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# Minimally locally 1-connected graphs

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The local connectivity, $vk(G)$ , of a graph $G$ is the minimum of the connectivities of neighbourhoods of the vertices of $G$ . $G$ is minimally locally $n$-connected if $vk(G) = n$ and for every edge $x$ of $G$ , $vk(G-x) = n - 1$ . A necessary and sufficient condition for a locally connected graph to be minimally locally 1-connected is given, and it is shown that for $n \geq 7$ , $C_n^2$ is minimally locally 1-connected.

## 1. Introduction

Our terminology is in conformity with that of Behzad and Chartrand [1]. For a vertex $v$ of a graph $G$ , let $N(v)$ denote the set of all vertices of $G$ adjacent with $v$ . The *neighbourhood of* $v$ , denoted by $G(v)$ , is the subgraph of $G$ induced by $N(v)$ . $G$ is said to be *locally connected* if the neighbourhood of every vertex of $G$ is connected. $G$ is said to be *locally* $n$*-connected* if the neighbourhood of every vertex of $G$ is $n$-connected. The *local connectivity*, $vk(G)$ , of $G$ is the maximum $n$ such that $G$ is locally $n$-connected. Similarly, $G$ is said to be *locally* $n$*-edge connected* if the neighbourhood of every vertex of $G$ is $n$-edge connected. The *local edge-connectivity*, $vk_1(G)$ , of $G$ is the maximum $n$ for which $G$ is locally $n$-edge connected. Hence it follows that the local connectivity (edge-connectivity) of a graph is the minimum of the connectivities (edge-connectivities) of the neighbourhoods of its

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vertices. A graph may be locally connected without being connected, and conversely. For example, $2K_3$ is locally connected while it is not connected; $K(1, 3)$, which is connected is not locally connected. A graph is locally connected if and only if each of its components is locally connected.

The concept of local connectivity has been introduced by Chartrand and Pippert [2]. Just like critically and minimally $n$-connected graphs, critically and minimally locally $n$-connected graphs can also be defined. A graph $G$ is *critically locally n-connected* if $vk(G) = n$ and for every vertex $v$ of $G$, $vk(G-v) = n - 1$; $G$ is *minimally locally n-connected* if $vk(G) = n$ and $vk(G-x) = n - 1$ for every edge $x$ of $G$. Just as the properties critically $n$-connected and minimally $n$-connected are independent in the sense that neither implies the other, so also are the properties critically locally $n$-connected and minimally locally $n$-connected. For example, the graph obtained from $K(n, n, n)$ by joining a pair of nonadjacent vertices is critically locally $n$-connected but it is not minimally locally $n$-connected; $K(n, n, n+1)$ is minimally locally $n$-connected while it is not critically locally $n$-connected. $K(n, n, n)$ is both critically and minimally locally $n$-connected. $K(n, n+1, n+1)$ is locally $n$-connected; but it is neither critically nor minimally locally $n$-connected.

The only critically 1-connected graph is the complete graph of order 2. But critically locally 1-connected graphs of many orders exist. In [4] a characterisation of such graphs is given. In this paper a necessary and sufficient condition for a locally connected graph to be minimally locally 1-connected is given and it is shown that for $n \geq 7$, $C_n^2$ is minimally locally 1-connected.

A *wheel* $W$ is a cycle $C$ together with an additional vertex $w$ adjacent with every vertex of $C$. The cycle $C$ is called the *circumcycle* of $W$, and $W$ is called the *wheel about* $w$. If $P$ is a path in the cycle $C$, then $P$ is called a *part of the wheel* $W$. The connectivity and edge-connectivity of a graph $G$ will be denoted by $k(G)$ and $k_1(G)$, respectively.

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## 2. Necessary and sufficient condition

THEOREM. *A locally connected graph* $G$ *is minimally locally* 1*-connected if and only if for every edge* $x = uv$ *of* $G$ *there exists a vertex* $w$ *adjacent with both* $u$ *and* $v$ *such that* $x$ *is not a part of any wheel about* $w$ *in* $G$ .

Proof. Assume that $G$ is minimally locally 1-connected. Consider an edge $x = uv$ of $G$ . Let $H$ denote the graph $G - x$ . Then $vk(H) = 0$ . Now, for every vertex $w'$ of $G$ which is not adjacent with both $u$ and $v$ , $H(w') = G(w')$ ; hence $k(H(w')) = k(G(w')) \geq 1$ . Hence there exists a vertex $w$ adjacent with both $u$ and $v$ such that $k(H(w)) = 0$ . Since $H(w)$ contains at least two vertices, namely, $u$ and $v$ , it is disconnected. Also, $H(w) = G(w) - x$ and $G(w)$ is connected imply that $x$ is a bridge of $G(w)$ . Suppose $G$ contains a wheel $W$ about $w$ such that $x$ is a part of $W$ . Then every vertex of the circum-cycle of $W$ is in $G(w)$ . The part of this cycle not containing the edge $x$ constitutes a $u - v$ path in $G(w)$ . This implies that $x$ is not a bridge of $G(w)$ , which is a contradiction. Hence $x$ is not a part of any wheel about $w$ in $G$ .

Conversely, assume that $G$ satisfies the hypothesis of the theorem. Consider an edge $x = uv$ of $G$ . Let $w$ be a vertex of $G$ adjacent with both $u$ and $v$ such that $x$ is not a part of any wheel about $w$ in $G$ . Let $G - x$ be denoted by $H$ . Then $H(w) = G(w) - x$ . Suppose $H(w)$ is connected. Then there exists a path $P$ in $H(w)$ joining $u$ and $v$ . This path together with the edge $x$ constitutes a cycle $C$ every vertex of which is adjacent with $w$ in $G$ . Hence $x$ is a part of a wheel about $w$ in $G$ which is a contradiction. Hence $H(w)$ is disconnected. Therefore $k(H(w)) = 0$ . Now $H(w) = G(w) - x$ and $G(w)$ is connected imply that $x$ is a bridge of $G(w)$ . Hence $vk_1(G) = 1$ , and hence $vk(G) = 1$ . This proves that $G$ is minimally locally 1-connected.

The following corollary is immediate.

COROLLARY 1. *If* $G$ *is a locally connected graph not containing a wheel, then* $G$ *is minimally locally* 1*-connected.*

COROLLARY 2. *For* $n \geq 7$ , $C_n^2$ *is minimally locally* 1*-connected.*

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Proof. Let $n \geq 7$ be arbitrary and $C_n : v_0, v_1, \ldots, v_{n-1}, v_0$. Let $C_n^2$ be denoted by $G$. For a vertex $v_i$ of $G$, $N(v_i) = \{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\}$, where the indices of the vertices are taken modulo $n$. Hence $G(v_i) = P_3$, for $n = 0, 1, \ldots, n-1$. Hence $G$ is locally connected. Also, since $n \geq 7$, $G$ does not contain any wheel. Hence, by Corollary 1, $G$ is minimally locally 1-connected.

## References

[1] Mehdi Behzad and Gary Chartrand, *Introduction to the theory of graphs* (Allyn and Bacon, Boston, 1971).

[2] Gary Chartrand and Raymond E. Pippert, "Locally connected graphs", *Časopis Pěst. Mat.* 99 (1974), 158-163.

[3] K.H. Kulkarni, "Some topics in graph theory" (PhD thesis, Karnatak University, Dharwar, 1976).

[4] K.H. Kulkarni, "A characterization of critically locally 1-connected graphs", submitted.

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