A LIMIT THEOREM FOR THE TAILS
OF DISCRETE INFINITELY DIVISIBLE LAWS
WITH APPLICATIONS TO FLUCTUATION THEORY

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Abstract

Suppose that (p_n) is an infinitely divisible distribution on the non-negative integers having Lévy measure (ν_n). In this paper we derive a necessary and sufficient condition for the existence of the limit $\lim_{n \to \infty} p_n/\nu_n$. We also derive some other results on the asymptotic behaviour of the sequence (p_n) and apply some of our results to the theory of fluctuations of random walks. We obtain a necessary and sufficient condition for the first positive ladder epoch to belong to the domain of attraction of a spectrally positive stable law with index α, $\alpha \in (1,2)$.

1. Introduction

This paper is primarily concerned with infinitely divisible distributions (p_n) on the non-negative integers. In this case the Lévy-Hinčin representation takes the form

$$(1) \qquad \hat{p}(z) = \sum_{n \geqslant 0} p_n z^n = \exp\left\{ - \sum_{j \geqslant 1} (1 - z^j)\nu_j \right\}$$

where (ν_j) is the Lévy measure and $\lambda = \Sigma \nu_j$ is finite. It will be convenient to write (1) in the form

$$(2) \qquad \sum_{n \geq 0} p_n z^n = \exp\left\{ -\lambda\left(1 - \sum_{j \geq 1} \alpha_j z^j\right)\right\}$$

so that $\alpha_j = \nu_j/\lambda$ is a probability measure with $\alpha_0 = 0$. Let $d = \text{g.c.d.}\{j: \alpha_j > 0\}$. Then (p_n) is supported by the set of multiples of d, and p_{kd} is positive for all sufficiently large k. Thus, by considering $\bar{\alpha}_j = \alpha_{jd}/d$ and $\bar{p}_n = p_{nd}$ we may suppose that $d = 1$ and that p_n is positive for all sufficiently large n. This assumption is made throughout this paper. The identity (2) can also be written as the infinite system of equations

$$(3) \qquad np_n = \lambda \sum_{1 \leq j \leq n} j\alpha_j p_{n-j}, \qquad n = 1, 2, \ldots,$$

with $p_0 = e^{-\lambda}$.

There are a number of results that relate the behaviour of the tails of the measures (p_n) and (ν_n). For example Embrechts *et al.* (1979), Theorem 1 implies the equivalence of the statements:

(i) $\Sigma_{k \leq [x]} p_k$ belongs to $\mathbb{S}$;

(ii) $\Sigma_{k \leq [x]} \alpha_k$ belongs to $\mathbb{S}$; and

(iii) $\Sigma_{k \geq n} p_k \sim \lambda \Sigma_{k \geq n} \alpha_k$ $(n \to \infty)$.

(Here $\mathbb{S}$ denotes the class of subexponential distribution functions on $[0, \infty)$, that is those F satisfying $\lim_{x \to \infty} (1 - F^{(2)}(x))/(1 - F(x)) = 2$). The theorem of Embrechts *et al.*, holding for tails of general infinitely divisible probability measures on $[0, \infty)$, generalized work of Cohen (1973), Feller (1971), Pakes (1975) and Teugels (1975).

In this paper our main concern is with the asymptotic behaviour of p_n/α_n, as opposed to that of $(\Sigma_{k \geq n} p_k)/(\Sigma_{k \geq n} \alpha_k)$. Special cases have already been considered by Hawkes *et al.* (1978) (in particular in Theorem 3.1). E. M. Wright (1967) works with the system of equations (3) but with the side conditions $p_0 = 1, \lambda = 1$ and $\Sigma \alpha_i = \infty$, he proved that the following statements are equivalent:

(i) $p_n \sim \alpha_n$ $(n \to \infty)$;

(ii) $\Sigma_{0 < i < n} \alpha_i \alpha_{n-i} = o(1)\alpha_n$ $(n \to \infty)$; and

(iii) $\Sigma_{0 < i < n} p_i p_{n-i} = p(1)p_n$ $(n \to \infty)$.

To make the comparison with Wright's work more pertinently, consider $q_n = e^\lambda p_n$, $\beta_n = \lambda\alpha_n$ hence the recurrence relation (3) becomes

$$q_0 = 1, \qquad nq_n = \sum_{j=1}^{n} j\beta_j q_{n-j}$$

as in Wright (1967). Our Theorem 1 gives conditions such that

$$\lim_{n \to \infty} (q_n/\beta_n) = \exp\left(\sum \beta_n\right)$$

when $\beta_{n+1}/\beta_n \to 1$ and $\Sigma\beta_n$ is finite. So all limit values > 1 are possible in this case, while by Wright (1967) $q_n/\beta_n \to 1$ can occur only if $\Sigma\beta_n = \infty$. Since Wright's work arose out of an enumeration problem, it was natural to take $q_0 = 1$. While this is not natural in the probabilisitic context, it does emphasize the difference between the two limits (namely $\exp(\Sigma\beta_n)$ and 1).

We let $(a*b)_n = \Sigma_{0 \leqslant k \leqslant n} a_k b_{n-k}$ denote the convolution product of two sequences and a_n^{*k} be the kth convolution power of the sequence (a_n). Our principal result is the following:

THEOREM 1. *The following three statements are equivalent*:
(i) $\alpha_n^{*2} \sim 2\alpha_n$ and $\alpha_{n+1} \sim \alpha_n \ (n \to \infty)$;
(ii) $p_n^{*2} \sim 2p_n$ and $p_{n+1} \sim p_n \ (n \to \infty)$; *and*
(iii) $p_n \sim \lambda\alpha_n$ and $\alpha_{n+1} \sim \alpha_n \ (n \to \infty)$.

The convolution condition should be compared with that in the result due to E. M. Wright, quoted above. One consequence of our result is that (p_n) is regularly varying if and only if (ν_n) is the same. In view of the above result by Embrechts, Goldie and Veraverbeke a similar statement can be made about the tails of the measures (p_n) and (ν_n). It follows that by constructing a suitable Lévy measure one can produce an example of an infinitely divisible distribution (p_n) such that $\Sigma_{k \geqslant n} p_k$ has regular variation but that (p_n) itself is not regularly varying.

The following (superficially stronger) result is equivalent to that of Theorem 1.

THEOREM 2. *Suppose that* $R \geqslant 1$. *Then the following three statements are equivalent*:
(i) $\hat{\alpha}(R)$ *is finite*, $\alpha_n^{*2} \sim 2\hat{\alpha}(R)\alpha_n$, *and* $\alpha_n \sim R\alpha_{n+1} \ (n \to \infty)$;
(ii) $\hat{p}(R)$ *is finite*, $p_n^{*2} \sim 2\hat{p}(R)p_n$, *and* $p_n \sim Rp_{n+1} \ (n \to \infty)$; *and*
(iii) $p_n \sim \lambda\hat{p}(R)\alpha_n$, *and* $\alpha_n \sim R\alpha_{n+1} \ (n \to \infty)$.

Theorem 2 is proved by applying Theorem 1 to the infinitely divisible probability measure $p_n R^n/\hat{p}(R)$, which corresponds to the Lévy measure $\nu_j R^j$.

When $\hat{p}(z)$ has radius of convergence R, $R \geqslant 1$, we can use Theorem 2 in the case where $\hat{p}(R) < \infty$. When $\hat{p}(R) = \infty$ we cannot make use of the Chover, Ney and Wainger results. In this situation the results of Hawkes *et al.* (1978) can be used to give information about the behaviour of p_n.

To translate the results of that paper to the present situation one lets $b_n = R^n p_n/p_0$ and $a_j = jR^j\nu_j$. One can, for example, apply Theorem 4.2 to obtain the following criteria for p_n to be asymptotically geometric.

LEMMA 1. *If $R > 1$, each of the following statements implies its successor*:
(i) $\Sigma(R^j \nu_j - 1/j)$ *converges absolutely to* L;
(ii) $p_n \sim p_0 e^L R^{-n}$ *as* $n \to \infty$; *and*
(iii) $\Sigma(R^j \nu_j - 1/j)$ *converges to* L.

The following beautiful result, due to Chover, Ney and Wainger plays a crucial rôle in our investigations. Since its proof relies on a deep argument involving Banach algebra techniques it is in some ways unsatisfactory from the point of view of real analysis. We therefore prefer to minimize the number of occasions on which we appeal to it.

PROPOSITION 1. *Suppose that the probability measure* (μ_n) *satisfies the three conditions*:
 (α) $\lim_{n \to \infty} \mu_n^{*2}/\mu_n = c$ *exists and is finite*;
 (β) $\lim_{n \to \infty} \mu_{n+1}/\mu_n = 1/R$ *exists and is positive*; *and*
 (γ) $d = \hat{\mu}(R)$ *is finite*.
Assume also that $\Phi(w)$ *is a function analytic in a region containing the range of* $\hat{\mu}(z)$ *for* $|z| \leq R$.
 Then there exists a measure $(\Phi(\mu)_n)$ *with*

$$(\Phi(\mu))\hat{}(z) \equiv \sum_{n \geq 0} \Phi(\mu)_n z^n = \Phi(\hat{\mu}(z)) \quad for \ |z| \leq R,$$

and for which

$$\lim_{n \to \infty} \Phi(\mu)_n/\mu_n = \Phi'(d).$$

If moreover $\Phi(w) = \sum_{j \geq 0} c_j w^j$ *for* $|w| \leq 1$, *where* $\sum_{j \geq 0} |c_j|$ *is finite, then we can identify the measure* $\Phi(\mu)$ *with* $\bar{\mu} = \sum_{j \geq 0} c_j \mu^{*j}$. *Finally we must have* $c = 2d$ *in* (α).

For details and examples of sequences satisfying (α), (β) and (γ) see Chover *et al.* (1973a), Theorem 1 and Remark 1. A similar result is proved in Borovkov (1976), Appendix.

2. Proof of Theorem 1

For simplicity we will only consider the case where $\lambda = 1$. First we show that (iii) implies (ii). Let N be a fixed positive integer and define

$$I = I(N, n) = \{j: n - N \leq j \leq n\}.$$

Then, since $p_{n+1} \sim p_n$, we have

$$\sum_{j \in I} p_j p_{n-j} = p_n (1 + o(1)) \sum_{j \leqslant N} p_j$$

$$= p_n \left\{ 1 - \sum_{j > N} p_j \right\} (1 + o(1)).$$

Hence for some integer $n_0 = n_0(N)$ we have

(4) $$\left| \sum_{j \in I} p_j p_{n-j} - p_n \right| \leqslant 2 p_n \sum_{j > N} p_j \quad \text{for } n \geqslant n_0.$$

Now, by (3),

(5) $$n p_n = \sum j \alpha_j p_{n-j} = \left(\sum_{j \in I} + \sum_{j \notin I} \right) j \alpha_j p_{n-j}.$$

We have

$$\sum_{j \in I} j \alpha_j p_{n-j} = n(1 + o(1)) \sum_{j \in I} \alpha_j p_{n-j}$$

$$= n(1 + o(1)) \sum_{j \in I} p_j p_{n-j} \quad (\text{as } \alpha_n \sim p_n)$$

$$= n p_n (1 + o(1)) \left\{ 1 - \sum_{j > N} p_j \right\}.$$

Thus, by (5),

(6) $$\sum_{j \notin I} j \alpha_j p_{n-j} = n p_n (1 + o(1)) \sum_{j > N} p_j$$

$$\leqslant 2 n p_n \sum_{j > N} p_j \quad \text{for } n \geqslant n_1 = n_1(N) \text{ say.}$$

Let n_2 be such that for $j \geqslant n_2$ we have $p_j \leqslant 2 \alpha_j$. Define

$$J = J(N, n) = \left\{ j : \frac{1}{2} n \leqslant j \leqslant n - N \right\}$$

so that if $n \geqslant n_0 + n_1 + 2 n_2$

(7) $$\sum_{j \in J} p_j p_{n-j} \leqslant 2 n^{-1} \sum_{j \in J} j p_j p_{n-j} \leqslant 4 n^{-1} \sum_{j \in J} j \alpha_j p_{n-j}$$

$$\leqslant 4 n^{-1} \sum_{j \notin I} j \alpha_j p_{n-j}$$

(8) $$\leqslant 8 p_n \sum_{j > N} p_j, \quad \text{by (6).}$$

If we combine the inequalities (4) and (8) we obtain

$$\left|p_n^{*2} - 2p_n\right| \leqslant 20 p_n \sum_{j>N} p_j$$

for sufficiently large n. It follows that $p_n^{*2} \sim 2 p_n$ $(n \to \infty)$. Thus (iii) implies (ii).
We now show that (ii) implies (iii). Let $\hat{c}(z)$ be the power series defined by

$$\hat{c}(z) = (1 - \hat{p}(z))/\hat{p}(z) = \sum_{n \geqslant 0} c_n z^n.$$

Then applying Proposition 1 with the function $\Phi(w) = (1 - w)/w$ we see that Σc_i is absolutely convergent with $\Sigma c_i = 0$, $c_n \sim -p_n$ $(n \to \infty)$, so that $|c_n| \leqslant 2 p_n$ for $n \geqslant n_3$, say. It follows from (2) and the definition of $\hat{c}$ that

$$\hat{\alpha}' = \hat{p}' + \hat{p}'\hat{c}.$$

Thus if we define $\hat{d} = \hat{p}'\hat{c} = \Sigma_{k \geqslant 0} d_k z^k$ we have

$$d_n = \sum (j + 1) p_{j+1} c_{n-j}.$$

Now fix $N \geqslant n_3$. Then

$$\left| \sum_{j \leqslant N} (j + 1) p_{j+1} c_{n-j} \right| \leqslant 2(N + 1) \sum_{j \leqslant N} p_{j+1} p_{n-j}$$

(9)
$$= O(1) p_n = o(1) n p_n,$$

$$\left| \sum_{N \leqslant j \leqslant n-N} (j + 1) p_{j+1} c_{n-j} \right| \leqslant 2n \sum_{N \leqslant j \leqslant n-N} p_{j+1} p_{n-j}$$

(10)
$$\leqslant 8 n p_n \sum_{j>N} p_j$$

for sufficiently large n, and

$$\left| \sum_{j \in I(N, n)} (j + 1) p_{j+1} c_{n-j} \right| = \left| \sum_{j \in I} \{(j + 1) p_{j+1} - n p_n\} c_{n-j} + n p_n \sum_{j \leqslant N} c_j \right|$$

(11)
$$\leqslant \sum_{j \in I} \left|(j + 1) p_{j+1} - n p_n\right| |c_{n-j}| + n p_n \left| \sum_{j \leqslant N} c_j \right|$$

$$= o(1) n p_n + n p_n \left| \sum_{j \leqslant N} c_j \right|.$$

If we combine (9), (10) and (11) we obtain

$$\limsup_{n \to \infty} |d_n|/n p_n \leqslant 8 \sum_{j>N} p_j + \left| \sum_{j \leqslant N} c_j \right|.$$

As the latter can be made arbitrarily small we have $d_n = o(1) n p_n$. Since

$$(n + 1)\alpha_{n+1} = (n + 1) p_{n+1} + d_n$$

and $p_{n+1} \sim p_n$ it follows that $\alpha_n \sim p_n$, with the result that statements (ii) and (iii) are equivalent. The argument of the first part of the proof shows that (iii), together with its equivalent form (ii), implies (i).

There remains the problem of showing that statement (i) implies (ii). First we state a preliminary lemma.

LEMMA 2. *Suppose that α_n is a probability measure on the non-negative integers such that $\alpha_n \sim \alpha_{n+1}$ and $\alpha_n^{*2} \sim 2\alpha_n$ as $n \to \infty$. Then if $c > 1$ there is a positive constant A such that*

$$\alpha_n^{*k} \leqslant Ac^k \alpha_n \quad \text{for each } k \text{ and } n.$$

The proof of this lemma is very similar to that of Lemma 1 of Chover *et al.* (1973b), so we just give an outline. If $1 < d < c$ we can choose integers N and K such that

$$\sum_{j \notin I(N, n)} \alpha_j \alpha_{n-j} \leqslant d\alpha_n \quad \text{if } n \geqslant K.$$

Next we let $M_k = \sup_{n \geqslant 0} \alpha_n^{*k}/\alpha_n$, define

$$B = \sup_{0 \leqslant n \leqslant K} \alpha_n^{-1} + \sup_{n \geqslant K} \alpha_n^{-1} \sum_{j \in I(N, n)} \alpha_j$$

and show that $M_{k+1} \leqslant B + M_k d$. Then if $A \geqslant 1 + Bc/(c - d)$ one sees by induction that $M_k \leqslant Ac^k$.

We now complete the proof of the theorem. It follows from (2) that

(12) $$p_n/\alpha_n = e^{-1} \sum_{k \geqslant 1} \left(\alpha_n^{*k}/\alpha_n\right)/k!.$$

The hypotheses on α imply that $\alpha_n^{*k} \sim k\alpha_n$ (see Chover *et al.* (1973a), Lemma 5). Thus by Lemma 2 we can use dominated convergence to deduce that $p_n \sim \alpha_n$. The Theorem is thus proved.

We remark that the last part of the proof could have been derived by applying Proposition 1 with the measure α_n and the function $\Phi(w) = e^{-\lambda(1-w)}$. However our approach has avoided the implicit use of Banach algebra methods. One can in fact use Lemma 2 to show that if $\hat{\Phi}(w) = \Sigma\Phi_j w^j$ has radius of convergence R, $R > 1$, then

$$\Phi(\hat{\alpha}(s)) = \sum \psi_n s^n \quad \text{where } \psi_n = \sum_k \Phi_k \alpha_n^{*k}$$

and $\psi_n/\alpha_n \to \Sigma k\Phi_k = \Phi'(1)$. So the whole situation is much simpler for these functions. This observation does not help us in the second part of the proof where we worked with $\Phi(w) = (1 - w)/w$.

Regarding the conditions in Theorem 1 we have not been able to construct a probability measure (α_n) satisfying $\alpha_n^{*2} \sim 2\alpha_n$ but not $\alpha_{n+1} \sim \alpha_n$. Conversely Rudin (1973) gives an example of a log-convex sequence (α_n) with $\Sigma n \alpha_n$ infinite which satisfies $\alpha_{n+1} \sim \alpha_n$ but not $\alpha_n^{*2} \sim 2\alpha_n$.

3. Fluctuation theory

Let (X_i) be a sequence of independent identically distributed random variables and let $S_0 = 0$ and $S_n = \Sigma_{1 < i < n} X_i$. We now quote some of the definitions and results from the fluctuation theory of random walks. (For more details the reader should consult Feller (1971).) The first ladder epochs are defined by $L^+ = \min\{k: S_k > 0\}$ and $L^- = \min\{k \geqslant 1: S_k \leqslant 0\}$. The number of positive and non-positive terms in the random walk are given by

$$N_n^+ = \#\{k \leqslant n: S_k > 0\} \quad \text{and} \quad N_n^- = \#\{k: 1 \leqslant k \leqslant n, S_k \leqslant 0\}$$

respectively. If we define

$$r_n = P(N_n^+ = n) = P(L^- > n), \quad \text{and} \quad q_n = P(N_n^- = n) = P(L^+ > n)$$

then the generating functions satisfy

$$\hat{r}(s) = \exp\left\{ \sum_{n \geqslant 1} (s^n/n)P(S_n > 0) \right\}$$

and

$$\hat{q}(s) = \exp\left\{ \sum_{n \geqslant 1} (s^n/n)P(S_n \leqslant 0) \right\}. \tag{13}$$

Also it is known that $P(N_n^- = k) = r_{n-k}q_k$. Finally the following statements are equivalent:

(i) $EL^+ < \infty$;

(ii) $\Sigma(P(S_n \leqslant 0)/n) < \infty$;

(iii) S_n drifts to $+\infty$ almost surely; and

(iv) $\sup_n N_n^- < \infty$ almost surely.

We remark that in the case where EL^+ is finite the relation (13) is precisely the equation (2) with $\alpha_j = P(S_j \leqslant 0)/(j \log EL^+)$, $\lambda = \log EL^+$, and $p_n = q_n/EL^+$. In these circumstances we will be able to apply Theorem 1.

The following result due to Rogozin (1971) and Spitzer (1956) summarizes the known properties concerning the asymptotic behaviour of the distributions of the random variables L^+ and N_n^-.

PROPOSITION 2. *Suppose that $EL^+ = \infty$ and that $0 < \alpha < 1$. Then the following statements are equivalent*:

(i) L^+ *belongs to the domain of attraction of a spectrally positive stable law with index α*;

(14)   (ii) $$\lim_{n \to \infty} \frac{1}{n} \sum_{1 \leqslant i \leqslant n} P(S_i \leqslant 0) = 1 - \alpha;$$

and

(iii) $$\lim_{n \to \infty} P(N_n^- \leqslant nx) = F_\alpha(x)$$

where F_α is the generalized arc-sine distribution function (that is the Beta $(\alpha, 1 - \alpha)$ distribution).

Rogozin has other results for the case where (14) holds with $\alpha = 1$. Doney (1977) has several results on the condition (14) in particular in the case where $\alpha = 0$. We now apply our result to obtain more information about the domains of attraction of L^+.

THEOREM 3. *Suppose that $EL^+ < \infty$ and let $N^- = \lim_{n \to \infty} N_n^-$. If $1 < \alpha < 2$ the following statements are equivalent*:

(i) L^+ *belongs to the domain of attraction of a spectrally positive stable law with index α*;

(ii) $P(S_n \leqslant 0) \sim n^{1-\alpha}L(n)$ *as $n \to \infty$ where L is a slowly varying function, and*

(iii) N^- *belongs to the domain of attraction of a spectrally positive stable law with index $\alpha - 1$*.

PROOF. The equivalence of (i) and (ii) follows by applying Theorem 1 to sequences of regular variation and the necessary and sufficient criteria for a distribution to belong to the domain of attraction of a stable law (see Feller (1971), 574–581 for details).

Since $P(N_n^- = k) = r_{n-k}q_k$ and $r_n \to (EL^+)^{-1}$ (apply Theorem 4.2 of Hawkes *et al.* (1978)) we have $P(N^- = k) = q_k(EL^+)^{-1}$. As $q_k = P(L^+ > k)$ is decreasing $P(N^- > k) \sim k^{1-\alpha}L(k)$ is equivalent to

$$P(N^- = k) = P(L^+ > k)(EL^+)^{-1} \sim (\alpha - 1)k^{-\alpha}L(k).$$

The equivalence of (i) and (iii) follows immediately, and the theorem is proved.

We have not been able to determine a necessary and sufficient condition for L^+ to belong to the domain of attraction of a stable law of index 1. A necessary condition is that (14) holds with $\alpha = 1$. There remains the problem of showing that the conditions of our theorem are satisfied in nontrivial situations.

EXAMPLE. Suppose that $1 < \alpha < 2$ and that (Y_i) is a sequence of independent random variables each having the symmetric stable distribution of index α. Let $X_i = Y_i + 1$ and let S_n be the random walk corresponding to X_i. Then $(S_n - n)n^{-1/\alpha}$ has the symmetric stable distribution of index α. Thus

$$P(S_n \leqslant 0) = P\big((S_n - n)n^{-1/\alpha} \leqslant -n^{1-1/\alpha}\big)$$
$$= P(Y \leqslant -n^{1-1/\alpha})$$
$$\sim C(n^{1-1/\alpha})^{-\alpha} = Cn^{1-\alpha}$$

as $n \to \infty$ (where C is a positive constant). By the strong law of large numbers S_n tends to $+\infty$ and so the theorem is indeed non vacuous.

It is possible to derive other information about the behaviour of the tail $P(L^+ > n)$. We now consider random walks generated by X_i with $\mu = EX_1 > 0$ and var $X_1 = 1$. Let $T_n = S_n - n\mu$. Nagaev (1979) has many results which are sufficient to ensure that

$$P(S_n \leqslant 0) = P(T_n \leqslant -n\mu) \sim nP(X \leqslant -(n-1)\mu).$$

Thus provided in addition $P(X \leqslant -(n-1)\mu)$ has sufficiently smooth asymptotic behaviour (as defined by Theorem 1)

(15) $P(L^+ > n) \sim EL^+ P(X \leqslant -(n-1)\mu)$ as $n \to \infty$.

The reader will be able to show that (15) is valid if
   (i) $P(X \leqslant -x) \sim \exp(-x^\alpha)$ $(x \to \infty)$, $0 < \alpha < 1/2$, if
   (ii) $E|X|^{2+\delta} < \infty$ for some $\delta > 0$ and $P(X \leqslant -x) \sim x^{-\alpha}L(x)$ as $x \to \infty$, where $\alpha > 2$ and L is slowly varying; or under some of the other conditions given by Nagaev.

If (15) holds it follows that for each $k \geqslant 1$

(16) $E(L^+)^k < \infty$ if and only if $E(X \wedge 1)^k < \infty$.

Bingham and Goldie (private communication) have shown that (16) always holds provided one has $EX^2 < \infty$.

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