JOINT NUMERICAL RANGES FOR UNBOUNDED NORMAL OPERATORS

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1. Introduction

For bounded operators, the theory of the joint numerical range has been developed by various authors [1, 2, 3, 4, 5]. Especially, the properties of commuting normal n-tuples are discussed in detail. Our purpose here is to show that many results in the above references still hold in the case of unbounded normal operators (see Theorem 2.3, Corrollary 3.5, Theorem 4.1, Theorem 4.2). Besides, the operator algebras are closely related to the theory of joint spectrum and joint numerical ranges in the bounded case (cf. [1, 3]). How about unbounded operators? It seems that one must consider unbounded operator algebras. Some work has been done in this direction for the joint spectrum of unbounded normal operators [9]. In the last section of this paper, we provide some intimate relations between the joint numerical range and the unbounded operator algebras for unbounded normal operators.

Throughout this note, H will be a complex Hilbert space with the scalar product (,). A closed operator T on H is said to be normal if T is densely defined and $T^*T = TT^*$ (cf. [10]). We say an n-tuple of normal operators $A = (A_1, \ldots A_n)$ is commuting if their spectral measures are commutative. The joint spectrum $\mathrm{Sp}(A)$ of A is defined as the support of $E = \prod_{i=1}^{n} E_i$, where E_i is the spectral measure of A_i, $i = 1, \ldots, n$.

The definitions here have their background in the multiparameter spectral theory (Sleeman [6]), and are closely connected with Taylor's joint spectrum of A (see [9] and Section 5).

2. Convexity of the joint numerical range

Definition 2.1. Let $A = (A_1, \ldots, A_n)$ be a commuting n-tuple of normal operators, the joint numerical range of A is the set $W(A)$ in $\mathbb{C}^n$ such that

$$W(A) = \left\{ ((A_1 x, x), \ldots, (A_n x, x)) : x \in D(A) = \bigcap_{i=1}^{n} D(A_i), \quad \|x\| = 1 \right\}$$

where $D(A_i)$ is the domain of A_i, $i = 1, \ldots, n$.

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We remark that $D(A)$ is a dense subspace of H since it contains $E(\Delta)H$, where E is the product spectral measure of A and Δ is any compact subset of $\mathbb{C}^n$.

As far as we know, $W(A)$ is convex when A is a commuting n-tuple of bounded normal operators or an n-tuple of Toeplitz operators [2,5]. In the following, we show that this is also true for a commuting n-tuple of unbounded normal operators.

Lemma 2.2. *Suppose that $A = (A_1, \ldots, A_n)$ is a commuting n-tuple of normal operators. Then there exists a family $\{(\mathbb{C}^n, \rho_\alpha) : \alpha \in \Lambda\}$ of measure spaces such that each A_k is unitary equivalent to the maximal operator of multiplication by z_k on $\bigoplus_{\alpha \in \Lambda} L_2(\mathbb{C}^n; \rho_\alpha)$, $k = 1, \ldots, n$. That is*

$$D(A_k) = \left\{ (f_\alpha) \in \bigoplus_{\alpha \in \Lambda} L_2(\mathbb{C}^n; \rho_\alpha) : (z_k f_\alpha) \in \bigoplus_{\alpha \in \Lambda} L_2(\mathbb{C}^n, \rho_\alpha) \right\},$$

and

$$A_k(f_\alpha) = (z_k f_\alpha), \quad \text{for} \quad (f_\alpha) \in D(A_k), \ k = 1, \ldots, n.$$

Proof. There exists the product spectral measure E of A since $A = (A_1, \ldots, A_n)$ is commuting and normal. It easily follows that

$$A_k = \int_{\mathbb{C}^n} z_k \, dE, \quad k = 1, \ldots, n.$$

The detailed proof is similar to that for single self-adjoint operators, we omit it here (cf. [10]).

Theorem 2.3. *Let $A = (A_1, \ldots, A_n)$ be a commuting n-tuple of normal operators. Then the joint numerical range $W(A)$ is convex.*

Proof. Without loss of generality, we may assume $A = (A_1, \ldots, A_n)$ to be the maximal operators of multiplication by $z = (z_1, \ldots, z_n)$ on $\bigoplus_{\alpha \in \Lambda} L_2(\mathbb{C}^n; \rho_\alpha)$ by using Lemma 2.2.

Suppose that $f = (f_\alpha)$, $g = (g_\alpha) \in D(A) = \bigcap_{i=1}^n D(A_i)$ and $\|f\| = \|g\| = 1$, then for any $t \in [0, 1]$ fixed, let $h = (h_\alpha) = ((t|f_\alpha|^2 + (1-t)|g_\alpha|^2)^{1/2})$. Since for any $k = 1, \ldots, n$ and $\alpha \in \Lambda$,

$$\sum_\alpha \int |z_k h_\alpha|^2 \, d\rho_\alpha = \sum_\alpha t \int |z_k f_\alpha|^2 \, d\rho_\alpha + \sum_\alpha (1-t) \int |z_k g_\alpha|^2 \, d\rho_\alpha < +\infty,$$

we have $h = (h_\alpha) \in D(A)$. Furthermore, it is clear that

$$\|h\|^2 = \sum_\alpha \|h_\alpha\|^2 = t \sum_\alpha \|f_\alpha\|^2 + (1-t) \sum_\alpha \|g_\alpha\|^2 = 1,$$

which implies

$$t((A_1 f, f), \ldots, (A_n f, f)) + (1-t)((A_1 g, g), \ldots, (A_n g, g))$$

$$= \left(\sum_\alpha \int z_1 (t|f_\alpha|^2 + (1-t)|g_\alpha|^2) \, d\rho_\alpha, \ldots, \sum_\alpha \int z_n (t|f_\alpha|^2 + (1-t)|g_\alpha|^2) \, d\rho_\alpha \right)$$

$$= ((A_1 h, h), \ldots, (A_n h, h)) \in W(A).$$

Hence $W(A)$ is a convex set in $\mathbb{C}^n$. This completes the proof.

3. A characterization of the joint numerical range

Definition 3.1. A point $z = (z_1, \ldots, z_n)$ of $\mathbb{C}^n$ is in the joint approximate point spectrum $\sigma_\pi(A)$ of A if there exists a sequence $\{x_i\}$ of unit vectors in $D(A) = \bigcap_{i=1}^n D(A_i)$ such that

$$\|(z_k - A_k)x_i\| \to 0 \qquad (i \to \infty), \; k = 1, 2, \ldots, n.$$

A point $z = (z_1, \ldots, z_n)$ of $\mathbb{C}^n$ is in the joint point spectrum $\sigma_p(A)$ of A if there exists a nonzero vector x in $D(A)$ such that

$$A_k x = z_k x, \quad k = 1, \ldots, n.$$

Lemma 3.2. *Let $A = (A_1, \ldots, A_n)$ be a commuting n-tuple of normal operators. Then $\mathrm{Sp}(A) = \sigma_\pi(A)$ and $z_0 \in \sigma_p(A)$ if and only if $E(\{z_0\}) \neq 0$ (cf. [9]).*

Lemma 3.3. *If Δ is a convex set in $\mathbb{C}^n$ and μ is a nonnegative Borel measure defined on Δ such that $\mu(\Delta) = 1$, then*

$$\int_\Delta z \, d\mu(z) \in \Delta \; (\text{cf. [7]}).$$

Now we proceed to prove the main result of this section.

Theorem 3.4. *Let E be the product spectral measure of a commuting n-tuple of normal operators $A = (A_1, \ldots, A_n)$, and let S be the family of all the convex Borel sets Δ in $\mathbb{C}^n$ satisfying $E(S) = I$. Then*

$$W(A) = \bigcap_{\Delta \in S} \Delta.$$

Proof. We denote $V(A) = \bigcap_{\Delta \in S} \Delta$.

(i) For any $\Delta \in S$ and $x \in D(A)$, $\|x\| = 1$, $\int_\Delta z \, d\|E(z)x\|^2$ is a Bochner integration about the probability measure $\mu_x = \|E(\cdot)x\|^2$. By Lemma 3.3 we have

$$((A_1 x, x), \ldots, (A_n x, x)) = \left(\int_{\mathrm{Sp}(A_1)} z_1 \, d(E_1(z_1)x, x), \ldots, \int_{\mathrm{Sp}(A_n)} z_n \, d(E_n(z_n)x, x) \right)$$

$$= \left(\int_\Delta z_1 \, d(E(z)x, x), \ldots, \int_\Delta z_n \, d(E(z)x, x) \right)$$

$$= \int_\Delta z \, d\|E(z)x\|^2 \in \Delta.$$

This proves that $W(A) \subset V(A)$.

(ii) Now we shall show that $V(A) \subset W(A)$.

It is known that $\mathrm{Sp}(A) = \sigma_\pi(A)$ (Lemma 3.2) and $W(A)$ is convex (Theorem 2.3), so we have $\mathrm{conv}(\mathrm{Sp}(A)) \subset \overline{W(A)}$. Suppose $z_0 \in V(A)$, it is clear that $z_0 \in \mathrm{conv}(\mathrm{Sp}(A))$.

If $E(\{z_0\}) \neq 0$, then $z_0 \in \sigma_p(A) \subset W(A)$.

If $z_0 \in \mathrm{int}(\mathrm{conv}(\mathrm{Sp}(A)))$, then $z_0 \in \mathrm{int}\,\overline{W(A)} \subset W(A)$ since $W(A)$ is convex.

It remains to consider the case when z_0 is a boundary point of $\mathrm{conv}(\mathrm{Sp}(A))$ and $E(\{z_0\}) = 0$.

Considering $\mathbb{C}^n$ as a real linear space of dimension $2n$, we know that there exists a hyperplane π of dimension $2n - 1$ which supports $\mathrm{conv}(\mathrm{Sp}(A))$ at z_0. Taking z_0 as origin, we shall establish a real rectangular coordinates system for $\mathbb{R}^{2n}$ as follows.

First we can fix the x_1-axis such that the equation of π is $x_1 = 0$ and $\mathrm{conv}(\mathrm{Sp}(A))$ is at the side of $x_1 \geqq 0$.

In $x_1 = 0$, if for every open half-space e one has $E(e) \neq 0$, then there must exist a compact convex set $\Delta \subset e$ such that $E(\Delta) \neq 0$. Hence we can pick a unit vector $x \in E(\Delta)H \subset D(A)$. By Lemma 3.3 we have

$$\lambda_e = ((A_1 x, x), \ldots, (A_n x, x)) = \int_\Delta z \, d\|E(z)x\|^2 \in \Delta \subset e.$$

Since e is an arbitrary half-space in π, we can choose some points of kind λ_e circles z_0. It follows from Theorem 2.3 that $z_0 \in W(A)$.

On the other hand, if there exists a half-space e in π for which $E(e) = 0$, then we fix the x_2-axis such that $e = \{x_1 = 0, x_2 < 0\}$. Replacing $\{x_1 = 0\}$ by $\{x_1 = 0, x_2 = 0\}$, we continue the procedure above. Then either for some step we obtain $z_0 \in W(A)$ or eventually we get

$$E(\{x_1 < 0\}) = 0,$$

$$E(\{x_1 = 0, x_2 < 0\}) = 0,$$

$$\ldots\ldots\ldots,$$

$$E(\{x_1 = x_2 = \cdots = x_{2n-2} = 0, \quad x_{2n-1} < 0\}) = 0.$$

Let $S_1 = \{x_1 = \cdots = x_{2n-1} = 0, x_{2n} < 0\}$, $S_2 = \{x_1 = \cdots = x_{2n-1} = 0, x_{2n} > 0\}$. We may assert that $E(S_i) = 0$, $i = 1, 2$. Indeed, notice that $E(\{z_0\}) = 0$, $E(S_1) = 0$ would imply $E(M) = I$, where

$$M = \{x_1 > 0\} \cup \{x_1 = 0, x_2 > 0\} \cup \cdots \cup \{x_1 = \cdots = x_{2n-1} = 0, x_{2n} > 0\}.$$

This is impossible since M is convex and $z_0 = (0, \ldots, 0) \notin M$. So we have proved $E(S_1) \neq 0$. Similarly, $E(S_2) \neq 0$. Then there must exist two compact convex sets $\Delta_1 \subset S_1$ and $\Delta_2 \subset S_2$ such that $E(\Delta_i) = 0$, $i = 1, 2$. Thus we can pick unit vectors $x_i \in E(\Delta_i)H \subset D(A)$ and by Lemma 3.3 we have

$$\eta_i = ((A_1 x_i, x_i), \ldots, (A_n x_i, x_i)) \in \Delta_i \subset S_i, \quad i = 1, 2.$$

Therefore, $z_0 \in \operatorname{conv}\{\eta_1, \eta_2\} \subset W(A)$. The proof is complete.

Corollary 3.5. *If $A = (A_1, \ldots, A_n)$ is a commuting n-tuple of normal operators, then*

$$\overline{W(A)} = \overline{\operatorname{conv}(\operatorname{Sp}(A))}.$$

Proof. It immediately follows from the proof of Theorem 3.4.

4. Boundary points of the joint numerical range

By using Theorem 3.4, we can generalize some properties of the boundary points of $W(A)$ which were described in [4, 5] for the bounded case.

Theorem 4.1. *Let $A = (A_1, \ldots, A_n)$ be a commuting n-tuple of normal operators. If λ belongs to $\operatorname{Ext} \overline{W(A)}$ (the extreme points of $\overline{W(A)}$), then $\lambda \in \operatorname{Sp}(A)$.*

Proof. Suppose $\lambda \in \operatorname{Ext} \overline{W(A)}$. If λ does not belong to $\operatorname{Sp}(A)$, then there is a ball neighbourhood 0_λ of $\{\lambda\}$ such that $E(0_\lambda) = 0$. One easily sees that $M' = \overline{\operatorname{conv}}(\overline{W(A)} \backslash 0_\lambda)$ is a convex Borel set of $\mathbb{C}^n$ and $E(M') = I$. By Theorem 3.4 we have $\lambda \in \overline{W(A)} \subset \overline{M'} = M'$. But the facts $\lambda \in 0_\lambda$ and $\lambda \in \operatorname{Ext} \overline{W(A)}$ imply λ does not belong to M'. This is a contradiction.

Theorem 4.2. *Let $A = (A_1, \ldots, A_n)$ be as above. If $\lambda \in W(A) \cap \operatorname{Ext} \overline{W(A)}$, then $\lambda \in \sigma_p(A)$.*

Proof. Suppose $\lambda \in W(A) \cap \operatorname{Ext} \overline{W(A)}$. If λ does not belong to $\sigma_p(A)$, then $E(\{\lambda\}) = 0$. It follows that $E(\overline{W(A)} \backslash \{\lambda\}) = I$. We see that $\overline{W(A)} \backslash \{\lambda\}$ is still convex since $\lambda \in \operatorname{Ext} \overline{W(A)}$ and $\overline{W(A)}$ is convex. It follows from Theorem 3.4 that $\lambda \in \overline{W(A) \backslash \{\lambda\}}$. But this is a contradiction. This completes the proof.

5. Connections with unbounded operator algebras

In this section, we shall consider some relations between the joint numerical range and the unbounded operator algebras for normal operators.

As before $A = (A_1, \ldots, A_n)$ is a commuting n-tuple of normal operators and E the product spectral measure of A. Denote by $\mathscr{F}$ the family of all continuous functions on $Sp(A)$. Consider the following set of operators:

$$\mathscr{A} = \left\{ E(f) = \int_{Sp(A)} f \, dE : f \in \mathscr{F} \right\},$$

We know already from [9] that $\mathscr{A}$ is a GB*-algebra as defined in G. Allan [8], whose topology is given by a collection of seminorms:

$$T \to \| E(\Delta_n) T \|, \quad n = 1, 2, \ldots,$$

where $T \in \mathscr{A}$ and $\{ \Delta_1 \subset \Delta_2 \subset \cdots \}$ is a sequence of compact subsets in $\mathbb{C}^n$ which monotonically increase to $\mathbb{C}^n$. Let $\mathscr{D} = \bigcap_{f \in \mathscr{F}} \mathrm{Dom}(E(f))$, we also showed in [9] that $\mathscr{D}$ is a common core for $\mathscr{A}$ and $\mathscr{U} = \mathscr{A}|_{\mathscr{D}}$ is an EC#-algebra defined by A. Inoue [12]. We shall call $\mathscr{A}$ and $\mathscr{U}$ the GB*-algebra and the EC#-algebra generated by $A = (A_1, \ldots, A_n)$ respectively.

In [9] we proved the following

$$\mathrm{Sp}_E(A) = \{ (\chi(A_1), \ldots, \chi(A_n)) : \quad \chi \text{ is the extended multiplicative} \\ \text{linear functional on } \mathscr{A} \},$$

and $\mathrm{Sp}(A) = \mathrm{Sp}_E(A) \cap \mathbb{C}^n$, where $\mathrm{Sp}_E(A)$ denotes the extended Taylor's joint spectrum of A.

Correspondingly, we have the following results referring to the joint numerical range of A.

First we recall that a functional ϕ on a GB*-algebra $\mathscr{A}$ is called a state if ϕ is positive in the sense $\phi(T^*T) \geq 0$ for any $T \in \mathscr{A}$ and $\phi(I) = 1$. Now we have

Theorem 5.1. *Let $A = (A_1, \ldots, A_n)$ be a commuting n-tuple of normal operators and $\mathscr{A}$ the GB*-algebra generated by A. Let*

$$U(A) = \{ (\phi(A_1), \ldots, \phi(A_n)) : \phi \text{ is state on } \mathscr{A} \},$$

then we have $\overline{W(A)} = \overline{U(A)}$.

Proof. Let $\mathscr{D} = \bigcap_{f \in \mathscr{F}} \mathrm{Dom}(E(f))$ as above, one easily sees from the proof of 3.4 that

$$W_{\mathscr{D}}(A) \overset{\mathrm{def}}{=} \{ ((A_1 x, x), \ldots, (A_n x, x)) : x \in \mathscr{D}, \| x \| = 1 \}$$

$$= \bigcap_{\Delta \in S} = W(A).$$

Since for any $x \in \mathscr{D}$, $\|x\| = 1$, $T \to (Tx, x)$ is a vector state on $\mathscr{A}$, we have $W(A) = W_{\mathscr{D}}(A) \subset U(A)$.

Conversely, if there exists a state ϕ on $\mathscr{A}$ such that $\lambda_0 = \phi(A) \notin \overline{W(A)}$, where $\phi(A) = (\phi(A_1), \ldots, \phi(A_n))$, then we can find a real hyperplane π in $\mathbb{R}^{2n} = \mathbb{C}$ which strictly separates λ_0 and $\overline{W(A)}$ since, from Theorem 2.3, $\overline{W(A)}$ is a closed convex subset of $\mathbb{C}^n$.

Now let $\{\Delta_m\}$ be a sequence of compact sets in $\mathbb{C}^n$ which monotonically increase to $\mathbb{C}^n$ (see the first paragraph of this section). Obviously $A_m = E(\Delta_m)A = (E(\Delta_m)A_1, \ldots, E(\Delta_m)A_n)$ is a commuting n-tuple of bounded normal operators and

$$\text{Sp}(A_m) = (\Delta_m \cap \text{Sp}(A)) U \{(0, \ldots, 0)\}.$$

There is no loss of generality in assuming that $(0, \ldots, 0) \in \text{Sp}(A)$, since $\overline{W}(A)$, $\overline{U}(A)$ and $\text{Sp}(A)$ are all linear in A. Thus we have $\text{Sp}(A_m) \subset \text{Sp}(A)$, $m = 1, 2, \ldots$. It follows by Corollary 3.5 that

$$\overline{W(A_m)} = \text{conv}(\text{Sp}(A_m)) \subset \overline{\text{conv}(\text{Sp}(A))} = \overline{W(A)}.$$

Let $C^*(A_m)$ be the unital C^*-subalgebra of $\mathscr{A}$ generated by A_m. Since Theorem 5.1 is known to be true for the bounded case [3] and ϕ is still a state when acting on $C^*(A_m)$, $m = 1, 2, \ldots$, we have ϕ is still a state when acting on $C^*(A_m)$, $m = 1, 2, \ldots$, and $\phi(A_m) \in \overline{W(A_m)} \subset W(A)$. Besides, we can see that $\lim_{m \to \infty} E(\Delta_m)A_i = A_i$, $i = 1, \ldots, n$, under the topology in $\mathscr{A}$ defined above. Since any positive linear functional on a complete symmetric unital algebra, whose topology is given by a countable family of seminorms, is continuous [13], we have $\phi(A_m) \to \phi(A) = \lambda_0$. But this is impossible because that λ_0 and $\overline{W(A)}$ are separated strictly by the hyperplane π in $\mathbb{C}^n$. Thus we have proved $U(A) \subset \overline{W(A)}$ and combining this with $W(A) \subset U(A)$ we get $\overline{U(A)} = \overline{W(A)}$. The proof is complete.

Remark. Theorem 5.1 generalizes the result in [3] since $U(A)$ is clearly a compact set when $A = (A_1, \ldots, A_n)$ are bounded.

Finally, we shall characterize $W(A)$ by means of the weakly continuous linear functional on $\mathscr{U}$ which is induced by A. Inoue [12].

Lemma 5.3. *Let ϕ be a positive linear functional on the $EC^\#$-algebra $\mathscr{U}$ on $\mathscr{D}$. Then the following conditions are equivalent.*

(1) *ϕ is weakly continuous;*
(2) *$\phi(T) = \sum_{i=1}^m (T\xi_i, \xi_i)$ for some $\xi_i \in \mathscr{D}$, $i = 1, \ldots, m$, and $T \in \mathscr{U}$.*

This lemma comes from A. Inoue [12] and immediately leads to the following consequence.

Theorem 5.4. *Let $A = (A_1, \ldots, A_n)$ be a commuting n-tuple of normal operators. $\mathscr{U}$ is the $EC^\#$-algebra generated by A. Let*

$$\mathscr{K} = \{\phi : \phi \text{ is weakly continuous positive functional on } \mathscr{U}, \phi(I) = 1\}.$$

Then $W(A) = \{(\phi(A_1), \ldots, \phi(A_n)) : \phi \in \mathcal{K}\}$. *(Strictly, $\phi(A_i)$ should be written as $\phi(A_i|_{\mathcal{D}})$.)*

Proof. By Lemma 4.3, any $\phi \in \mathcal{K}$ can be written in the form $\phi(T) = \sum_{i=1}^{m}(T\xi_i, \xi_i)$ for some $\xi_i \in \mathcal{D}$, $i = 1, \ldots, m$, where $T \in \mathcal{U}$.

Since $\phi(I) = \sum_{i=1}^{m} \|\xi_i\|^2 = 1$, one easily sees that

$$(\phi(A_1), \ldots, \phi(A_n))$$

$$= \sum_{i=1}^{m} \|\xi_i\|^2 \left(\left(A_1 \frac{\xi_i}{\|\xi_i\|}, \frac{\xi_i}{\|\xi_i\|} \right), \ldots, \left(A_n \frac{\xi_i}{\|\xi_i\|}, \frac{\xi_i}{\|\xi_i\|} \right) \right) \in W(A).$$

So we have $\{(\phi(A_1), \ldots, \phi(A_n)) : \phi \in \mathcal{K}\} \subset W(A)$. The inverse inclusion is obvious. Thus the proof is complete.

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