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DIFFERENT METHODS OF COMPLEX INTERPOLATION

LAURA SERVIDEI

We prove that the classic interpolation spaces of Calderón can be defined using spaces of functions that satisfy weaker conditions. For Calderón's second space we use a space of functions defined by Cwickel and Janson; we then modify their definition to find another space of functions which defines Calderón's first space.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Complex interpolation methods give, for any couple $\overline{A} = (A_0, A_1)$ of compatible Banach spaces, a family of intermediate spaces as a quotient of a space of functions. These functions satisfy some holomorphy condition on the strip $S = \{z \in \mathbb{C} \mid 0 \leqslant \mathrm{Re}(z) \leqslant 1\}$. Different choices of the space of functions lead to different families of intermediate spaces. Since Calderón's classic work, several interpolation methods have been studied. In this paper we compare a complex interpolation method introduced by Cwickel and Janson with the two methods of Calderón. Throughout this paper we will use the notation of [**3**, Chapter 4] which is devoted to the complex interpolation methods as described by Calderón in [**1**].

Cwickel and Janson in [**2**] define a complex interpolation method in the following context: let $\overline{A} = (A_0, A_1)$ be a couple of Banach spaces continuously embedded in a Banach space A. Let A^+ be a linear subspace of the dual space A^* that determines the norms A_0, A_1 and A, that is there are subsets Γ_0, Γ_1 and Γ of A^+ such that:

(i) $\|a\|_A = \sup\{|\langle a, a^+\rangle| \mid a^+ \in \Gamma\}$, $a \in A$;
(ii) if $a \in A$ and $\sup\{|\langle a, a^+\rangle| \mid a^+ \in \Gamma_j\} < \infty$ then $a \in A_j$; $\|a\|_{A_j} = \sup\{|\langle a, a^+\rangle| \mid a^+ \in \Gamma_j\}$ $(j = 0, 1)$.

By the bipolar theorem, as noted in [**2**], the existence of such Γ_0, Γ_1 and Γ is equivalent to the unit balls of A_0, A_1 and A being closed in the topology $\sigma(A, A^+)$ and A^+ induces on A. Then define $\mathcal{D}(\overline{A})$ to be the space of all functions $f\colon S \to A$ such that:

(i) for every $a^+ \in A^+$ $z \mapsto \langle f(z), a^+\rangle$ is a bounded, continuous function on S, holomorphic on its interior;
(ii) for almost every $t \in \mathbf{R}$ and for $j = 0, 1$ $f(j + it) \in A_j$ and $t \mapsto \|f(j + it)\|_{A_j}$ is an essentially bounded function.

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$\mathcal{D}(\overline{A})$ is a Banach space with respect to the norm:

$$\|f\|_{\mathcal{D}} = \max\{\operatorname{supess}_{t\in\mathbf{R}} \|f(it)\|_{A_0}, \operatorname{supess}_{t\in\mathbf{R}} \|f(1+it)\|_{A_1}\}.$$

If $\theta \in (0, 1)$ we define $\overline{A}_{\{\theta\}}$ to be the space $\{f(\theta) \mid f \in \mathcal{D}(\overline{A})\}$ equipped with the norm: $\|a\|_{\{\theta\}} = \inf\{\|f\|_{\mathcal{D}} \mid f(\theta) = a\}$.

The definition of the space $\overline{A}_{\{\theta\}}$ is different from the definition of Calderón's first space, $\overline{A}_{[\theta]}$, for several reasons:

(i) the functions in $\mathcal{D}(\overline{A})$ satisfy an apparently weaker holomorphy condition:

(ii) the continuity conditions on S are weaker.

Cwickel and Janson proved that $\overline{A}_{\{\theta\}}$ is continuously embedded in Calderón's second space, $\overline{A}^{[\theta]}$; we will prove that, if an extra hypothesis holds, the space $\overline{A}_{\{\theta\}}$ coincides with the space $\overline{A}^{[\theta]}$.

THEOREM 1. *If one of the following hypotheses holds:*

(i) *the unit ball of A is $\sigma(A^{**}, A^+)$-closed in A^{**},*

(ii) *the unit balls of A_0 and A_1 are $\sigma(A^{**}, A^+)$-closed in A^{**},*

then $\overline{A}_{\{\theta\}} = \overline{A}^{[\theta]}$ with equality of norms.

In Section 3 we consider a variant of this method where we relax the continuity conditions only on one side of the boundary of S, that is $\operatorname{Re}(z) = 0$, while on $\operatorname{Re}(z) = 1$ strong continuity is requested. In fact we define $\mathcal{C}(\overline{A})$ to be the space of all functions $f\colon S \to A$ such that:

(i) for every $a^+ \in A^+$ $z \mapsto \langle f(z), a^+\rangle$ is a bounded, continuous function on S, holomorphic on its interior;

(ii) for almost every $t \in \mathbf{R}$ $f(it) \in A_0$ and $t \mapsto \|f(it)\|_{A_0}$ is an essentially bounded function;

(iii) for every $t \in \mathbf{R}$ $f(1+it) \in A_0$ and $t \mapsto f(1+it)$ is a A_1-norm-continuous and bounded function.

$\mathcal{C}(\overline{A})$ is a Banach space with respect to the norm:

$$\|f\|_c = \max\{\operatorname{supess}_{t\in\mathbf{R}} \|f(it)\|_{A_0}, \sup_{t\in\mathbf{R}} \|f(1+it)\|_{A_1}\}.$$

It $\theta \in (0, 1)$ we define $\overline{A}_{\{\theta]}$ to be the space $\{f(\theta) \mid f \in \mathcal{C}(\overline{A})\}$ equipped with the norm: $\|a\|_{\{\theta]} = \inf \|f\|_c \mid f(\theta) = a\}$. We will prove that this definition leads to Calderón's first space $\overline{A}_{[\theta]}$.

THEOREM 2. $\overline{A}_{\{\theta]} = \overline{A}_{[\theta]}$ *with equality of norms.*

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2. Proof of Theorem 1

We prove only the inclusion $\overline{A}^{[\theta]} \subset \overline{A}_{\{\theta\}}$, as the converse inclusion was proved in [2].

Let $\mathcal{G}(\overline{A})$ be the space of holomorphic functions on S in $\sum(\overline{A})$ (where $\sum(\overline{A}) = A_0 + A_1$) needed in the definition of the space $\overline{A}^{[\theta]}$. If $a \in \overline{A}^{[\theta]}$ and $\varepsilon > 0$ we can choose $F \in \mathcal{G}(\overline{A})$ so that $F'(\theta) = a$ and $\|F\|_{\mathcal{G}} \leqslant \|a\|^{[\theta]} + \varepsilon$. As $\sum(\overline{A})$ is naturally embedded in A, with continuous embedding, F is an A-norm continuous function on S, A-norm-holomorphic on $S_0 = \{z \in \mathrm{C} \mid 0 < \mathrm{Re}(z) < 1\}$ and $\|F'(z)\|_A \leqslant c\|F\|_{\mathcal{G}}$ for some $c > 0$, for all $z \in S_0$.

Therefore for every $a^+ \in A^+$ $z \mapsto \langle F'(z), a^+\rangle$ is a bounded, holomorphic function on S_0 and

$$\left|\langle F'(z), a^+\rangle\right| \leqslant c\|F\|_{\mathcal{G}} \left\|a^+\right\|_{A^*}, \tag{1}$$

so we can define

$$\varphi_j(t) = \lim_{s \to j} \langle F'(s + it), a^+\rangle \quad j = 0, 1 \tag{2}$$

as the limit above exists for almost every $t \in \mathbf{R}$. Also $\varphi_j \in L^\infty(\mathbf{R})$ and

$$\|\varphi_j\|_\infty \leqslant c\|F\|_{\mathcal{G}} \left\|a^+\right\|_{A^*} \qquad j = 0, 1. \tag{3}$$

On the other hand, if $j = 0, 1$ $t \mapsto \langle F(j + it), a^+\rangle$ is a Lipschitz function, with Lipschitz constant not exceeding $c\|F\|_G \|a^+\|_{A^*}$. Then we can define, for $j = 0, 1$

$$\begin{aligned}\psi_j(t) &= \lim_{h \to 0} \frac{\langle F(j + it + ih), a^+\rangle - \langle F(j + it), a^+\rangle}{ih} \\ &= -i\frac{d}{dt}\langle F(j + it), a^+\rangle\end{aligned}$$

as the limit above exists for almost every $t \in \mathbf{R}$, and we have

$$\langle F(j + it), a^+\rangle = \langle F(j), a^+\rangle + i\int_0^t \psi_j(u)du \quad j = 0, 1, t \in \mathbf{R}. \tag{4}$$

Now we prove that $\psi_j(t) = \varphi_j(t)$ almost everywhere; for this purpose we consider the function

$$h(z) = \begin{cases} \langle F'(z), a^+\rangle & z \in S_0 \\ \varphi_j(t) & z = j + it. \end{cases}$$

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By Cauchy's integration formula, the boundedness of h and Lebesgue's dominated convergence theorem, we have

$$\oint_\gamma h(a)dz = 0 \tag{5}$$

where γ is the curve given by the boundary of the rectangle whose vertices are: 0, it, $s+it$, s; $s>0$, $t\in \mathbb{R}$. From (4) and (5) it is easily proved that, for every $s\in(0,1)$

$$\begin{aligned}
&\left|\int_0^t (\varphi_0(u)-\psi_0(u))dr\right| \\
&\leqslant \int_0^s \left|\langle F'(v)-F'(v+it),\, a^+\rangle\right| dv \\
&\qquad + \left|\langle F(s+it)-F(s),\, a^+\rangle - \langle F(it)-F(0),\, a^+\rangle\right| \\
&\leqslant 2c\|F\|_{\mathcal{G}}\left\|a^+\right\|_{A^*}|s| \\
&\qquad + \left|\langle F(s)-F(0),\, a^+\rangle\right| + \left|\langle F(s+it)-F(it),\, a^+\rangle\right|
\end{aligned}$$

By the continuity of F on S the right hand side of the inequality above tends to 0 when $s\to 0$. So $\int_0^t (\varphi_0(u)-\psi_0(u))du = 0$ for every $t\in\mathbb{R}$, thus $\varphi_0=\psi_0$ almost everywhere.

In the same way we can prove that $\varphi_1(t)=\psi_1(t)$ almost everywhere. Now (2) and (3) show that for almost every t there exists $\Phi_j(t)\in A^{**}$, $j=0,1$, such that

$$\langle \Phi_j(t),\, a^+\rangle = \varphi_j(t) = \lim_{s\to j}\langle F'(s+it),\, a^+\rangle \qquad j=0,1. \tag{6}$$

If the unit ball of A is $\sigma(A^{**},A^+)$-closed in A^{**}, then (1) and (6) imply that $\Phi_j(t)\in A$. On the other hand for almost every $t\in\mathbb{R}$ we have

$$\langle \Phi_j(t),\, a^+\rangle = \psi_j(t) = \lim_{h\to 0}\frac{1}{ih}\langle F(j+it+ih)-F(j+it),\, a^+\rangle \tag{7}$$

and then as the unit ball of A_j is $\sigma(A^{**},A^+)$-closed, we also have that $\Phi_j(t)\in A_j$, $j=0,1$, for almost every $t\in\mathbb{R}$.

If we assume hypothesis (ii), that is the unit balls of A_0 and A_1 are $\sigma(A^{**},A^+)$-closed instead of the unit ball of A, we can conclude that $\Phi_j(t)\in A_j$ from (7) and

$$\left|\frac{1}{ih}\langle F(j+it+ih)-F(j+it),\, a^+\rangle\right| \leqslant \|F\|_{\mathcal{G}}\left\|a^+\right\|_{A_j^*}. \tag{8}$$

In any case from (7) and (8) we also have $\|\Phi_j(t)\|_{A_j}\leqslant\|F\|_{\mathcal{G}}$, $j=0,1$. This proves that the function

$$DF(z) = \begin{cases} F'(z) & z\in S_0 \\ \Phi_j(t) & z=j+it \end{cases}$$

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is in $\mathcal{D}(\overline{A})$ and $\|DF\|_{\mathcal{D}} \leqslant \|F\|_{\mathcal{G}}$. Therefore $a = DF(\theta) \in \overline{A}_{\{\theta\}}$ and $\|a\|_{\{\theta\}} \leqslant \|DF\|_{\mathcal{D}} \leqslant \|a\|^{[\theta]} + \varepsilon$. ∎

From Theorem 1 we also obtain that, if one of the hypotheses holds, $\overline{A}_{\{\theta\}}$ is an interpolation space with respect to the couple $\overline{A}$. Actually it is easy to see that $\overline{A}_{\{\theta\}}$ is an interpolation space with respect to $\overline{A}$ even without hypothesis (i) or (ii). Then the mapping $(A_0, A_1) \mapsto \overline{A}_{\{\theta\}}$ is an interpolation functor. The domain of this functor is the category whose objects are the couples of Banach spaces embedding in another Banach space (not the same for every couple) such that there is a linear subspace A^+ of A^* that determines the norms A_0, A_1 and A.

3. Proof of Theorem 2

We need the following Lemma:

Lemma 3. *Let $P_j(z, t)$, $j = 0, 1$, be the Poisson kernel for the strip S. If $f \in \mathcal{C}(\overline{A})$ and $\theta \in [0, 1]$, then*

$$\begin{aligned} &\|f(\theta)\|_{\{\theta]} \\ &\leqslant \left(\frac{1}{1-\theta}\int_{\mathbf{R}} \|f(it)\|_{A_0} P_0(\theta, t)dt\right)^{1-\theta} \cdot \left(\frac{1}{\theta}\int_{\mathbf{R}} \|f(1+it)\|_{A_1} P_1(\theta, t)dt\right)^{\theta}. \end{aligned}$$

The proof of this Lemma is analogous to the proof of [**3**, Lemma 4.3.2].

Obviously the space $\mathcal{F}(\overline{A})$ is contained in the space $\mathcal{C}(\overline{A})$, so we have $\overline{A}_{[\theta]} \subset \overline{A}_{\{\theta]}$, and $\|a\|_{\{\theta]} \leqslant \|a\|_{[\theta]}$, $a \in \overline{A}_{[\theta]}$. To prove the converse inclusion it is sufficient to prove that the unit ball of $\overline{A}_{[\theta]}$ is dense in the unit ball of $\overline{A}_{\{\theta]}$ (in fact this gives that the inclusion $\overline{A}_{[\theta]} \to \overline{A}_{\{\theta]}$ is open, thus surjective).

In order to prove this density we consider $f \in \mathcal{C}(\overline{A})$ and define $g(z) = e^{\varepsilon(z^2-\theta^2)}f(z)$, $z \in S$, $\varepsilon > 0$. Now we regularise g by convolving it on the vertical lines with the Weierstrass kernel $W_t(x) = (4\pi t)^{-1/2}e^{-x^2/4t}$, $x \in \mathbf{R}$, $t > 0$, that is we define

$$g_t(z) = \int_{\mathbf{R}} g(x + i(y-u))W_t(u)du \qquad z = x + iy, \quad t > 0.$$

We will prove that $g_t \in \mathcal{F}(\overline{A})$.

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The function $y \mapsto g_t(iy)$ is A_0-norm-continuous and bounded, in fact:

$$\begin{aligned}
\|g_t(iy)\|_{A_0} &\leqslant \int_{\mathbf{R}} \|g(iy-iu)\|_{A_0} W_t(u)du \\
&\leqslant \sup_{y\in\mathbf{R}} \|g(iy)\|_{A_0} \\
&= \sup_{y\in\mathbf{R}} \left|e^{-\varepsilon(y^2+\theta^2)}\right| \|f(iy)\|_{A_0} \\
&\leqslant \sup_{y\in\mathbf{R}} \|f(iy)\|_{A_0} \\
&\stackrel{\text{def}}{=} M_0
\end{aligned}$$

so

$$\|g_t(iy)\|_{A_0} \leqslant M_0. \tag{1}$$

Furthermore, since $g_t(iy) = \int_{\mathbf{R}} g(iu)W_t(y-u)du$, we have, as $y \to y'$,

$$\begin{aligned}
&\|g_t(iy) - g_t(iy')\|_{A_0} \\
&= \frac{1}{\sqrt{4\pi t}} \left\| \int_{\mathbf{R}} g(iu)\Big(\exp\left(-(y-u)^2/4t\right) - \exp\left(-(y'-u)^2/4t\right)\Big)du \right\|_{A_0} \\
&\leqslant \frac{M_0}{\sqrt{4\pi t}} \int_{\mathbf{R}} \left|\exp\left(-(y-u)^2/4t\right) - \exp\left(-(y'-u)^2/4t\right)\right| du \to 0.
\end{aligned}$$

Similarly it can be proved that $y \mapsto g_t(1+iy)$ is A_1-norm-continuous and

$$\sup_{y\in\mathbf{R}} \|g_t(1+iy)\|_{A_t} \leqslant e^{\varepsilon} \sup_{y\in\mathbf{R}} \|f(1+iy)\|_{A_1} = e^{\varepsilon} M_1. \tag{2}$$

Since $y \mapsto g_j(j+it)$, $j = 0, 1$, are A_j-norm-continuous, they are continuous functions with respect to the norm in $\sum(\overline{A})$. Let h be the Poisson integral of the restriction of the function g_t to the boundary of S; h is a $\sum(\overline{A})$-valued function which is strongly continuous. Then, if $a^+ \in A^+$,

$$\begin{aligned}
\langle h(z), a^+\rangle &= \int_{\mathbf{R}} \langle g_t(iy), a^+\rangle P_0(z,t)dy + \int_{\mathbf{R}} \langle g_t(1+iy), a^+\rangle P_1(z,t)dy \\
&= \langle g_t(z), a^+\rangle
\end{aligned}$$

because $\langle g_t(z), a^+\rangle$ is holomorphic and bounded, and thus admits a Poisson integral representation. Then, since A^+ separates points in A, $g_t(z) = h(z)$, $z \in S$, that is $g_t\colon S \to \sum(\overline{A})$ is a strongly continuous function. If γ is a piecewise regular, closed curve in S_0 and if $a^+ \in A^+$ we have

$$\langle \oint_\gamma g_t(z), a^+\rangle = \oint_\gamma \langle g_t(z), a^+\rangle dz = 0$$

then $\oint_\gamma g_t(z)dz = 0$. So, by Morera's theorem, we obtain that $g_t(z)$ is a $\sum(\overline{A})$-norm-holomorphic function on S_0. Furthermore from (1) and (2)

$$\|g_t(z)\|_{\sum(\overline{A})} = \|h(z)\|_{\sum(\overline{A})} \leqslant e^\varepsilon \|f\|_c, \quad z \in S$$

then g_t is bounded in $\sum(\overline{A})$-norm.

Thus we have proved that $g_t \in \mathcal{F}(\overline{A})$, $t > 0$. As $g(\theta) = f(\theta)$, applying Lemma 3 to the function $(g - g_t)$, $t > 0$, we have

$$\begin{aligned}\|g_t(\theta) - f(\theta)\|_{\{\theta]} &= \|(g_t - g)(\theta)\|_{\{\theta]} \\ &\leqslant \left(\frac{1}{1-\theta}\int_{\mathbf{R}} \|(g_t - g)(iy)\|_{A_0} P_0(\theta, y)dy\right)^{1-\theta} \cdot \\ &\qquad \left(\frac{1}{\theta}\int_{\mathbf{R}} \|(g_t - g)(1+iy)\|_{A_1} P_1(\theta, y)dy\right)^{\theta} \\ &\leqslant \sup_{y\in\mathbf{R}} \|(g_t - g)(iy)\|_{A_0}^{1-\theta} \cdot \sup_{y\in\mathbf{R}} \|(g_t - g)(1+iy)\|_{A_1}^{\theta}\end{aligned}$$

that is, by (1),

$$\|g_t(\theta) - f(\theta)\|_{\{\theta]} \leqslant 2M_0^{1-\theta} \sup_{y\in\mathbf{R}} \|(g_t - g)(1+iy)\|_{A_1}^{\theta}. \tag{3}$$

Now we have

$$\|(g_t - g)\|_{A_t} \leqslant \int_{\mathbf{R}} \|g(1+iy+iu) - g(1+iy)\|_{A_1} W_t(u)du.$$

As W_t is a mollifier and the function $y \mapsto g(1+iy)$ is uniformly continuous in A_1-norm, we have $\|g_t(1+iy) - g(1+iy)\|_{A_1} \to 0$ as $t \to 0$. Then from (3) we obtain $\|g_t(\theta) - f(\theta)\|_{\{\theta]} \to 0$ as $t \to 0$. Furthermore, if $\|f\|_c \leqslant 1$, from (1) and (2) we have $\|g_t\|_{\mathcal{F}} \leqslant e^\varepsilon$ where ε is arbitrary. This proves our theorem. ∎

References

[1] A.P. Calderón, 'Intermediate spaces and interpolation, the complex method', *Studia Math* **24** (1964), 113–190.

[2] M. Cwickel and S. Janson, 'Interpolation of analytic families of operators', *Studia Math.* **79** (1984), 61–71.

[3] J. Bergh and J. Löfstöm, *Interpolation Spaces An Introduction* (Springer-Verlag, Berlin, Heidelberg, New York, 1976).

Department of Mathematics
University of Chicago
5734 S. University Ave
Chicago IL 60637
United States of America.