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Sharp Sobolev regularity of restricted X-ray transforms

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We study L^p-Sobolev regularity estimates for the restricted X-ray transforms generated by nondegenerate curves. Making use of the inductive strategy in the recent work by the authors, we establish the sharp L^p-regularity estimates for the restricted X-ray transforms in $\mathbb{R}^{d+1}$, $d \geqslant 3$. This extends the result due to Pramanik and Seeger in $\mathbb{R}^3$.

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1. Introduction

Let γ be a smooth curve from $I = [-1, 1]$ to $\mathbb{R}^d$. We consider

$$\mathfrak{R}f(x, s) = \psi(s) \int f(x + t\gamma(s), t)\chi(t)\,\mathrm{d}t, \quad f \in \mathcal{S}(\mathbb{R}^{d+1}),$$

where ψ and χ are smooth functions supported in the interiors of the intervals I and $[1, 2]$, respectively. The operator $\mathfrak{R}f$ is referred to as the restriction of X-ray transform to the line complex generated by the directions $(\gamma(s), 1)$, $s \in \operatorname{supp}\psi$. We say γ is nondegenerate if

$$\det(\gamma'(s), \ldots, \gamma^{(d)}(s)) \neq 0, \quad \forall s \in I. \tag{1.1}$$

The operator $\mathfrak{R}f$ is a model case of the general class of restricted X-ray transforms (see [**12**, **16**–**19**]). Especially in $\mathbb{R}^3$, under the nondegeneracy assumption (1.1), $\mathfrak{R}f$ is a typical example of Fourier integral operators with one-sided fold singularity [**13**]. Regularity properties of $\mathfrak{R}f$ have been studied in terms of L^p improving and L^p Sobolev regularity estimates. The L^p improving property of $\mathfrak{R}$ is well understood by now [**14**, **15**, **24**, **25**]. The problem was, in fact, considered in a more general framework: $L^p - L^q(L^r)$ estimates for $\mathfrak{R}$ were studied by some authors (see, e.g. [**7**, **11**, **34**]) and the estimates on the optimal range of p, q, r were established except for some endpoint cases. (See also [**8**–**10**, **20**, **33**] for related results.)

$L^2 - L^2_{1/(2d)}$ bound on $\mathfrak{R}$ is easy to obtain via TT^* argument and van der Corput's lemma [**16**] (see also [**13**, **19**] for the sharp L^2 Sobolev estimates for general class of operators). Interpolation between this and the trivial L^∞ estimate shows that $\mathfrak{R}$ is bounded from L^p to $L^p_{1/(pd)}$ for $p \geqslant 2$. This is optimal in that $L^p - L^p_\alpha$ estimate fails if $\alpha > 1/(pd)$ (see Proposition 5.1). However, when $p < 2$, the sharp L^p regularity estimate is less straightforward. Such an estimate has not known until recently. When $d = 2$, the optimal $L^p - L^p_{1/p'}$ estimate was established for $1 < p < 4/3$ by Pramanik and Seeger's conditional result [**31**] and the sharp decoupling inequality for the cone $\subset \mathbb{R}^3$ due to Bourgain and Demeter [**5**]. Those estimates and interpolation give the sharp $L^p - L^p_{1/(2d+)}$ estimate for $4/3 \leqslant p < 2$, but the endpoint $L^p - L^p_{1/(2d)}$ estimate remains open. (See Conjecture 1.1.) In $\mathbb{R}^3$ the result has been extended to more general operators. In fact, Pramanik and Seeger [**30**] obtained the sharp L^p regularity estimates for the Fourier integral operator with folding canonical relation. Bentsen [**4**] (also see [**3**]) extended the result to a class of radon transforms with fold and blowdown singularities.

However, in higher dimensions ($d \geqslant 3$) the sharp L^p regularity estimates for $\mathfrak{R}$ have remained open for $1 < p < 2$. Set $p_d = 2d/(2d-1)$ and

$$\alpha(p) = \begin{cases} 1 - \frac{1}{p}, & 1 \leqslant p < p_d, \\ \frac{1}{2d}, & p_d \leqslant p \leqslant 2. \end{cases}$$

It is natural to conjecture the following.

Conjecture 1.1. *Let $d \geqslant 3$ and $1 < p < 2$. Suppose γ is a smooth nondegenerate curve. Then, $\mathfrak{R}$ boundedly maps L^p to L^p_α for $\alpha \leqslant \alpha(p)$.*

Failure of $L^p - L^p_\alpha$ boundedness for $\alpha > \alpha(p)$ can be shown by a slight modification of the examples in [**31**]. (See Proposition 5.1.) The following is our main result which verifies the conjecture except for some endpoint cases in every dimension $d \geqslant 3$.

Theorem 1.2. *Let $d \geqslant 3$ and $1 \leqslant p < p_d$. Suppose γ is nondegenerate. Then,*

$$\|\mathfrak{R}f\|_{L^p_\alpha} \leqslant C\|f\|_p \tag{1.2}$$

holds if and only if $\alpha \leqslant 1 - 1/p$.

When $p \in [p_d, 2)$, interpolation with the $L^2 - L^2_{1/(2d)}$ estimate yields (1.2) for $\alpha < \alpha(p)$ but estimate (1.2) with the endpoint regularity $\alpha = \alpha(p)$, which looks to be a subtle problem, remains open. By a standard scaling argument [**29**, **31**] the result in Theorem 1.2 can be extended to the curves of finite type.

A curve $\gamma : I \mapsto \mathbb{R}^d$ is said to be of finite type if there is an $L = L(s)$ such that $\operatorname{span}\{\gamma^{(1)}(s), \dots, \gamma^{(L)}(s)\} = \mathbb{R}^d$ for each $s \in I$, and the smallest of such $L(s)$ is called the type at s. The supremum of the type over $s \in I$ is called the maximal type of γ (see, e.g. [**21**, **29**]).

COROLLARY 1.3. *Let $d \geqslant 3$, $1 \leqslant p < 2$ and $L > d$. Suppose γ is a curve of maximal type L. Then, $\mathfrak{R}f$ is bounded from L^p to L^p_α for $\alpha \leqslant \min(\alpha(p), 1/(Lp))$ if $p \neq (L+1)/L$ when $L \geqslant 2d-1$, and if $p \in (1, p_d) \cup (2d/L, 2)$ when $d < L < 2d-1$.*

For $p \in [2, \infty]$, it is easy to show the sharp $L^p - L^p_{1/(Lp)}$ estimate, which can be shown by using the $L^2 - L^2_{1/(2L)}$ estimate and interpolation in a similar manner as above. Corollary 1.3 and Proposition 5.1 give the optimal Sobolev regularity estimate for $\mathfrak{R}$ if $L \geqslant 2d-1$ when $p \neq (L+1)/L$. However, some endpoint cases remain open not to mention such estimates for the nondegenerate curves.

In this paper, we make use of the inductive strategy in the recent work of the authors [**23**], where smoothing properties of the (convolution) averaging operator over curves were studied (see [**2**, **22**, **26**, **27**, **32**] for previous works). Exploiting similarity between $\mathfrak{R}^*f$ and the averaging operator, we adapt our previous argument. The main new feature of the current paper is use of the decoupling inequality associated with the conical sets generated by curves (see Definition 2.5 and Theorem 3.1). Compared with our previous work where the averaging operator was decoupled by a class of symbols adjusted to short subcurves, our new decoupling inequality allows us to dispense with some technicalities related to the symbols. The decoupling inequality can also be used to simplify the argument in [**23**].

Organization. In § 2, we reduce the proof of Theorem 1.2 to obtain Proposition 2.4. We prove a decoupling inequality associated with a nondegenerate curve (Theorem 3.1) in § 3 which is crucial for the proof of Proposition 2.4. The proofs of Proposition 2.4 and Theorem 1.2 are given in § 4 and § 5, respectively. We discuss the sharpness of the smoothing order α in § 5.

Notation. For positive constants A, D, we denote $A \lesssim D$ if there exists a (independent) constant C such that $A \leqslant CD$, where the constant C may vary from line to line depending on the context.

2. Estimates with localized frequency

In this section, we reduce the proof of Theorem 1.2 to show an inductive statement (see Proposition 2.4). Afterwards, we obtain some preliminary results which are needed to prove Proposition 2.4.

Let us consider the operator

$$\mathcal{R}f(x,t) = \chi(t) \int f(x - t\gamma(s), s)\psi(s)\,\mathrm{d}s,$$

which is the dual operator of $\mathfrak{R}$. By duality estimate (1.2) is equivalent to

$$\|\mathcal{R}f\|_{L^p(\mathbb{R}^{d+1})} \lesssim \|f\|_{L^p_{-1/p}}, \quad 2d < p < \infty. \tag{2.1}$$

For the purpose, we closely follow the line of arguments in our previous paper [**23**]. So, there is a significant overlap between the current paper and [**23**]. This can be avoided by omitting some shared details. However, we decide to include them so that the paper is self-contained and more easily accessible.

2.1. Frequency localized estimate

We begin with defining a class of curves in order to prove (2.1) in an inductive manner. For an integer $1 \leqslant L \leqslant d$, by $\mathrm{Vol}(v_1, \dots, v_L)$ we denote the L-dimensional volume of the parallelepiped generated by vectors $v_1, \dots, v_L \in \mathbb{R}^d$.

Definition 2.1. *Let* $B \geqslant 1$. *We say* $\gamma \in \mathfrak{V}^d(L, B)$ *if* $\gamma \in C^{3d+1}(I)$ *satisfies*

$$\max_{s\in I} |\gamma^{(j)}(s)| \leqslant B, \quad 0 \leqslant j \leqslant 3d+1, \tag{2.2}$$

$$\min_{s\in I} \mathrm{Vol}\left(\gamma^{(1)}(s), \dots, \gamma^{(L)}(s)\right) \geqslant B^{-1}. \tag{2.3}$$

For a smooth function $a(s, t, \xi)$ on $I \times [1, 2] \times \mathbb{R}^d$, we define

$$\mathcal{R}[a]f(x, t) = (2\pi)^{-d} \iint \mathrm{e}^{i(x - t\gamma(s))\cdot\xi} a(s, t, \xi)\, \mathcal{F}_x f(\xi, s)\, \mathrm{d}s\, \mathrm{d}\xi.$$

Here, $\mathcal{F}_x$ denotes Fourier transform in x. Note that $\mathcal{R}f = \mathcal{R}[a]f$ if $a(s, t, \xi) = \psi(s)\chi(t)$. We prove estimate (2.1) by induction on L for $\gamma \in \mathfrak{V}^d(L, B)$ under the localized nondegeneracy assumption:

$$\sum_{\ell=1}^{L} |\langle \gamma^{(\ell)}(s), \xi \rangle| \geqslant B^{-1}|\xi| \tag{2.4}$$

which holds if $(s, t, \xi) \in \operatorname{supp} a$ for some t. When $L < d$, (2.4) cannot be true in general even if γ is nondegenerate. However, an appropriate decomposition in the frequency domain makes it possible that (2.4) holds. To do this, we consider a class of symbols a.

Definition 2.2. *Let* $\mathbb{A}_k = \{\xi \in \mathbb{R}^d : 2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}\}$ *for* $k \geqslant 0$, *and* $\mathcal{I}_L = \{(j, \alpha) : 0 \leqslant j \leqslant 2L, |\alpha| \leqslant d + L + 2\}$. *We say a symbol* $a \in C^{d+L+2}(\mathbb{R}^{d+2})$ *is of type* $(2^k, L, B)$ *if* $\operatorname{supp} a \subset I \times [2^{-1}, 2^2] \times \mathbb{A}_k$

$$|\partial_t^j \partial_\xi^\alpha a(s, t, \xi)| \leqslant B|\xi|^{-|\alpha|}, \quad (j, \alpha) \in \mathcal{I}_L,$$

and (2.4) *holds on* $\operatorname{supp}_{s,\xi} a$. *Here, as in* [**23**], *we denote* $\operatorname{supp}_{s,\xi} a = \cup_t \operatorname{supp} a(\cdot, t, \cdot)$. *We simply say a statement* $S(s, \xi)$, *depending on* s, ξ, *holds on* $\operatorname{supp} a$ *if* $S(s, \xi)$ *holds for* $s, \xi \in \operatorname{supp}_{a,\xi} a$. *We also use the same convention with other variables.*

Estimate (2.1) (and hence Theorem 1.2) follows from the next theorem via a standard argument using Fefferman–Stein #-function (e.g. see [**28**]). See § 5.1 for details.

Theorem 2.3. *Suppose that* $\gamma \in \mathfrak{V}^d(L, B)$ *and* a *is a symbol of type* $(2^k, L, B)$. *Then, for* $p > 2L$

$$\|\mathcal{R}[a]f\|_p \leqslant C 2^{-k/p} \|f\|_p. \tag{2.5}$$

As mentioned above, we prove Theorem 2.3 by induction on L. Theorem 2.3 with $L = 1$ is easy to prove. Indeed, setting $\tilde{\mathcal{R}}f = \mathcal{F}_x(\mathcal{R}[a]\mathcal{F}_x^{-1}f)$, we note that

$$\tilde{\mathcal{R}}^*\tilde{\mathcal{R}}f(\xi, s) = \int \mathcal{K}(s, s', \xi) f(\xi, s')\, \mathrm{d}s',$$

where

$$\mathcal{K}(s, s', \xi) = \int \mathrm{e}^{it(\gamma(s)-\gamma(s'))\cdot\xi}\overline{a}(s, t, \xi)a(s', t, \xi)\, \mathrm{d}t.$$

Since (2.4) holds with $L = 1$ on $\operatorname{supp} a$, integration by parts gives $|\mathcal{K}(s, s', \xi)| \leqslant C(1 + 2^k|s - s'|)^{-2}$. By Young's convolution inequality it follows that $\|\tilde{\mathcal{R}}^*\tilde{\mathcal{R}}f\|_2 \lesssim 2^{-k}\|f\|_2$. Thus, we get $\|\mathcal{R}[a]f\|_2 \lesssim 2^{-k/2}\|f\|_2$ by Plancherel's theorem. Interpolation with the trivial estimate $\|\mathcal{R}[a]f\|_\infty \lesssim \|f\|_\infty$ gives (2.5) with $L = 1$.

Consequently, Theorem 2.3 for $L \geqslant 2$ follows from the next proposition (cf. [**23**, Proposition 2.3]).

Proposition 2.4. *Let $2 \leqslant N \leqslant d$. Suppose Theorem 2.3 holds with $L = N - 1$. Then, Theorem 2.3 holds with $L = N$.*

We prove the proposition through the rest of this section, § 3 and § 4. Fixing $2 \leqslant N \leqslant d$, we assume that Theorem 2.3 holds with $L = N - 1$. Additionally, assuming that $\gamma \in \mathfrak{V}^d(N, B)$ and a is of type $(2^k, N, B)$, we prove (2.5) for $p > 2N$. For the purpose, composing the symbol a, we may further assume that

$$|\gamma^{(N)}(s) \cdot \xi| \geqslant (2B)^{-1}|\xi| \tag{2.6}$$

holds on $\operatorname{supp} a$. Otherwise, (2.4) holds with $L = N - 1$, so the hypothesis (Theorem 2.3 with $L = N - 1$) yields (2.5) for $p > 2(N - 1)$.

We prove Proposition 2.4 in § 4 using the associated decoupling inequality which is obtained in § 3. The rest of the section is devoted to proving two lemmas (2.6 and 2.8) which play crucial roles in proving Proposition 2.4.

2.2. Symbols adapted to γ

We define a class of symbols adapted to the curve γ. From now on, we assume that δ satisfies

$$2^{-k/N} \leqslant \delta \leqslant (2^2B)^{-N}. \tag{2.7}$$

Let γ satisfy (2.3) with $L = N - 1$. For $s \in I$, set $\mathrm{V}_s^{\gamma,\ell} = \operatorname{span}\{\gamma^{(j)}(s) : j = 1, \dots, \ell\}$. Consider a linear map $\tilde{\mathcal{L}}_s^\delta : \mathbb{R}^d \mapsto \mathbb{R}^d$ given as follows:

$$(\tilde{\mathcal{L}}_s^\delta)^\intercal\gamma^{(j)}(s) = \delta^{N-j}\gamma^{(j)}(s), \quad j = 1, \dots, N - 1,$$
$$(\tilde{\mathcal{L}}_s^\delta)^\intercal v = v, \quad v \in \left(\mathrm{V}_s^{\gamma,N-1}\right)^\perp.$$

We also consider a linear map $\mathcal{L}_s^\delta : \mathbb{R}^{d+1} \mapsto \mathbb{R}^{d+1}$ given by

$$\mathcal{L}_s^\delta(\tau, \xi) = \left(\delta^N\tau - \gamma(s) \cdot \tilde{\mathcal{L}}_s^\delta\xi,\ \tilde{\mathcal{L}}_s^\delta\xi\right), \quad (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d.$$

Denoting $G(s) = (1, \gamma(s))$, we set

$$\Lambda_k(\delta, s) = \bigcap_{0\leqslant j\leqslant N-1} \left\{(\tau, \xi) \in \mathbb{R} \times \mathbb{A}_k : |\langle G^{(j)}(s), (\tau, \xi)\rangle| \leqslant B2^{k+5}\delta^{N-j}\right\},$$

which roughly corresponds to the Fourier support of the operator $\mathcal{R}[a]f$ with $\operatorname{supp}_s a$ included in an interval centred at s of length about δ. We define a class of symbols associated with $\Lambda_k(\delta, s)$.

Definition 2.5. *Let* $s_\circ \in (-1, 1)$ *and* $0 < \delta \leqslant 1$ *such that* $I(s_\circ, \delta) := [s_\circ - \delta, s_\circ + \delta] \subset I$. *We denote by* $\mathfrak{A}_k(\delta, s_\circ) = \mathfrak{A}_k(\delta, s_\circ, d, N, B, \gamma)$ *the set of smooth functions* $\mathfrak{a}$ *on* $\mathbb{R}^{d+3}$ *which satisfies the following*:

$$\operatorname{supp} \mathfrak{a} \subset I(s_\circ, \delta) \times [1, 2] \times \Lambda_k(\delta, s_\circ), \tag{2.8}$$

$$\left|\partial_t^j \partial_{\tau,\xi}^\alpha \mathfrak{a}\left(s, t, \mathcal{L}_{s_\circ}^\delta(\tau, \xi)\right)\right| \leqslant B|(\tau, \xi)|^{-|\alpha|}, \quad (j, \alpha) \in \mathcal{I}_N. \tag{2.9}$$

It should be noted that there is no s-differentiation in (2.9). Here, $\mathcal{I}_N$ is given in Definition 2.2. We set

$$\mathcal{F}(\mathcal{T}[\mathfrak{a}]f)(\xi, \tau) = \iint \mathrm{e}^{-it'(\tau+\gamma(s)\cdot\xi)}\mathfrak{a}(s, t', \tau, \xi)\, \mathrm{d}t'\, \mathcal{F}_x f(\xi, s)\, \mathrm{d}s. \tag{2.10}$$

Clearly, $\mathcal{R}[a]f = \mathcal{T}[a]f$ if $\mathfrak{a} = a(s, t, \xi)$. The following is an analogue of [**23**, Lemma 2.7].

Lemma 2.6. *Let* $\tilde\chi \in \mathrm{C}_0^\infty((2^{-2}, 2^2))$ *such that* $\tilde\chi = 1$ *on* $[3^{-1}, 3]$. *Let* $\mathfrak{a}$ *be a smooth function which satisfies* (2.8) *and* (2.9) *with* $j \leqslant 2$ *and* $|\alpha| \leqslant d + 3$. *Then, we have*

$$\|\mathcal{T}[\mathfrak{a}]f\|_p \leqslant C\delta^{1-1/p}\|f\|_p \tag{2.11}$$

for $p \geqslant 2$, *and*

$$\|(1 - \tilde\chi(t))\mathcal{T}[\mathfrak{a}]f\|_p \leqslant C\delta^{1-1/p-N}2^{-k}\|f\|_p, \quad p > 1. \tag{2.12}$$

Proof. Note that $\mathcal{T}[\mathfrak{a}]f(x, t) = \int K[\mathfrak{a}](s, t, \cdot) * f(\cdot, s)(x)\, \mathrm{d}s$ where

$$K[\mathfrak{a}](s, t, x) = \frac{1}{(2\pi)^{d+1}} \iiint \mathrm{e}^{i(t-t')\tau+i(x-t'\gamma(s))\cdot\xi}\mathfrak{a}(s, t', \tau, \xi)\, \mathrm{d}\xi\, \mathrm{d}\tau\, \mathrm{d}t'.$$

It is easy to show that $|(\mathcal{L}_{s_\circ}^\delta)^{-1}\mathcal{L}_s^\delta(\tau, \xi)| \sim |(\tau, \xi)|$ provided $|s - s_\circ| \leqslant \delta$ (cf. [**23**, Lemma 2.6]). Since (2.8) and (2.9) hold with $j = 0$ and $|\alpha| \leqslant d + 3$, it follows that $\operatorname{supp} \mathfrak{a}(s, t, 2^k\mathcal{L}_s^\delta\cdot) \subset \{(\tau, \xi) : |(\tau, \xi)| \lesssim 1\}$ and $|\partial_{\tau,\xi}^\alpha(\mathfrak{a}(s, t, 2^k\mathcal{L}_s^\delta(\tau, \xi)))| \lesssim 1$, $|\alpha| \leqslant d + 3$. By changing variables $(\tau, \xi) \to 2^k\mathcal{L}_s^\delta(\tau, \xi)$ followed by repeated integration by parts, we have

$$|K[\mathfrak{a}](s,t,x)| \lesssim \delta^{N(N+1)/2} 2^{k(d+1)} \int_1^2 \left(1 + 2^k |(\delta^N(t-t'), (\tilde{\mathcal{L}}_s^\delta)^\intercal (x - t\gamma(s)))|\right)^{-d-3} \mathrm{d}t'.$$

This gives $\|K[\mathfrak{a}](s,t,\cdot)\|_{L^1_x} \lesssim 1$. From (2.8), note $\mathcal{T}[\mathfrak{a}]f(x,t) = \int_{I(s_\circ,\delta)} K[\mathfrak{a}](s,t,\cdot) * f(\cdot,s)(x)\,\mathrm{d}s$. Thus, we get

$$\|\mathcal{T}[\mathfrak{a}]f\|_\infty \leqslant C\delta \|f\|_\infty.$$

Recall (2.10). By Plancherel's theorem, integration by parts in t', and Hölder's inequality, we get

$$\|\mathcal{T}[\mathfrak{a}]f\|_2^2 \lesssim \delta \int_{I(s_\circ,\delta)} \|\mathcal{F}_x f(\cdot,s)\|_2^2 \,\mathrm{d}s \lesssim \delta \|f\|_2^2.$$

Thus, interpolation gives (2.11). To show (2.12), we note from the above estimate for $K[\mathfrak{a}](s,t,x)$ that $\|(1-\tilde{\chi}(t))K[\mathfrak{a}](s,t,\cdot)\|_{L^1_x} \lesssim \mathfrak{K}(t) =: 2^{-k}\delta^{-N}|t-1|^{-1}(1-\tilde{\chi}(t))$. By (2.8), using Hölder's and Young's convolution inequalities, as before, we see that $\|(1-\tilde{\chi})\mathcal{T}[\mathfrak{a}]f\|_p^p$ is bounded above by constant times

$$\delta^{p-1} \int \mathfrak{K}^p(t) \int_{I(s_\circ,\delta)} \|f(\cdot,s)\|_{L^p_x}^p \,\mathrm{d}s\,\mathrm{d}t \lesssim C\delta^{p-1-pN} 2^{-pk} \|f\|_p^p.$$

This gives (2.12). □

2.3. Rescaling

Let $I(s_\circ,\delta) \subset I$. For $\gamma \in \mathfrak{V}^d(N,B)$ we consider a rescaled curve

$$\gamma_{s_\circ}^\delta(s) := \delta^{-N} (\tilde{\mathcal{L}}_{s_\circ}^\delta)^\intercal \left(\gamma(\delta s + s_\circ) - \gamma(s_\circ)\right).$$

Lemma 2.7. *Let $\gamma \in \mathfrak{V}^d(N,B)$. If $0 < \delta < \delta_*$ for a δ_* small enough, $\gamma_{s_\circ}^\delta \in \mathfrak{V}^d(N,3B)$ and $\gamma_{s_\circ}^\delta \in \mathfrak{V}^d(N-1,B')$ for some B'.*

Proof. Taylor series expansion of $\gamma^{(j)}(\delta s + s_\circ)$ at $s = 0$ yields:

$$(\gamma_{s_\circ}^\delta)^{(\ell)}(s) = \sum_{0 \leqslant j \leqslant N-1-\ell} \gamma^{(\ell+j)}(s_\circ)\frac{s^j}{j!} + (\tilde{\mathcal{L}}_{s_\circ}^\delta)^\intercal \gamma^{(N)}(s_\circ)\frac{s^{N-\ell}}{(N-\ell)!} + O(B\delta)$$

for $1 \leqslant \ell \leqslant N-1$ and $(\gamma_{s_\circ}^\delta)^{(N)}(s) = (\tilde{\mathcal{L}}_{s_\circ}^\delta)^\intercal \gamma^{(N)}(s_\circ) + O(B\delta)$. Writing $\gamma^{(N)}(s_\circ) = v_1 + v_2 \in \mathrm{V}_{s_\circ}^{\gamma,N-1} \oplus (\mathrm{V}_{s_\circ}^{\gamma,N-1})^\perp$, we have $(\tilde{\mathcal{L}}_{s_\circ}^\delta)^\intercal \gamma^{(N)}(s_\circ) = (\tilde{\mathcal{L}}_{s_\circ}^\delta)^\intercal v_1 + v_2 = v_2 + O(B\delta)$. Since $\gamma \in \mathfrak{V}^d(N,B)$, we see $\gamma_{s_\circ}^\delta \in \mathfrak{V}^d(N,3B)$ if $0 < \delta < \delta_*$ for a sufficiently small $\delta_* > 0$. Consequently, $\gamma_{s_\circ}^\delta \in \mathfrak{V}^d(N-1,B')$ for some B'. □

The following lemma, which is an analogue of [**23**, Lemma 2.8], is important for our inductive argument. Let us set

$$\mathcal{R}[\gamma_{s_\circ}^\delta, a]f(x,t) = (2\pi)^{-d} \iint \mathrm{e}^{i(x - t\gamma_{s_\circ}^\delta(s))\cdot\xi} a(s,t,\xi)\mathcal{F}_x f(\xi,s)\,\mathrm{d}s\,\mathrm{d}\xi.$$

LEMMA 2.8. *Let* $s_\circ \in (-1,1)$, $\mathfrak{a} \in \mathfrak{A}_k(\delta, s_\circ)$, *and* $\gamma \in \mathfrak{V}^d(N,B)$. *Suppose*

$$\sum_{j=1}^{N-1} \delta^j |\langle \gamma^{(j)}(s), \xi\rangle| \geqslant B^{-1} 2^k \delta^N \tag{2.13}$$

for $(s,\xi) \in I(s_\circ, \delta) \times \operatorname{supp}_\xi \mathfrak{a}$. *Then, there exist constants* C, $\tilde{B}$, $\delta_* = \delta_*(B,N,d)$, *and* $\tilde{f}$ *and a symbol* $\tilde{a}$ *such that*

$$\|\tilde{\chi}(t)\mathcal{T}[\mathfrak{a}]f\|_p = \delta^{1-1/p} \|\mathcal{R}[\gamma^\delta_{s_0}, \tilde{a}]\tilde{f}\|_p \tag{2.14}$$

for $0 < \delta < \delta_*$, $\|\tilde{f}\|_p = \|f\|_p$, $|\partial_t^j \partial_\xi^\alpha \tilde{a}(s,t,\xi)| \leqslant \tilde{B}|\xi|^{-|\alpha|}$ *for* $(j,\alpha) \in \mathcal{I}_{N-1}$, *and*

$$\operatorname{supp} \tilde{a} \subset I \times [2^{-2}, 2^2] \times \{\xi \in \mathbb{R}^d : C^{-1}\delta^N 2^k \leqslant |\xi| \leqslant C\delta^N 2^k\}. \tag{2.15}$$

Proof. Let $\mathfrak{a}_\delta(s,t,\tau,\xi) = \mathfrak{a}(\delta s + s_\circ, t, \tau, \xi)$. By Fourier inversion and (2.10), changing variables $s \to \delta s + s_\circ$, $(\tau,\xi) \to (\tau - \gamma(s_\circ)\cdot\xi, \xi)$ gives

$$\mathcal{T}[\mathfrak{a}]f(x,t) = (2\pi)^{-d}\delta \iint \mathrm{e}^{i\langle x - t\gamma(s_\circ), \xi\rangle} b(s,t,\xi)\mathcal{F}_x f(\xi, \delta s + s_\circ)\,\mathrm{d}s\,\mathrm{d}\xi, \tag{2.16}$$

where

$$b(s,t,\xi) = \frac{1}{2\pi}\iint \mathrm{e}^{it\tau}\,\mathrm{e}^{-it'(\tau + \langle\gamma(\delta s + s_\circ) - \gamma(s_\circ), \xi\rangle)}\mathfrak{a}_\delta\left(s, t', \tau - \gamma(s_\circ)\cdot\xi, \xi\right)\,\mathrm{d}t'\,\mathrm{d}\tau.$$

We observe that

$$\tilde{\chi}(t) b(s,t,\delta^{-N}\tilde{\mathcal{L}}^\delta_{s_\circ}\xi) = \mathrm{e}^{-it\gamma^\delta_{s_\circ}(s)\cdot\xi}\,\tilde{a}(s,t,\xi),$$

where

$$\tilde{a}(s,t,\xi) = \frac{1}{2\pi}\iint \mathrm{e}^{-it'(\tau + \gamma^\delta_{s_\circ}(s)\cdot\xi)}\tilde{\chi}(t)\mathfrak{a}_\delta(s, t'+t, \delta^{-N}\mathcal{L}^\delta_{s_\circ}(\tau,\xi))\,\mathrm{d}t'\,\mathrm{d}\tau. \tag{2.17}$$

It is clear that (2.15) holds for some $C \geqslant 1$. Since $\mathfrak{a} \in \mathfrak{A}_k(\delta, s_\circ)$, it is not difficult to see $|\partial_t^j\partial_\xi^\alpha \tilde{a}(s,t,\xi)| \leqslant \tilde{B}|\xi|^{-|\alpha|}$ for $(j,\alpha) \in \mathcal{I}_{N-1}$ (see (2.25) in [**23**]).

Set $\mathcal{C}_p = \mathcal{C}_p(\delta) := \delta^{1/p}|\det \delta^{-N}\tilde{\mathcal{L}}^\delta_{s_\circ}|^{1-1/p}$. Let $\tilde{f}$ be given by $\mathcal{F}_x\tilde{f}(\xi,s) = \mathcal{C}_p\mathcal{F}_x f(\delta^{-N}\tilde{\mathcal{L}}^\delta_{s_\circ}\xi, \delta s + s_\circ)$, thus $\|\tilde{f}\|_p = \|f\|_p$. Recalling (2.16) and changing variables $\xi \to \delta^{-N}\tilde{\mathcal{L}}^\delta_{s_\circ}\xi$, we now have

$$\tilde{\chi}(t)\mathcal{T}[\mathfrak{a}]f(x,t) = \frac{\mathcal{C}_{p'}}{(2\pi)^d}\iint \mathrm{e}^{i\langle x - t\gamma(s_\circ), \delta^{-N}\tilde{\mathcal{L}}^\delta_{s_\circ}\xi\rangle}\,\mathrm{e}^{-it\gamma^\delta_{s_\circ}(s)\cdot\xi}\,\tilde{a}(s,t,\xi)\mathcal{F}_x\tilde{f}(\xi,s)\,\mathrm{d}s\,\mathrm{d}\xi.$$

This gives $\tilde{\chi}(t)\mathcal{T}[\mathfrak{a}]f(x,t) = \mathcal{C}_{p'}\mathcal{R}[\gamma^\delta_{s_\circ}, \tilde{a}]\tilde{f}(y,t)$ where $y = \delta^{-N}(\tilde{\mathcal{L}}^\delta_{s_\circ})^\intercal(x - t\gamma(s_\circ))$. Therefore, changing variable $x \to \delta^N(\tilde{\mathcal{L}}^\delta_{s_\circ})^{-\intercal}x + t\gamma(s_\circ)$, we obtain (2.14). ☐

Combining Lemma 2.8 and the hypothesis (Theorem 2.3 with $L = N - 1$), we obtain the following.

Corollary 2.9. *Suppose that Theorem 2.3 holds with* $L = N - 1$, *and* $\mathfrak{a}$, γ *and* δ_* *are the same as in Lemma 2.8. Then, if* $p > 2(N-1)$, *for* $0 < \delta < \delta_*$ *we have*

$$\|\mathcal{T}[\mathfrak{a}]f\|_p \lesssim 2^{-k/p}\delta^{1-(N+1)/p}\|f\|_p.$$

Proof. By (2.14) and dyadic decomposition (of $\tilde{a}$ in the Fourier side), we have

$$\|\tilde{\chi}\mathcal{T}[\mathfrak{a}]f\|_p \leqslant C\delta^{1-1/p}\sum_{0\leqslant \ell\leqslant C}\big\|\mathcal{R}[\gamma^{\delta}_{s_\circ}, a_\ell]f_\ell\big\|_p, \tag{2.18}$$

for some constant C where $\|f_\ell\|_p = \|f\|_p$, and a_ℓ are symbols of type $(2^j, N-1, \tilde{B})$ with $C^{-1}2^k\delta^N \leqslant 2^j \leqslant C2^k\delta^N$. Once we have this, the proof is straightforward. By Lemma 2.7, $\gamma^{\delta}_{s_\circ} \in \mathfrak{V}^d(N-1, B')$ for some $B' > 0$. Since $\|f_l\|_p = \|f\|_p$, applying Theorem 2.3 with $L = N - 1$, we have

$$\|\tilde{\chi}\mathcal{T}[\mathfrak{a}]f\|_p \leqslant C\sum_l \delta^{1-1/p}(2^k\delta^N)^{-1/p}\|f_l\|_p \lesssim 2^{-k/p}\delta^{1-(N+1)/p}\|f\|_p$$

for $p > 2(N-1)$. Recalling (2.7), we combine this and (2.12) to get the desired bound.

It remains to show (2.18). In fact, after applying Lemma 2.8 we only need to adjust the support of the consequent symbol $\tilde{a}$ via by moderate decomposition and scaling. We omit details. (See the proof of [**23**, Lemma 2.8].) □

3. Decoupling inequalities for curves

In this section, we prove the decoupling inequality, which is to be used to decompose the operator $\mathcal{T}[\mathfrak{a}]f$. In our earlier work [**23**], the averaging operator was decoupled by making use of decomposition based on a class of symbols that are adjusted to short subcurves. The same approach also works to prove Proposition 2.4. However, instead of following the previous strategy, we directly obtain a decoupling inequality associated with the conic sets:

$$\Lambda_k(\delta, s_l), \quad 1 \leqslant l \leqslant L,$$

while $\{s_1, \ldots, s_L\} \subset I$ is a collection of δ-separated points contained in I. More precisely, we have the following.

Theorem 3.1. *Let* $0 < \delta \leqslant 1$ *and* $S := \{s_1, \ldots, s_L\} \subset I$ *be a collection of* δ*-separated points. Then, if* $2 \leqslant p \leqslant N(N+1)$, *for any* $\epsilon > 0$ *there is a constant* $C_\epsilon = C_\epsilon(B)$, *independent of* S, *such that*

$$\Big\|\sum_{1\leqslant l\leqslant L} f_l\Big\|_{L^p(\mathbb{R}^{d+1})} \leqslant C_\epsilon\delta^{-\epsilon}\Big(\sum_{1\leqslant l\leqslant L}\|f_l\|^2_{L^p(\mathbb{R}^{d+1})}\Big)^{1/2} \tag{3.1}$$

holds whenever $\operatorname{supp}\widehat{f_l} \subset \Lambda_k(\delta, s_l)$.

Hölder's inequality gives $\|\sum_{1\leqslant l\leqslant L} f_l\|_p \leqslant C_\epsilon\delta^{-\epsilon}\delta^{1/p-1/2}(\sum_{1\leqslant l\leqslant L}\|f_l\|_p^p)^{1/p}$. Interpolation with the trivial $L^\infty-\ell^\infty L^\infty$ estimate yields the inequality:

$$\Big\|\sum_{1\leqslant l\leqslant L} f_l\Big\|_{L^p(\mathbb{R}^{d+1})} \leqslant C_\epsilon\delta^{-1+(N+1)/p+\epsilon}\Big(\sum_{1\leqslant l\leqslant L}\|f_l\|^p_{L^p(\mathbb{R}^{d+1})}\Big)^{1/p} \tag{3.2}$$

for $p>2N$ whenever $\operatorname{supp}\widehat{f_l}\subset\Lambda_k(\delta,s_l)$.

3.1. Decoupling inequalities for curves

Fixing $N\geqslant 2$, we now consider the slabs given by an anisotropic neighbourhood of the moment curve

$$\gamma_\circ(s) := \big(s,\ s^2/2!,\dots,s^{N+1}/(N+1)!\big)\,.$$

Definition 3.2. *Let $0<\delta\leqslant 1$ and $B\geqslant 1$. For $s\in I$, let $\mathbf{S}(s,\delta,B)$ denote the set of $(\tau,\xi)\in\mathbb{R}\times\mathbb{R}^N$ such that*

$$B^{-1}\leqslant|\langle\gamma_\circ^{(N+1)}(s),(\tau,\xi)\rangle|\leqslant B;\quad |\langle\gamma_\circ^{(j)}(s),(\tau,\xi)\rangle|\leqslant\delta^{N+1-j},\quad j=1,\dots,N.$$

We now recall the decoupling inequality for such slabs as above which was shown in [**1**, **6**] (see also [**23**, Corollary 2.15]).

Theorem 3.3. *Let $0<\delta\leqslant 1$ and $\{s_1,\dots,s_L\}\subset I$ be a collection of δ-separated points contained in I. Denote $\mathbf{S}_l=\mathbf{S}(s_l,\delta,B)$. Then, if $2\leqslant p\leqslant N(N+1)$, for any $\epsilon>0$ there is a constant $C_\epsilon=C_\epsilon(B)$ such that*

$$\Big\|\sum_{1\leqslant l\leqslant L} F_l\Big\|_{L^p(\mathbb{R}^{N+1})}\leqslant C_\epsilon\delta^{-\epsilon}\Big(\sum_{1\leqslant l\leqslant L}\|F_l\|^2_{L^p(\mathbb{R}^{N+1})}\Big)^{1/2}$$

holds whenever $\operatorname{supp}\widehat{F_l}\subset\mathbf{S}_l$.

To show Theorem 3.1, we apply the decoupling inequality after projecting the sets $\Lambda_0(\delta,s_l)$ to the subspace V_μ which is spanned by $\{G^{(0)}(\mu),\dots,G^{(N)}(\mu)\}$. To do so, for $\mu\in I$ we consider a coordinate system $\mathbf{y}_\mu=\mathbf{y}_\mu(\tau,\xi)$ given by

$$\mathbf{y}_\mu=(y^0_\mu,\cdots,y^N_\mu)=(\langle G^{(0)}(\mu),(\tau,\xi)\rangle,\dots,\langle G^{(N)}(\mu),(\tau,\xi)\rangle). \tag{3.3}$$

Recall that $\gamma\in\mathfrak{V}^d(N,B)$, so $\mathrm{Vol}(\langle G^{(0)}(\mu),\dots,G^{(N)}(\mu))\geqslant 1/B$. Let δ,δ' be positive numbers satisfying

$$0<\delta<\delta'\leqslant\delta^{N/(N+1)}\leqslant 1. \tag{3.4}$$

Then, it is easy to see that

$$(\delta')^{\ell+1}\leqslant\delta^\ell,\quad \ell=1,\dots,N. \tag{3.5}$$

The following lemma shows that the projections of the sets $\Lambda_0(\delta,s_l)$ form a reverse δ/δ'-adapted cover after a proper linear change of variables (cf. [**23**, Lemma 3.3]) if s_l are contained in an interval of length δ'. Let D_δ denote the $(N+1)\times(N+1)$ diagonal matrix given by

$$\mathrm{D}_\delta = (\delta^{-N}e_1, \delta^{-N+1}e_2, \dots, \delta^0 e_{N+1}).$$

Lemma 3.4. *Let δ, δ' be positive numbers satisfying (3.4) and $s' \in [\mu-\delta', \mu+\delta']$. Suppose $(\tau,\xi)\in\Lambda_0(\delta,s')$. Then, we have*

$$(4B)^{-1} \leqslant |\langle \mathrm{D}_{\delta'}\mathbf{y}_\mu, \gamma_\circ^{(N+1)}\rangle| \leqslant 4B, \tag{3.6}$$

$$\left|\left\langle \mathrm{D}_{\delta'}\mathbf{y}_\mu, \gamma_\circ^{(j)}\left(\frac{s'-\mu}{\delta'}\right)\right\rangle\right| \lesssim B\,(\delta/\delta')^{N+1-j}, \quad 1\leqslant j\leqslant N. \tag{3.7}$$

Proof. Note that (3.6) is clear from (2.6). To prove (3.7), we first note that $\langle \mathbf{y}_\mu, \gamma_\circ^{(j)}(s)\rangle = (\delta')^{N+1-j}\langle \mathrm{D}_{\delta'}\mathbf{y}_\mu, \gamma_\circ^{(j)}(s/\delta')\rangle$. Thus, it is sufficient to show that

$$|\langle \mathbf{y}_\mu, \gamma_\circ^{(j)}(s'-\mu)\rangle| \lesssim B\delta^{N+1-j} \tag{3.8}$$

for $1\leqslant j\leqslant N$. Recalling (3.3), we observe

$$\langle \mathbf{y}_\mu, \gamma_\circ^{(j)}(s'-\mu)\rangle = \left\langle \sum_{\ell=j-1}^{N} G^{(\ell)}(\mu)\frac{(s'-\mu)^{\ell-j+1}}{(\ell-j+1)!}, (\tau,\xi)\right\rangle.$$

Taylor's theorem gives

$$\left| G^{(j-1)}(s') - \sum_{\ell=j-1}^{N} G^{(\ell)}(\mu)\frac{(s'-\mu)^{\ell-j+1}}{(\ell-j+1)!}\right| \leqslant B|s'-\mu|^{N-j+2}$$

for $j=1,\dots,N$. Since $|s'-\mu|\leqslant\delta'$ and $(\tau,\xi)\in\Lambda_0(\delta,s')$, (3.8) follows by (3.5). □

By Lemma 3.4 and Theorem 3.3, we can show that (3.1) holds if a δ-separated set $\{s_1,\dots,s_L\}$ is contained in an interval of length $\lesssim \delta^{N/(N+1)}$. More precisely, we have the following.

Lemma 3.5. *Let $0<\delta\leqslant 1$ and $\delta\leqslant\delta'\leqslant\delta^{N/(N+1)}$. Let $\{s_1,\dots,s_L\}\subset[\mu-\delta',\mu+\delta']$ be a collection of δ-separated points. Then, if $2\leqslant p\leqslant N(N+1)$, for any $\epsilon>0$ there is a constant $C_\epsilon = C_\epsilon(B)$ such that (3.1) holds whenever* $\operatorname{supp}\widehat{f_l}\subset\Lambda_k(\delta,s_l)$.

Proof. Set $\mathrm{V}_\mu = \operatorname{span}\{\gamma'(\mu),\dots,\gamma^{(N)}(\mu)\}$ and let $\{v_{N+1},\dots,v_d\}$ be an orthonormal basis of $\mathrm{V}_\mu^\perp$. Recalling that (2.3) holds with $L=N$, we write $\xi=\overline{\xi}+$

$\sum_{j=N+1}^{d} y_j(\xi)v_j$ for $\overline{\xi} \in \mathrm{V}_\mu$. Changing variables

$$(\tau,\xi) \to \mathrm{Y}_\mu(\tau,\xi) := (\mathbf{y}_\mu(\tau,\xi), y_{N+1}(\xi), \dots, y_d(\xi))$$

(see (3.3)), we may work with the coordinate system given by $\{\mathbf{y}_\mu, y_{N+1}, \dots, y_d\}$ instead of (τ,ξ). We consider the linear map

$$\mathrm{Y}_\mu^{\delta'}(\tau,\xi) = (\mathrm{D}_{\delta'}\mathbf{y}_\mu(\tau,\xi), y_{N+1}(\xi), \dots, y_d(\xi)).$$

Since $\{s_1,\dots,s_L\} \subset [\mu-\delta', \mu+\delta']$ and $\delta' \leqslant \delta^{N/(N+1)}$, by Lemma 3.4 it follows that

$$\mathrm{Y}_\mu^{\delta'}(\Lambda_0(\delta,s_l)) \subset \mathbf{S}_l \times \mathbb{R}^{d-N} := \mathbf{S}\left(\frac{s_l-\mu}{\delta'}, C\frac{\delta}{\delta'}, 4B\right) \times \mathbb{R}^{d-N} \tag{3.9}$$

for some $C>0$ depending only on B. Applying Theorem 3.3 with δ replaced by $C\delta/\delta'$ and slabs $\mathbf{S}_l, 1 \leqslant l \leqslant L$, and then using a trivial extension via Minkowski's inequality, we have

$$\Big\| \sum_{1\leqslant l\leqslant L} f_l \Big\|_p \leqslant C_\epsilon \delta^{-\epsilon} \Big(\sum_{1\leqslant l\leqslant L} \|f_l\|_p^2 \Big)^{1/2}$$

for $2 \leqslant p \leqslant N(N+1)$ whenever $\operatorname{supp} \widehat{f_l} \subset \mathbf{S}_l \times \mathbb{R}^{d-N}$. Since the decoupling inequality is invariant under affine changes of variables, by undoing the change of variables $(\tau,\xi) \to \mathrm{Y}_\mu(\tau,\xi)$ and rescaling $(\tau,\xi) \to 2^{-k}(\tau,\xi)$, we obtain (3.1) whenever $\operatorname{supp} \widehat{f_l} \subset \Lambda_k(\delta,s_l)$. ☐

3.2. Proof of Theorem 3.1

We now prove Theorem 3.1. Let $2 \leqslant p \leqslant N(N+1)$. For the purpose, for some $\alpha > 0$ we assume that

$$\Big\| \sum_{1\leqslant l\leqslant L} f_l \Big\|_{L^p(\mathbb{R}^{d+1})} \leqslant C\delta^{-\alpha} \Big(\sum_{1\leqslant l\leqslant L} \|f_l\|_{L^p(\mathbb{R}^{d+1})}^2 \Big)^{1/2} \qquad \mathfrak{D}(\alpha) \tag{3.1}$$

holds for $0 < \delta \leqslant \delta_0 := (2^2B)^{-N-1}$ with a constant C, independent of S, whenever $\operatorname{supp} \widehat{f_l} \subset \Lambda_k(\delta,s_l)$, $1 \leqslant l \leqslant L$. Of course, ($\mathfrak{D}(\alpha)$) holds true if $\alpha \geqslant 1/2$ by Minkowski's and Hölder's inequalities. We set

$$\delta' = \delta^{N/(N+1)}.$$

Let us denote I_ν, $1 \leqslant \nu \leqslant M$, be disjoint intervals of length $\rho \in (2^{-3}\delta', 2^{-2}\delta']$ which partition I. Let s'_ν be a point contained in I_ν such that $s'_1,\dots,s'_M$ are separated at least by $2^{-4}\delta'$. We now claim that

$$\Lambda_k(\delta,s_l) \subset \Lambda_k(\delta',s'_\nu) \tag{3.10}$$

if $s_l \in I_\nu$. Indeed, by scaling it is sufficient to show $\Lambda_0(\delta,s_l) \subset \Lambda_0(\delta',s'_\nu)$. Let $(\tau,\xi) \in \Lambda_0(\delta,s_l)$. Then, it follows that $|\langle G^{(\ell)}(s_l), (\tau,\xi)\rangle| \leqslant 2^5 B\delta^{1/(N+1)}(\delta')^{N-\ell}$. By Taylor's theorem we have

$$\langle G^{(j)}(s'_\nu),(\tau,\xi)\rangle = \sum_{\ell=j}^{N-1}\langle G^{(\ell)}(s_l),(\tau,\xi)\rangle\frac{(s'_\nu-s_l)^{\ell-j}}{(\ell-j)!}+\mathcal{E},$$

where $|\mathcal{E}|\leqslant 2B|s'_\nu-s_l|^{N-j}$. Therefore, we see that $(\tau,\xi)\in\Lambda_0(\delta',s'_\nu)$.

Let $\operatorname{supp}\widehat{f_l}\subset\Lambda_k(\delta,s_l)$, $1\leqslant l\leqslant L$. We write $\sum_{1\leqslant l\leqslant L}f_l=\sum_{1\leqslant\nu\leqslant M}\sum_{s_l\in I_\nu}f_l$. By (3.10), the Fourier support of $\sum_{s_l\in I_\nu}f_l$ is included in $\Lambda_k(\delta',s'_\nu)$. Since s'_ν are separated by $2^{-4}\delta'$, ($\mathfrak{D}(\alpha)$) implies

$$\|\sum_{1\leqslant l\leqslant L}f_l\|_{L^p(\mathbb{R}^{d+1})}\leqslant C\delta^{-N\alpha/(N+1)}\Big(\sum_{1\leqslant\nu\leqslant M}\|\sum_{s_l\in I_\nu}f_l\|^2_{L^p(\mathbb{R}^{d+1})}\Big)^{1/2}$$

for a constant C. Since the length of interval I_ν is less than $\delta^{N/(N+1)}$, by Lemma 3.5 we have $\|\sum_{s_l\in I_\nu}f_l\|_p\leqslant C_\epsilon\delta^{-\epsilon}(\sum_{s_l\in I_\nu}\|f_l\|_p^2)^{1/2}$. Therefore, combining this and the above inequality, we obtain

$$\|\sum_{1\leqslant l\leqslant L}f_l\|_{L^p(\mathbb{R}^{d+1})}\leqslant C_\epsilon\delta^{-N\alpha/(N+1)-\epsilon}\Big(\sum_{1\leqslant l\leqslant L}\|f_l\|^2_{L^p(\mathbb{R}^{d+1})}\Big)^{1/2}$$

for a constant C_ϵ. This establishes the implication $\mathfrak{D}(\alpha)\to\mathfrak{D}(\epsilon+N\alpha/(N+1))$. Iteration of this implication suppresses α arbitrarily small.

4. Proof of Proposition 2.4

In this section, we prove Proposition 2.4 by making use of the decoupling inequality (3.2). As mentioned in § 2.1 (below Proposition 2.4), in order to prove Proposition 2.4, it suffices to show Theorem 2.3 with $L=N$. We first reduce the matter to obtaining estimates for $\mathcal{T}[\mathfrak{a}_0]$ with a suitable $\mathfrak{a}_0$.

4.1. Reduction

We begin by recalling $\gamma\in\mathfrak{V}^d(N,B)$ and a is of type $(2^k,N,B)$. Let δ_* be the small number given in Lemma 2.8 and set

$$\delta_\circ=\min\{\delta_*,(2^2B)^{-N}\}. \tag{4.1}$$

Let $\beta_0\in \mathrm{C}_0^\infty([-1,1])$ such that $\beta_0=1$ on $[-1/2,1/2]$. We set

$$a_N(s,t,\xi)=a(s,t,\xi)\prod_{1\leqslant j\leqslant N-1}\beta_0\left(100\,dB2^{-k}\delta_\circ^{-N}\langle\gamma^{(j)}(s),\xi\rangle\right).$$

Clearly, (2.4) holds on $\operatorname{supp}(a-a_N)$ with $L=N-1$ and B replaced by $100\,dB\delta_\circ^{-N}$. Since a is of type $(2^k,N,B)$, it is easy to see $(a-a_N)$ is a symbol of type $(2^k,N-1,B')$ for some B'. Thus, the hypothesis (Theorem 2.3 with $L=N-1$ and $B=B'$)

gives the estimate

$$\|\mathcal{R}[a-a_N]f\|_p \lesssim 2^{-(k/p)}\|f\|_p$$

for $p>2(N-1)$. So, we need only to consider $\mathcal{R}[a_N]$ instead of $\mathcal{R}[a]$. Furthermore, by a moderate decomposition of a_N, we assume

$$\operatorname{supp}_s a_N \subset [s_\circ-\delta_\circ, s_\circ+\delta_\circ]$$

for some $s_\circ \in (-1,1)$. We may assume that $s_\circ=\delta_\circ\nu$ for $\nu\in\mathbb{Z}$.

It is not difficult to see that the contribution of the frequency part $\{(\tau,\xi): |\tau+\gamma(s)\cdot\xi| \gtrsim 2^{k+1}\delta_\circ^N, \forall s\in I\}$ is not significant. To see this, let us set

$$\mathfrak{a}_0(s,t,\tau,\xi) = a_N(s,t,\xi)\beta_0\left(\delta_\circ^{-2N}2^{-2k}|\tau+\gamma(s)\cdot\xi|^2\right)$$

and $\mathfrak{a}_1=\mathfrak{a}_0-a_N$. Recalling (2.10), by Fourier inversion we have

$$\mathcal{R}[a_N]f = \mathcal{T}[\mathfrak{a}_0]f+\mathcal{T}[\mathfrak{a}_1]f.$$

The operator $\mathcal{T}[\mathfrak{a}_1]$ is easy to handle. Let us set $\mathfrak{a}=-i2^k\delta_\circ^N(\tau+\langle\gamma(s),\xi\rangle)^{-1}\partial_t\mathfrak{a}_1$. Then, by integration by parts in t' and (2.10) we see $\mathcal{T}[\mathfrak{a}_1]=(2^k\delta_\circ^N)^{-1}\mathcal{T}[\mathfrak{a}]$. Note that $|\tau+\gamma(s)\cdot\xi|\gtrsim 2^k\delta_\circ^N$ on $\operatorname{supp}\mathfrak{a}_1$ and so on $\operatorname{supp}\mathfrak{a}$. It is clear that $\mathfrak{a}$ satisfies (2.8) and (2.9) with $\delta=\delta_\circ$ and $B=C_1\delta_\circ^{-C}$ for some large C, C_1. Thus, Lemma 2.6 gives $\|\mathcal{T}[\mathfrak{a}_1]f\|_p\lesssim 2^{-k}\|f\|_p$ for $p\geqslant 2$.

Therefore, the proof of Theorem 2.3 with $L=N$ is now reduced to show that

$$\|\mathcal{T}[\mathfrak{a}_0]f\|_p \leqslant C2^{-(k/p)}\|f\|_p, \quad p>2N. \tag{4.2}$$

4.2. Decomposition

For $n\geqslant 0$, let us set $\delta_n=2^n2^{-k/N}$ and

$$\mathfrak{J}_n=\delta_n\mathbb{Z}\cap I. \tag{4.3}$$

We consider

$$\mathfrak{G}_N(s,\tau,\xi)=\sum_{0\leqslant j\leqslant N-1}(2^{-k}|\langle G^{(j)}(s),(\tau,\xi)\rangle|)^{2N!/(N-j)},$$

by which we can decompose $\mathfrak{a}_0$ into the symbols contained in $\mathfrak{A}_k(\delta_n,s)$ for $s\in\mathfrak{J}_n$.

Set $\beta_*=\beta_0-\beta_0(2^{2N!}\cdot)$. Note that $\beta_0+\sum_{n\geqslant 1}\beta_*(2^{-2N!n}\cdot)=1$. Let $\zeta\in \mathrm{C}_0^\infty([-1,1])$ such that $\sum_{\nu\in\mathbb{Z}}\zeta(\cdot-\nu)=1$. We set

$$\mathfrak{a}_\nu^n=\mathfrak{a}_0\times\begin{cases}\beta_0(\delta_0^{-2N!}\,\mathfrak{G}_N)\,\zeta(\delta_0^{-1}s-\nu), & \nu\in\mathfrak{J}_0,\ n=0,\\ \beta_*(\delta_n^{-2N!}\,\mathfrak{G}_N)\,\zeta(\delta_n^{-1}s-\nu), & \nu\in\mathfrak{J}_n,\ n\geqslant 1.\end{cases}$$

Then, it follows that

$$\mathfrak{a}_0(s,t,\tau,\xi)=\sum_{n\geqslant 0}\sum_{\nu\in\mathfrak{J}_n}\mathfrak{a}_\nu^n(s,t,\tau,\xi). \tag{4.4}$$

Since $\delta_\circ$ is the fixed constant, it is clear that $C^{-1}\mathfrak{a}_0\in\mathfrak{A}_k(\delta_\circ,s_\circ)$ for a large constant $C>0$. So, $\operatorname{supp}\mathfrak{a}_0\subset\Lambda_k(\delta_\circ,s_\circ)$ and $\mathfrak{G}_N\lesssim 1$ for $(\tau,\xi)\in\operatorname{supp}\mathfrak{a}_\nu^n$. Obviously, we may assume $\delta_n\lesssim 1$ since $\mathfrak{a}_\nu^n=0$ otherwise.

The following tells that $\mathfrak{a}_\nu^n$ is contained in a proper symbol class.

Lemma 4.1 (cf. [**23**, Lemma 3.2]). *For* $n \geqslant 0$, *there exists a constant* C *such that* $C^{-1}\mathfrak{a}_\nu^n \in \mathfrak{A}_k(\delta_n, \delta_n\nu)$.

Proof. Condition (2.8) trivially holds for $\mathfrak{a} = \mathfrak{a}_\nu^n$. So, we only need to show (2.9) for $\delta = \delta_n$ and $s = \delta_n\nu$.

It is not difficult to see that $\mathfrak{a}_0$ satisfies (2.9) (see [**23**, (3.35)]). So it suffices to show (2.9) for $\beta_N(\delta_n^{-2N!}\mathfrak{G}_N(s, \tau, \xi))$. By Leibniz's rule, it is enough to prove that

$$\big|\nabla_{\tau,\xi}\delta_n^{-(N-j)}2^{-k}\big\langle G^{(j)}(s), \mathcal{L}^{\delta_n}_{\delta_n\nu}(\tau,\xi)\big\rangle\big| \lesssim 2^{-k}, \tag{4.5}$$

for $j = 0, \ldots, d-1$. Note that if $|\delta_n\nu - s| \leqslant \delta_n$, then

$$|(\mathcal{L}_s^{\delta_n})^{-1}\mathcal{L}^{\delta_n}_{\delta_n\nu}(\tau,\xi)| \sim |(\tau,\xi)| \tag{4.6}$$

(see [**23**, Lemma 2.6]). Note that $\nabla_{\tau,\xi}\langle G^{(j)}(s), \mathcal{L}^{\delta_n}_{\delta_n\nu}(\tau,\xi)\rangle = (\mathcal{L}^{\delta_n}_{\delta_n\nu})^{\intercal}G^{(j)}(s)$. Thus, by (4.6) we get (4.5). ☐

4.3. Proof of Proposition 2.4

By the reduction in § 4.1, it suffices to prove (4.2). Recalling (4.4) and applying the Minkowski inequality, we have

$$\|\mathcal{T}[\mathfrak{a}_0]f\|_p \leqslant \sum_{2^{-k/N} \leqslant \delta_n \lesssim 1} \Big\|\sum_{\nu \in \mathfrak{I}_n} \mathcal{T}[\mathfrak{a}_\nu^n]f\Big\|_p.$$

Using Lemma 4.1, one can easily see that $\operatorname{supp}\mathfrak{a}_\nu^n \subset \Lambda_k(\delta_n, \delta_n\nu)$. Thus, we may use the decoupling inequality (3.2). Combining this and the above inequalities gives

$$\|\mathcal{T}[\mathfrak{a}_0]f\|_p \leqslant C_\epsilon \sum_{2^{-k/N} \leqslant \delta_n \lesssim 1} \delta_n{}^{-1+(N+1)/p+\epsilon}\Big(\sum_{\nu \in \mathfrak{I}_n} \|\mathcal{T}[\mathfrak{a}_\nu^n]f\|_p^p\Big)^{1/p}$$

for $2N < p < \infty$. Hence, for estimate (4.2) it suffice to show that

$$\|\mathcal{T}[\mathfrak{a}_\nu^n]f\|_p \lesssim \delta_n^{1-(N+1)/p}2^{-k/p}\|f\|_p, \quad p > 2N. \tag{4.7}$$

Indeed, let $f_\nu(x, s) = \tilde{\zeta}(\delta_n^{-1}s - \nu)f(x, s)$ where $\tilde{\zeta} \in \mathrm{C}_0^\infty([-2, 2])$ such that $\tilde{\zeta} = 1$ on $\operatorname{supp}\zeta$. From (2.10) we see $\mathcal{T}[\mathfrak{a}_\nu^n]f = \mathcal{T}[\mathfrak{a}_\nu^n]f_\nu$. Combining this and (4.7), we have

$$\begin{aligned}\Big(\sum_{\nu \in \mathfrak{I}_n} \|\mathcal{T}[\mathfrak{a}_\nu^n]f\|_p^p\Big)^{1/p} &\lesssim \delta_n^{1-(N+1)/p}2^{-k/p}\Big(\sum_{\nu \in \mathfrak{I}_n} \|f_\nu\|_p^p\Big)^{1/p}\\ &\lesssim \delta_n^{1-(N+1)/p}2^{-k/p}\|f\|_p.\end{aligned}$$

Therefore, taking the sum over n, we get (4.2), which proves Proposition 2.4.

It remains to prove (4.7). By Lemma 4.1, we have $C^{-1}\mathfrak{a}_\nu^n \in \mathfrak{A}_k(\delta_n, \delta_n\nu)$ for a constant $C > 0$. For $n = 0$, it is easy to show (4.7). Since $\delta_0 = 2^{-k/N}$, applying

Lemma 2.6, we get

$$\|\mathcal{T}[\mathfrak{a}_\nu^0]f\|_p \lesssim \delta_0^{1-1/p}\|f\|_p = \delta_0^{1-(N+1)/p}2^{-k/p}\|f\|_p, \quad 2 \leqslant p \leqslant \infty.$$

For $n \geqslant 1$, we need to decompose $\mathfrak{a}_\nu^n$ further. Let us set

$$\mathfrak{a}_{\nu,1}^n(s,t,\tau,\xi) = \mathfrak{a}_\nu^n(s,t,\tau,\xi)(1-\beta_0)\left(10\delta_n^{-2N!}|\langle 2^{-k}G(s),(\tau,\xi)\rangle|^{2(N-1)!}\right)$$

and $\mathfrak{a}_{\nu,0}^n = \mathfrak{a}_\nu^n - \mathfrak{a}_{\nu,1}^n$, so we have $\mathfrak{a}_\nu^n = \mathfrak{a}_{\nu,1}^n + \mathfrak{a}_{\nu,0}^n$. We note that $C^{-1}\mathfrak{a}_{\nu,i}^n \in \mathfrak{A}_k(\delta_n, \delta_n\nu)$, $i = 0,1$ for some $C > 0$. This can be shown by following the proof of Lemma 4.1. So, we omit the detail.

We now decompose $\mathcal{T}[\mathfrak{a}_\nu^n]f = \mathcal{T}[\mathfrak{a}_{\nu,1}^n]f + \mathcal{T}[\mathfrak{a}_{\nu,0}^n]f$. For (4.7), it suffices to show

$$\|\mathcal{T}[\mathfrak{a}_{\nu,i}^n]f\|_p \leqslant C\delta_n^{1-(N+1)/p}2^{-k/p}\|f\|_p, \quad i = 0,1, \tag{4.8}$$

for $p > 2N-2$. It is clear that (2.13) holds with $\delta = \delta_n$, $s_\circ = \delta_n\nu$, and some large B on $\operatorname{supp}\mathfrak{a}_{\nu,1}^n$. By Corollary 2.9, we have (4.8) for $i = 1$ if $p > 2N-2$. The operator $\mathcal{T}[\mathfrak{a}_{\nu,0}^n]$ can be handled in the same manner as $\mathcal{T}[\mathfrak{a}_1]$ since

$$|\tau + \langle\gamma(s),\xi\rangle| \gtrsim \delta_n^N 2^k \tag{4.9}$$

holds on $\operatorname{supp}\mathfrak{a}_{\nu,0}^n$. We set $\mathfrak{a} = -i2^k\delta_n^N(\tau + \langle\gamma(s),\xi\rangle)^{-1}\partial_t\mathfrak{a}_{\nu,0}^n$. Integration by parts in t' and (2.10) yields $\mathcal{T}[\mathfrak{a}_{\nu,0}^n] = (2^k\delta_n^N)^{-1}\mathcal{T}[\mathfrak{a}]$. Using (4.9) and the fact that $C^{-1}\mathfrak{a}_{\nu,0}^n \in \mathfrak{A}_k(\delta_n, \delta_n\nu)$ for some $C > 0$, one can easily verify that (2.8) and (2.9) hold for $\mathfrak{a}$ with $\delta = \delta_n$, $s_\circ = \delta_n\nu$. Thus, by Lemma 2.6, we have

$$\|\mathcal{T}[\mathfrak{a}_{\nu,0}^n]f\|_p \lesssim \delta_n^{1-1/p}(\delta_n^N 2^k)^{-1}\|f\|_p \lesssim \delta_n^{1-(N+1)/p}2^{-k/p}\|f\|_p$$

for $p \geqslant 2$, which gives (4.8) for $i = 0$. For the second inequality we use the fact that $\delta_n \geqslant 2^{-k/N}$. ☐

5. Proof of Theorem 1.2

We first prove the sufficiency part, that is to say, estimate (1.2) with $\alpha = 1 - 1/p$ for $1 \leqslant p < p_d$ by making use of Theorem 2.3.

5.1. Proof of estimate (1.2) with $\alpha = 1 - 1/p$

We make use of the argument in [**28**, **31**]. As mentioned before, it suffices to prove (2.1) by duality. Let P_k denote the (Littlewood–Paley projection) operator defined by

$$\mathcal{F}(P_k g)(\xi,\tau) = \beta(2^{-k}|(\xi,\tau)|)\widehat{g}(\xi,\tau), \quad k \geqslant 1$$

for $\beta \in \mathrm{C}_0^\infty([1/2,2])$. Recall that $\beta_0 \in \mathrm{C}_0^\infty([-1,1])$ such that $\beta_0 = 1$ on $[-1/2,1/2]$ and set $\beta_*(t) = \beta_0(C_0^{-1}2^{-6}t) - \beta_0(C_0 2^6 t)$. Here, $C_0 = 1 + 2\sup\{|\gamma(s)| + |\gamma'(s)| :$

$s \in \operatorname{supp}\psi\}$. Let f_k be given by

$$\widehat{f_k}(\xi, u) = \beta_*(2^{-k}|(\xi, u)|)\widehat{f}(\xi, u).$$

We claim that

$$\Big\|\Big(\sum_{k\geqslant 1} |P_k\mathcal{R}f|^2\Big)^{1/2}\Big\|_p \lesssim \Big\|\Big(\sum_{k\geqslant 1} 2^{-2k/p}|f_k|^2\Big)^{1/2}\Big\|_p + \|f\|_{L^p_{-M}} \tag{5.1}$$

for $p > 2d$ and $M \gg 1$. Then, (2.1) follows by the Littlewood–Paley inequality.

Let $\tilde{\beta} = \beta_0(2^{-3}\,\cdot) - \beta_0(C_0 2^3\,\cdot)$ Considering an operator $\mathcal{R}_k$ given by

$$\mathcal{F}_x(\mathcal{R}_k f)(\xi, t) = \tilde{\beta}(|\xi|/2^k)\mathcal{F}_x(\mathcal{R}f)(\xi, t),$$

we decompose

$$P_k\mathcal{R}f = P_k\mathcal{R}_k f_k + P_k\mathcal{R}_k(f - f_k) + P_k(\mathcal{R} - \mathcal{R}_k)f. \tag{5.2}$$

In what follows, we show that the contributions from the second and third terms are negligible. In fact, for any $M \geqslant 1$ if $p \geqslant 1$, we have

$$\Big\|\Big(\sum_{k} |P_k\mathcal{R}_k(f - f_k)|^2\Big)^{1/2}\Big\|_p \lesssim \|f\|_{L^p_{-M}} \tag{5.3}$$

and (5.4).

To see (5.3), note $\mathcal{F}_x(\mathcal{R}_k g)(\xi, t') = \int m(\xi, t', u)\widehat{g}(\xi, u)\,\mathrm{d}u$ where

$$m(\xi, t', u) = (2\pi)^{-1}\chi(t')\tilde{\beta}(|\xi|/2^k)\int \mathrm{e}^{i(su - t'\gamma(s)\cdot\xi)}\psi(s)\,\mathrm{d}s.$$

Set $g = f - f_k$. Since $|(\xi, u)| \geqslant C_0 2^{k+5}$ or $|(\xi, u)| \leqslant C_0^{-1}2^{k-5}$ on $\operatorname{supp}\mathcal{F}(f - f_k)$, we have $|u| \geqslant C_0|\xi|$ if $C_0^{-1}2^{k-4} \leqslant |\xi| \leqslant 2^{k+3}$. Therefore, integration by parts gives

$$|\partial^\alpha_{\xi,u} m(\xi, t', u)| \lesssim 2^{-kN}(1 + |(\xi, u)|)^{-N}, \quad (\xi, u) \in \operatorname{supp}\widehat{g}$$

for any α and $N \geqslant 1$. Note that $P_k\mathcal{R}_k g(x, t) = \int K(x, y, t, s')g(y, s')\,\mathrm{d}y\mathrm{d}s'$ where

$$K(x, y, t, s') = \frac{1}{(2\pi)^{d+1}}\int \mathrm{e}^{i(x-y,t-t')\cdot(\xi,\tau)}\,\mathrm{e}^{-is'u}\beta\left(\frac{|(\xi, \tau)|}{2^k}\right)m(\xi, t', u)\,\mathrm{d}\xi\,\mathrm{d}\tau\,\mathrm{d}u\,\mathrm{d}t'.$$

Therefore, $|\partial^\alpha_{\xi,u} m(\xi, t', u)| \lesssim 2^{-kN}2^{-jN}$ for $(\xi, u) \in \operatorname{supp}\widehat{g}$ and integration by parts shows

$$|K(x, y, t, s')| \lesssim 2^{-kN}2^{-jN}(1 + |x - y| + |s'|)^{-N}(1 + |t|)^{-N}.$$

Decomposing $\mathcal{R}_k(f - f_k) = \sum_j \mathcal{R}_k P_j(f - f_k)$, we get (5.3) for any $M \geqslant 1$ and $p \geqslant 1$.

We now show

$$\Big\|\Big(\sum_k |P_k(\mathcal{R}-\mathcal{R}_k)f|^2\Big)^{1/2}\Big\|_p \lesssim \|f\|_{L^p_{-M}} \tag{5.4}$$

for $p \geqslant 1$ and $M \geqslant 1$. We write $\mathcal{F}(\mathcal{R}f-\mathcal{R}_k f)(\xi,\tau)=\int b(s,\xi,\tau)\mathcal{F}_x f(\xi,s)\,\mathrm{d}s$ where

$$b(s,\xi,\tau)=\frac{1}{2\pi}\int \mathrm{e}^{it'(\gamma(s)\cdot\xi-\tau)}\Big(1-\tilde\beta(|\xi|/2^k)\Big)\chi(t')\,\mathrm{d}t'\psi(s).$$

Since $|\xi| \leqslant C_0^{-1}2^{k-2}$ or $|\xi| \geqslant 2^{k+2}$ on $\operatorname{supp}\mathcal{F}_x(\mathcal{R}f-\mathcal{R}_k f)$, we have $|\tau| \geqslant C_0|\xi|$ if $2^{k-1} \leqslant |(\xi,\tau)| \leqslant 2^{k+1}$. Integration by parts gives $|\partial_\xi^\alpha b(s,\xi,\tau)| \lesssim 2^{-kN}$ for any α and N. Hence,

$$\|P_k(\mathcal{R}-\mathcal{R}_k)f\|_p \lesssim 2^{-kN}\|f\|_p, \quad p \geqslant 1 \tag{5.5}$$

for all $N \geqslant 1$. Since $|\xi| \leqslant C_0^{-1}2^{k-2}$ on $\operatorname{supp}\mathcal{F}(P_k(\mathcal{R}-\mathcal{R}_k)f)$, similarly as in the proof of (5.3), we have $\|P_k(\mathcal{R}-\mathcal{R}_k)P_j f\|_p \lesssim 2^{-jN}\|P_j f\|_p$ for $j \geqslant k+C'$ for some $C' \geqslant 1$. Estimate (5.5) gives $\|P_k(\mathcal{R}-\mathcal{R}_k)P_j f\|_p \lesssim 2^{-kN}\|P_j f\|_p$ for $j \leqslant k+C'$. Combining those estimates, we get (5.4).

Therefore, estimate (5.1) follows if we show

$$\Big\|\Big(\sum_k |P_k\mathcal{R}_k f_k|^2\Big)^{1/2}\Big\|_p \lesssim \Big\|\Big(\sum_{k\geqslant 1} 2^{-2k/p}|f_k|^2\Big)^{1/2}\Big\|_p \tag{5.6}$$

for $p > 2d$. This can be done by using [**28**, Theorem 1] and (2.5) (also see [**1**, **29**, **31**]). Indeed, let $\tilde\beta \in \mathrm{C}_c^\infty((1/4,4))$ such that $\tilde\beta\beta=\beta$. Consider the operator $\tilde P_k$ given by $\mathcal{F}(\tilde P_k g)(\xi,\tau)=\tilde\beta(2^{-k}|(\xi,\tau)|)\widehat{g}(\xi,\tau)$. Note that $P_k\mathcal{R}_k f_k = P_k\tilde P_k\mathcal{R}_k f_k$.

Let us denote the centre of a cube Q by (x_Q,t_Q) and set

$$\mathcal{E}_Q=\{(y,s):\mathrm{dist}\ (y-x_Q,t_Q\gamma(I)) \leqslant 10\mathrm{diam}(Q), s\in I\}.$$

Since $T_k=\tilde P_k\mathcal{R}_k$ and $\mathcal{E}_Q$ satisfy the assumptions in [**28**, Theorem 1], by using (2.5) we obtain (5.6). We omit the details.

5.2. Sharpness of smoothing order

In this section, we show upper bounds on the smoothing order α for which $L^p - L^p_\alpha$ estimate for $\mathfrak{R}f$ holds when γ is of maximal type L. In [**31**], those bounds were obtained for $d=2$. Modifying the examples in [**31**], we show the following.

PROPOSITION 5.1. *Let $d \geqslant 3$, $L \geqslant d$ and $1 \leqslant p \leqslant \infty$. Let ψ and χ be nontrivial, nonnegative continuous functions supported in the interiors of I and $[1,2]$, respectively. Suppose there is an $s_\circ$ such that $\psi(s_\circ)\neq 0$ and γ is of type L at $s_\circ$. Then, $\mathfrak{R}f$ maps L^p boundedly to L^p_α only if*

$$(i)\ \alpha \leqslant 1-p^{-1}, \quad (ii)\ \alpha \leqslant (2d)^{-1}, \quad (iii)\ \alpha \leqslant (Lp)^{-1}.$$

In particular, the upper bound (i) provides the necessity part of Theorem 1.2, thus, the proof Theorem 1.2 is completed. We prove the upper bounds (i), (ii) and (iii), separately.

Proof of (i). Let $t_0 \in (1,2)$ such that $\chi(t_0) > 0$. We choose $\zeta \in \mathcal{S}(\mathbb{R}^d)$ such that $\zeta \geqslant 1$ on $[-1,1]^d$, $\operatorname{supp}\widehat{\zeta} \subset [1/2,4]^d$, and $\widehat{\zeta} = 1$ on $[1,2]^d$. Let $\psi_0 \in \mathrm{C}_c^\infty((-1,1))$ satisfy $\psi_0 = 1$ on $[-1/2,1/2]$. We take

$$f(x,t) = \zeta(\lambda x)\psi_0(\lambda r_0|t-t_0|),$$

where $r_0 = 1 + \sup_{s\in I}|\gamma(s)|$. Note $\mathfrak{R}f(x,s) \gtrsim \lambda^{-1}$ if $|x + t_0\gamma(s)| \leqslant c\lambda^{-1}$ and $|s - s_\circ| < c$ for a small constant $c > 0$. Thus, $\|\mathfrak{R}f\|_p \gtrsim \lambda^{-1-d/p}$. Since

$$\mathcal{F}_x(\mathfrak{R}f(\cdot,s))(\xi) = \lambda^{-d}\psi(s)\int \widehat{\zeta}(\lambda^{-1}\xi)\,\mathrm{e}^{it\gamma(s)\cdot\xi}\psi_0(\lambda|t-t_0|)\chi(t)\,\mathrm{d}t,$$

it follows that $\operatorname{supp}_\xi \mathcal{F}_x(\mathfrak{R}f)$ is included in $\{\xi : |\xi| \sim \lambda\}$. Hence, $\|\mathfrak{R}f(\cdot,s)\|_{L^p_\alpha(\mathbb{R}^d;\mathrm{d}x)} \gtrsim \lambda^{\alpha-1-d/p}$, so we have $\|\mathfrak{R}f\|_{L^p_\alpha(\mathbb{R}^{d+1})} \gtrsim \lambda^{\alpha-1-d/p}$. Since $\|f\|_p \lesssim \lambda^{-(d+1)/p}$, we get $\alpha \leqslant 1 - 1/p$. $\square$

Proof of (ii). Let $\tilde{I} \subset (-1,1)$ be a nonempty compact interval such that (1.1) holds for $s \in \tilde{I}$. Also, we fix a constant $\rho \gg 1$ to be chosen later. Let $\{s_\ell\} \subset \tilde{I}$ be a collection of $\rho\lambda^{-1/d}$-separated points which are as many as $C\rho^{-1}\lambda^{1/d}$. Since $G(s_\ell), G'(s_\ell), \dots, G^{(d-1)}(s_\ell)$ are linearly independent in $\mathbb{R}^{d+1}$, there is a unit vector $\Xi_\ell \in (\operatorname{span}\{G^{(j)}(s_\ell) : j = 0,1,\dots,d-1\})^\perp$.

Let $\phi \in \mathcal{S}(\mathbb{R}^{d+1})$ such that $\phi \geqslant 1$ on $[-3r_0, 3r_0]^{d+1}$ and $\widehat{\phi}$ is supported in $[-1,1]^{d+1}$ where $r_0 = 1 + \sup_{s\in I}|\gamma(s)|$. Let $\varepsilon_\ell \in \{\pm 1\}$ be independent random variables. We consider

$$f(x,t) = \sum_\ell \varepsilon_\ell f_\ell(x,t) := \sum_\ell \varepsilon_\ell \phi(x,t)\mathrm{e}^{i\lambda\Xi_\ell\cdot(t,x)}.$$

Since $\langle \Xi_\ell, G^{(j)}(s_\ell)\rangle = 0$ for $j = 0,\dots,d-1$, by Taylor's theorem we have

$$\langle \Xi_\ell, G(s)\rangle = \langle \Xi_\ell, G^{(d)}(s_\ell)\rangle(s-s_\ell)^d/d! + O(|s-s_\ell|^{d+1}). \tag{5.7}$$

Thus, $|t\langle \Xi_\ell, G(s)\rangle| \leqslant 2^{-2}\lambda^{-1}$ whenever $s \in I_\ell := \{s \in \tilde{I} : |s - s_\ell| \leqslant c\lambda^{-1/d}\}$ for a $c > 0$ small enough. Noting that

$$\mathfrak{R}f_\ell(x,s) = \mathrm{e}^{i\lambda\Xi_\ell\cdot(0,x)}\psi(s)\int \phi(x + t\gamma(s), t)\mathrm{e}^{i\lambda t\Xi_\ell\cdot G(s)}\chi(t)\,\mathrm{d}t, \tag{5.8}$$

we see $|\mathfrak{R}f_\ell(x,s)| \gtrsim 1$ if $(x,s) \in B_\ell := [-c,c]^d \times I_\ell$. Thus, $\sum_\ell \|\mathfrak{R}f_\ell\|^p_{L^p(B_\ell)} \gtrsim \rho^{-1}$. Meanwhile, by (5.8), (5.7) and integration by parts in t we have $|\mathfrak{R}f_m(x,s)| \lesssim (1 + \lambda|s_\ell - s_m|^d)^{-N}$ for any $N \geqslant 1$ if $m \neq \ell$ and $s \in I_\ell$. Since $\{s_\ell\}$ are $\rho\lambda^{-(1/d)}$-separated, it is easy to see

$$\sum_\ell \Big\| \sum_{m\neq\ell} |\mathfrak{R}f_m| \Big\|^p_{L^p(B_\ell)} \lesssim \sum_\ell \sum_{m\neq\ell} (1 + \lambda|s_\ell - s_m|^d)^{-pN}\lambda^{-1/d} \lesssim \rho^{-pdN-1}.$$

Therefore, taking ρ, N sufficiently large, we have $\|\mathfrak{R}f\|_p^p \gtrsim \rho^{-1}$ for any choice of ε_ℓ.

By our choice of ϕ it follows that $\mathcal{F}_x(\mathfrak{R}f)$ is supported on $\{\xi : C_1\lambda \leqslant |\xi| \leqslant C_2\lambda\}$ for some positive constant C_1, C_2. Thus, $\|\mathfrak{R}f\|_{L^p_\alpha} \gtrsim \lambda^\alpha\|\mathfrak{R}f\|_p$. Combining this with the $L^p - L^p_\alpha$ estimate gives $\lambda^\alpha \leqslant C\|f\|_p$ for any choice of ε_ℓ. By Khintchine's inequality we have $\mathbb{E}(\|f\|_p^p) \sim \int (\sum_\ell |f_\ell|^2)^{p/2}\, \mathrm{d}x\, \mathrm{d}t \sim C_\rho \lambda^{p/2d}$. Therefore, we see $\lambda^\alpha \lesssim \lambda^{1/2d}$ and then $\alpha \leqslant 1/(2d)$ taking $\lambda \to \infty$. ☐

Proof of (iii). Since γ is of type L at $s_\circ$, by an affine transformation and taking ψ supported near $s_\circ$, we may assume

$$\gamma(s + s_\circ) = \gamma(s_\circ) + (s^{a_1}\varphi_1(s), \dots, s^{a_d}\varphi_d(s))$$

for $1 \leqslant a_1 < \cdots < a_d = L$ and smooth functions φ_j, $j = 1, \dots, d$, where $\|\varphi_j - 1/a_j!\|_{\mathrm{C}^{a_d+1}(I)} \leqslant c$ for a small constant $c > 0$. We may also assume $s_\circ = 0$ and furthermore $\gamma(0) = 0$ by replacing $f(x, t)$ by $f(x - t\gamma(0), t)$.

Let $\phi_1 \in \mathcal{S}(\mathbb{R})$ such that $\phi_1 \geqslant 1$ on $[-1, 1]$, and $\operatorname{supp} \widehat{\phi_1} \subset [1/2, 4]$ with $\widehat{\phi_1} = 1$ on $[1, 2]$. Let $\psi_0 \in \mathrm{C}_c^\infty((-1, 1))$ with $\psi_0 = 1$ on $[-1/2, 1/2]$. We consider

$$f(x, t) = \prod_{j=1}^{d-1} \psi_0(\lambda^{a_j/L} x_j)\phi_1(\lambda x_d)\chi(t).$$

Denoting $\|a\| = \sum_{j=1}^d a_j$, we have $\|f\|_p \lesssim \lambda^{-\|a\|/(Lp)}$. Set $E_\lambda = \{(x, s) \in \mathbb{R}^d \times I : |x_j| \leqslant c\lambda^{-a_j/L}, j = 1, \dots, d, \quad |s| \leqslant c\lambda^{-1/L}\}$ for a sufficiently small $c > 0$. Since $\gamma(s) = (s^{a_1}\varphi_1(s), \dots, s^{a_d}\varphi_d(s))$, $|\langle x + t\gamma(s), e_j\rangle| \leqslant 2^{-1}\lambda^{-a_j/L}$, $j = 1, \dots, d$, for $(x, s) \in E_\lambda$ and $t \in [1, 2]$. So, $\mathfrak{R}f(x, s) \gtrsim 1$ for $(x, s) \in E_\lambda$. This gives $\|\mathfrak{R}f\|_p \gtrsim \lambda^{-(\|a\|+1)/(Lp)}$. Since $\operatorname{supp} \mathcal{F}_{x_d}(\mathfrak{R}f) \subset \{\xi_d : |\xi_d| \sim \lambda\}$, $\|\mathfrak{R}f\|_{L^p_\alpha} \gtrsim \lambda^{\alpha-(\|a\|+1)/(Lp)}$. Therefore, we obtain $\alpha \leqslant 1/(Lp)$. ☐

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References

1 D. Beltran, S. Guo, J. Hickman and A. Seeger. Sobolev improving for averages over curves in $\mathbb{R}^4$. *Adv. Math.* **393** (2021), 108089.

2 D. Beltran, S. Guo, J. Hickman and A. Seeger. *Sharp L^p bounds for the helical maximal function.* To appear in Am. J. Math. arXiv:2102.08272.

3 G. Bentsen. L^p regularity for a class of averaging operators on the Heisenberg group. *Indiana Univ. Math. J.* **71** (2022), 819–855.

4 G. Bentsen. Regularity estimates for a class of integral operators with fold blowdown singularities. *J. Geom. Anal.* **32** (2022), 89.

5 J. Bourgain and C. Demeter. The proof of the ℓ^2 decoupling conjecture. *Ann. Math.* **182** (2015), 351–389.

6 J. Bourgain, C. Demeter and L. Guth. Proof of the main conjecture in Vinogradov's mean value theorem for degrees higher than three. *Ann. Math.* **184** (2016), 633–682.

7 M. Christ and B. M. Erdoğan. Mixed norm estimates for a restricted X-ray transform. *J. Anal. Math.* **87** (2002), 187–198.

8 M. Christ and B. M. Erdoğan. Mixed norm estimates for certain generalized Radon transforms. *Trans. Am. Math. Soc.* **360** (2008), 5477–5488.

9 S. Dendrinos and B. Stovall. Uniform estimates for the X-ray transform restricted to polynomial curves. *J. Funct. Anal.* **262** (2012), 4986–5020.

10 S. Dendrinos and B. Stovall. Uniform bounds for convolution and restricted X-ray transforms along degenerate curves. *J. Funct. Anal.* **268** (2015), 585–633.

11 M. B. Erdoğan. Mixed-norm estimates for a restricted X-ray transform in $\mathbb{R}^4$ and $\mathbb{R}^5$. *Int. Math. Res. Not.* **2001** (2001), 575–600.

12 I. M. Gelfand and M. I. Graev. Line complexes in the space $\mathbb{C}^n$. *Funct. Ann. Appl.* **2** (1968), 219–229.

13 A. Greenleaf and A. Seeger. Fourier integral operators with fold singularities. *J. Reine Angew. Math.* **455** (1994), 35–56.

14 A. Greenleaf and A. Seeger. Fourier integral operators with cusp singularities. *Am. J. Math.* **120** (1998), 1077–1119.

15 A. Greenleaf, A. Seeger and S. Wainger. On X-ray transforms for rigid line complexes and integrals over curves in $\mathbb{R}^4$. *Proc. Am. Math. Soc.* **127** (1999), 3533–3545.

16 A. Greenleaf and A. Seeger. *Oscillatory and Fourier integral operators with degenerate canonical relations*. In *Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations*, pp. 93–141 (El Escorial, 2000).

17 A. Greenleaf and G. Uhlmann. Nonlocal inversion formulas for the X-ray transform. *Duke Math. J.* **58** (1989), 205–240.

18 A. Greenleaf and G. Uhlmann. Estimates for singular Radon transforms and pseudodifferential operators with singular symbols. *J. Funct. Anal.* **89** (1990), 202–232.

19 A. Greenleaf and G. Uhlmann. Composition of some singular Fourier integral operators and estimates for restricted X-ray transforms I. *Ann. Inst. Fourier (Grenoble)* **40** (1990), 443–466.

20 P. T. Gressman. L^p-improving properties of averages on polynomial curves and related integral estimates. *Math. Res. Lett.* **16** (2009), 971–989.

21 S. Ham and S. Lee. Restriction estimates for space curves with respect to general measure. *Adv. Math.* **254** (2014), 251–279.

22 H. Ko, S. Lee and S. Oh. Maximal estimate for average over space curve. *Invent. Math.* **228** (2022), 991–1035.

23 H. Ko, S. Lee and S. Oh. Sharp smoothing properties of averages over curves. *Forum Math. Pi* **11** (2023), e4.

24 N. Laghi. A note on restricted X-ray transforms. *Math. Proc. Cambridge Philos. Soc.* **46** (2009), 719–729.

25 D. M. Oberlin. An estimate for a restricted X-ray transform. *Can. Math. Bull.* **43** (2000), 472–476.

26 D. Oberlin and H. Smith. A Bessel function multiplier. *Proc. Am. Math. Soc.* **127** (1999), 2911–2915.

27 D. Oberlin, H. Smith and C. D. Sogge. Averages over curves with torsion. *Math. Res. Lett.* **5** (1998), 535–539.

28 M. Pramanik, K. M. Rogers and A. Seeger. A Calderón–Zygmund estimate with applications to generalized Radon transforms and Fourier integral operators. *Stud. Math.* **202** (2011), 1–15.

29 M. Pramanik and A. Seeger. L^p regularity of averages over curves and bounds for associated maximal operators. *Am. J. Math.* **129** (2007), 61–103.

30 M. Pramanik and A. Seeger. L^p Sobolev estimates for a class of integral operators with folding canonical relations. *J. Geom. Anal.* **31** (2021), 6725–6765.

31 M. Pramanik and A. Seeger. *L^p Sobolev regularity of a restricted X-ray transform in $\mathbb{R}^3$*. Harmonic analysis and its applications, pp. 47–64 (Yokohama: Yokohama Publ., 2006).

32 T. Tao and A. Vargas. A bilinear approach to cone multipliers. II. Applications. *Geom. Funct. Anal.* **10** (2000), 216–258.

33 T. Tao and J. Wright. L^p improving bounds for averages along curves. *J. Am. Math. Soc.* **16** (2003), 605–638.

34 T. Wolff. A sharp bilinear cone restriction estimate. *Ann. Math. (2)* **153** (2001), 661–698.