_{g+1} \in \mathbb{C}$ and $c(\alpha)$, $d(\alpha)$ are appropriate functions.

Our study of three-periodic functions of two complex variables will be extended in our future publications, in particular, we plan to derive explicit form of addition law, and to find a special dynamical system solvable by these functions like the KdV equation is solved by the Weierstrass elliptic function.

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