FORMAL SYSTEM HAVING JUST ONE PRIMITIVE NOTION

KATUZI ONO

Introduction

In this paper, I would like to point out some characteristic features of formal systems having just one primitive notion. Most remarkable systems of this kind may be Zermelo's and Fraenkel's set-theories, both having just one primitive notion $\in$.

In my former work [1], I have proved that **J**-series logics (intuitionistic predicate logic **LJ**, Johansson's minimal predicate logic **LM**, and positive predicate logic **LP**) as well as **K**-series logics (fortified logics **LK**, **LN**, and **LQ** of **LJ**, **LM**, and **LP** by Peirce's rule that $(A \rightarrow B) \rightarrow A$ implies A, respectively; **LK** being the lower classical predicate logic) can be faithfully interpreted in the primitive logic **LO**. However, this does not mean that CONJUNCTION, DISJUNCTION, NEGATION (except for **LP** and **LQ**), and EXISTENTIAL QUANTIFICATION of these logics can be defined in **LO** in such ways that propositions behave with respect to the defined logical constants together with the original logical constants of **LO** just as propositions of **J**- and **K**-series logics concerning their *provability*. In reality, these logical constants are defined so only for a certain class of propositions in **LO**.

If we restrict ourselves only to *formal systems having just one primitive notion*, however, we can really define CONJUNCTION, DISJUNCTION, and EXISTENTIAL QUANTIFICATION together with one NEGATION for **LJ**, a series of NEGATIONS for **LM**, and naturally no NEGATION for **LP** in such way that propositions behave with respect to the *defined logical constants* together with the original logical constants of **LO** just as propositions of these **J**-series logics, respectively. I will discuss the matter in (**1**).

In **LK**, all the logical constants can be expressed exactly in terms of IMPLICATION, UNIVERSAL QUANTIFICATION, and NEGATION, but not in

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LJ. Can we expect similarly to express in **LQ** all the logical constants of the logic in terms of IMPLICATION and UNIVERSAL QUANTIFICATION? I can not answer the question in the affirmative in general. However, if we restrict ourselves only to *formal systems having just one primitive notion*, we can really express in **LQ** all the logical constants in terms of the two logical constants. It is very remarkable that the same situation arises for **LP** too.

For **LJ**, **LK**, **LM**, and **LN**, we can not express all the logical constants in terms of IMPLICATION and UNIVERSAL QUANTIFICATION, even for formal systems having just one primitive notion. However, we can do this for these logics under certain condition. I will discuss the matter in (**2**).

In my former work [2], I have proposed to establish formal theories standing on **LO** and starting from TABOOS instead of starting from AXIOMS and assuming highly brought up logics such as **LK** from the beginning. However, I have to admit that it is hard to describe TABOOS in **LO** because of its scanty logical vocabulary. Since fundamental theories of mathematics such as set theory can be expected to have just one primitive notion, the result of (**1**) would serve for describing TABOOS in **LO** for these fundamental theories. In (**3**), I will give some remarks concerning the matter.

(1) Definitions of CONJUNCTION, DISJUNCTION, EXISTENTIAL QUANTIFICATION, and NEGATION in LO for formal systems having just one primitive notion

Let S be a formal system having just one primitive notion R (n-ary relation) and standing on the logic **LO**. I will denote by $\mathfrak{a}, \mathfrak{b}, \ldots, \mathfrak{x}, \mathfrak{y}, \ldots$ series of variables of length n. When they are used as variables in a quantifier such as $\mathfrak{x}$ of $(\mathfrak{x})R(\mathfrak{x})$, it is tacitly assumed that they are series of n *distinct* variables. For example, let $\mathfrak{S}_0$ be a set-theory having $\in$ as its sole primitive notion. Then, $n=2$ for $\mathfrak{S}_0$, and $\mathfrak{a}, \mathfrak{b}, \ldots, \mathfrak{x}, \mathfrak{y}, \ldots$ denote ordered pairs of variables, and $(\mathfrak{x})R(\mathfrak{x})$ denotes $(u)(v)u\in v$.

In my work [1], I have introduced the notion R-closed as follows: Any proposition $\mathfrak{P}$ is called R-closed if and only if its R-closure $(\mathfrak{x})((\mathfrak{P}\to R(\mathfrak{x}))\to R(\mathfrak{x}))$ implies $\mathfrak{P}$. For the formal system $\mathfrak{S}$ having the sole primitive notion R, this turns out to be trivial because all the propositions $\mathfrak{P}$ of $\mathfrak{S}$ can be proved R closed. This can be shown by that any elementary proposition of $\mathfrak{S}$ is R-closed and that $\mathfrak{A}\to\mathfrak{B}$ as well as $(\mathfrak{x})\mathfrak{C}(\mathfrak{x})$ is R-closed as far as $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}(\mathfrak{x})$

are R-closed.[1)]

We define CONJUNCTION, DISJUNCTION, and EXISTENTIAL QUANTIFICATION by

(CD) $\mathfrak{A} \wedge \mathfrak{B} \leftrightharpoons (\mathfrak{x})((\mathfrak{A} \to (\mathfrak{B} \to R(\mathfrak{x}))) \to R(\mathfrak{x}))$,

(DD) $\mathfrak{A} \vee \mathfrak{B} \leftrightharpoons (\mathfrak{x})((\mathfrak{A} \to R(\mathfrak{x})) \to ((\mathfrak{B} \to R(\mathfrak{x})) \to R(\mathfrak{x})))$,

(ED) $(\exists u)\mathfrak{A}(u) \leftrightharpoons (\mathfrak{x})((u)(\mathfrak{A}(u) \to R(\mathfrak{x})) \to R(\mathfrak{x}))$.

We can easily show that *the inference rules of conjunction, disjunction, and existential quantification hold for these logical constants CONJUNCTION, DISJUNCTION, and EXISTENTIAL QUANTIFICATION, respectively.* This can be proved just as (4) of [1]. Hence, propositions of $\mathfrak{S}$ behave just as propositions in **LP** with respect to the *above defined logical constants* together with the original logical constants of **LO**.

If we further define NEGATION by

(ND) $\neg\mathfrak{A} \leftrightharpoons \mathfrak{A} \to (\mathfrak{x})R(\mathfrak{x})$,

then we can prove that $\mathfrak{A}$ and $\neg\mathfrak{A}$ imply any proposition of $\mathfrak{S}$. Hence, propositions of $\mathfrak{S}$ behave just as propositions in **LJ** with respect to the above defined logical constants together with the original logical constants of **LO**.

Lastly, we can also define $\mathfrak{c}$-NEGATION for any $\mathfrak{c}$ by

(N$\mathfrak{c}$D) $\neg_{\mathfrak{c}}\mathfrak{A} \leftrightharpoons \mathfrak{A} \to R(\mathfrak{c})$.

For any one ($\mathfrak{c}$-NEGATION) of this series of negations, we can prove that propositions of $\mathfrak{S}$ behave just as propositions in **LM** with respect to the above defined CONJUNCTION, DISJUNCTION, EXISTENTIAL QUANTIFICATION, and any one of $\mathfrak{c}$-NEGATIONS together with the original logical constants of **LO**.

(2) Some equivalences in respective logics for formal systems having just one primitive notion

In this section, I will discuss whether equivalences corresponding to the definitions (CD), (DD), (ED), (ND), and (N$\mathfrak{c}$D) hold for formal systems having just one primitive notion R (n-ary relation) and standing on the logics **LP**, **LO**, **LM**, **LN**, or **LK**.

(2.1 T) *Let $\mathfrak{S}$ be a formal system having just one primitive notion R and*

[1)] See (4) of [1].

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standing on any one of the logics **LP** *and* **LQ**. *Then, the formulas*

$$\text{(CF)}\quad \mathfrak{A}\wedge\mathfrak{B}\equiv(\mathfrak{x})((\mathfrak{A}\rightarrow(\mathfrak{B}\rightarrow R(\mathfrak{x})))\rightarrow R(\mathfrak{x})),$$

$$\text{(DF)}\quad \mathfrak{A}\vee\mathfrak{B}\equiv(\mathfrak{x})((\mathfrak{A}\rightarrow R(\mathfrak{x}))\rightarrow((\mathfrak{B}\rightarrow R(\mathfrak{x}))\rightarrow R(\mathfrak{x}))),$$

$$\text{(EF)}\quad (\exists u)\mathfrak{A}(u)\equiv(\mathfrak{x})((u)(\mathfrak{A}(u)\rightarrow R(\mathfrak{x}))\rightarrow R(\mathfrak{x}))$$

hold in $\mathfrak{S}$.

Because **LP** as well as **LQ** is stronger than **LO**, CONJUNCTION, DISJUNCTION, and EXISTENTIAL QUANTIFICATION defined by (**CD**), (**DD**), and (**ED**), respectively, satisfy the same inference rules as that of $\wedge$, $\vee$, and $(\exists\ )$ of the logic **LP** as well as of **LQ**. These equivalence relations can be easily proved by this fact.

Now, let us turn our attention to formal systems having just one primitive notion R but standing on logics having NEGATION notion. For these systems, we can not expect that (**CF**), (**DF**), and (**EF**) of (**2.1 T**) hold unconditionally. For, we can not expect even that any formula of the form $\rightharpoondown R(\mathfrak{x})$ is R-closed.

For these logics, we can denote NEGATION OF $\mathfrak{A}$ in the form $\mathfrak{A}\rightarrow\curlywedge$ instead of the form $\rightharpoondown\mathfrak{A}$ taking up a propositional constant $\curlywedge$. Speaking unconstrainedly, we would be able to modify R so that $R(\mathfrak{c})$ would denote $\curlywedge$ for a *special* series $\mathfrak{c}$ of *new object-constants* in place of the propositional constant $\curlywedge$ even when $R(\mathfrak{x})$ never becomes equivalent to $\curlywedge$ originally. More formally, we have

(**2.2 T**) *Let* $\mathfrak{S}$ *be a formal system having just one primitive notion* R *and standing on any one of logics* **LJ**, **LK**, **LM**, and **LN**. *Then, the formulas* (**CF**), (**DF**), and (**EF**) of (**2.1 T**) *together with the formula*

$$\text{(NcF)}\quad \rightharpoondown\mathfrak{A}\equiv\mathfrak{A}\rightarrow R(\mathfrak{c})$$

hold, assuming that $R(\mathfrak{c})$ *is equivalent to* $\curlywedge$. *For any formal system* $\mathfrak{S}$ *of this kind standing on* **LJ** *or* **LK**, *the formulas* (**CF**), (**DF**), *and* (**EF**) *of* (**2.1T**) *together with the formula*

$$\text{(NF)}\quad \rightharpoondown\mathfrak{A}\equiv\mathfrak{A}\rightarrow(\mathfrak{x})R(\mathfrak{x})$$

hold, assuming that $(\mathfrak{x})R(\mathfrak{x})\rightarrow\rightharpoondown R(\mathfrak{c})$ *holds for some* $\mathfrak{c}$.

The latter half of the theorem can be proved because $(\mathfrak{x})R(\mathfrak{x})\equiv R(\mathfrak{c})\wedge\rightharpoondown R(\mathfrak{c})$ is easily provable in $\mathfrak{S}$.

(3) Application for describing taboos of set theory

Main- difficulty in developing my program of establishing a fundamental theory of mathematics such as set theory standing on **LO** and starting from TABOOS seems to lie in expressing taboos relying on the scanty logical vocabulary of **LO**. For example, *taboo scheme* corresponding to the *abstraction principle* would be

(**AT**) $(p)(\exists x)(x \in p \equiv \mathfrak{A}(x))$.[2)]

However, it is hard to interpret this again in **LO**, because neither EXISTENTIAL QUANTIFICATION nor EQUIVALENCE can not be generally defined in **LO**. Fortunately enough, set theory can be described by making use of only one primitive notion $\in$. If we take up $\in$ as the sole primitive notion of our set theory, we can state the *taboo scheme* corresponding to the *abstraction principle* in the form of (**AT**) because we can define EXISTENTIAL QUANTIFICATION as well as EQUIVALENCE in **LO** for the set theory according to our theory described in (**1**).

Speaking frankly, I am dreaming that we can develop a fundamental mathematical theory from taboos in **LO** adopting just one primitive notion, I do not know whether it is $\in$ or not.

References

[1] Ono, K., On universal character of the primitive logic, Nagoya Math. J., **27-1** (1966), 331-353.

[2] Ono, K., A certain kind of formal theories, Nagoya Math. J., **25** (1965), 59-86.

Mathematical Institute,
Nagoya University

2) This taboo corresponds to the abstraction

$$(\exists p)(x)(x \in p \equiv \rightarrow \mathfrak{A}(x)).$$

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