H. Aritürk
Nagoya Math. J.
Vol. 77 (1980), 177–196

ON THE COMPOSITION SERIES OF PRINCIPAL SERIES REPRESENTATIONS OF A THREE-FOLD COVERING GROUP OF $SL(2, K)$[1]

HALUK ARITÜRK

Introduction

In this paper, we study the composition series of certain principal series representations of the three-fold metaplectic covering group of $SL(2, K)$, where K is a non-archimedean local field. These representations are parametrized by unramified characters $\mu(x)=|x|^s$ of $K^\times$, and characters ω of the group of third roots of unity. We study only the genuine representations corresponding to nontrivial ω parameter, as the case where $\omega = 1$ gives nothing but representations of $SL(2, K)$. We show that, outside the line $\mathrm{Re}\, s = 0$ (where the representations may decompose simply), the genuine principal series are irreducible except when $s = \pm 1/3$. We find the composition series at $s = \pm 1/3$, and obtain a unique quotient, r_ω, which is spherical.

The motivation for this study is a paper of Gelbart and Sally (cf. [4]) where it is proved that an irreducible component of the Weil representation appears as a quotient of the genuine principal series representation corresponding to $s = 1/2$ of the two-fold covering group of $SL(2, K)$; this is the only spherical quotient of the representations corresponding to $s = \pm 1/2$, and all other genuine principal series representations parametrized by nonunitary unramified characters are irreducible.

The author would like to extend his thanks to J. Igusa for suggesting this problem, and to P. Sally and J. Shalika for encouragement and several helpful conversations.

1. Metaplectic group

We fix once and for all a non-archimedean local field, K, of

Received March 2, 1979.

1) This work was partially supported by the Scientific and Technical Research Council of Turkey.

characteristic zero containing the cube roots of unity. We denote by $\mathcal{O}$ the ring of integers of K, τ a fixed generator of the prime ideal $\mathscr{P}$ of $\mathcal{O}$, $\mathcal{O}^\times$ its group of units, and q the order of $\mathcal{O}/\mathscr{P}$. We shall assume, for convenience, that q is odd.

The three-fold metaplectic group is defined by a two-cocycle on $G = SL(2, K)$ which involves the cubic power residue symbol of K. (This construction is given by Kubota for n-fold metaplectic groups in [7]). We shall, therefore, list some properties of the cubic power residue symbol, $(\ ,\ )_3$, which will be frequently used.

1.1. PROPOSITION.

i) $(\ ,\ )_3$ *is bilinear.*

ii) $(a, b)_3 = (b, a)_3^{-1}$

iii) $(\ ,\ )_3$ *is identically* 1 *on* $\mathcal{O}^\times \times \mathcal{O}^\times$

iv) *If* a *is a cube in* K, (a, b) *is identically* 1.

For proofs and more information cf. [7], [1].

Now, suppose $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $SL(2, K)$. We set $x(g)$ equal to c or d according as c is non-zero or not. The following theorem is proved in [7].

1.2. THEOREM. *The map* $\alpha\colon SL(2, K) \times SL(2, K) \to Z_3$ *defined by*:

$$\alpha(g_1, g_2) = (x(g_1), x(g_2))_3(-x(g_1)^{-1}x(g_2), x(g_1g_2))_3$$

is a cohomologically non-trivial two-cocycle on $SL(2, K)$.

We thus get a covering group, G', of G by Z_3 which is central as a group extension. This is the three-fold metaplectic group. The group law in G' is given by

$$(g_1, \tau_1)(g_2, \tau_2) = (g_1g_2, \alpha(g_1, g_2)\tau_1\tau_2)\ .$$

We denote by B the upper triangular subgroup of G; A is the diagonal subgroup, and N the subgroup $\left\{\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}\right\}$. We set $M = SL(2, \mathcal{O})$. If H is any subgroup of G, we shall denote its inverse image in G' by H'.

The cocycle α is trivial on $M \times M$ and $N \times N$. Therefore, M and N are isomorphic to subgroups of G' which we shall also denote by M and N. As a notational convenience, we shall write a for the element $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$ of A when the meaning is clear from the context. We can easily see that

$$\alpha(a, b) = (a, b^{-1})_3 .$$

It is also clear that α is trivial on $A_0 \times A_0$, where A_0 is the subgroup of the diagonal group consisting of elements with entries whose order is divisible by 3—the order of a nonzero element x in K is the unique integer $v(x)$ for which $x\tau^{-v(x)}$ is a unit. We therefore have $A'_0 = A_0 \times \boldsymbol{Z}_3$.

2. Principal series representations of G'

Any irreducible representation of A'_0 is clearly of the form $L_{\omega,\mu}$ with

$$L_{\omega,\mu}(a, \zeta) = \omega(\zeta)\mu(a)$$

where μ is a quasi-character of the multiplicative group $K^\times$ of nonzero elements in K, and ω is a character of $\boldsymbol{Z}_3$.

2.1. Proposition. *All finite dimensional irreducible representations of A' are obtained by inducing $L_{\omega,\mu}$ from A'_0.*

Proof. Let $L_0 = L_{\omega,\mu}$ be an arbitrary representation of A'_0, and $h' = (h, \eta)$ any element of A'. Since we have

$$h'(b, \zeta)h'^{-1} = (b, (h, b^{-1})_3^2\zeta)$$

L_0 and $L_0^{h'}$, its conjugation by h' are identical on A' if and only if $\omega((h, b^{-1})_3^2) = 1$ for all b in A_0. Hence the set

$$H = \{h' \in A' : L_0^{h'} = L\}$$

is either A' or A'_0 depending on whether ω is trivial or not. So, from the theory of representations for groups with normal subgroups of finite index (cf. [3], Lemma 5.2), we can see that all finite dimensional representations of A' are obtained by inducing from A'_0.

We put $\sigma_{\omega,\mu} = \operatorname{Ind}(A'_0, A', L_{\omega,\mu})$. $\sigma_{\omega,\mu}$ acts by right translations on the space of $\boldsymbol{C}$-valued functions f on A' satisfying

$$f(x'_0, y') = L_{\omega,\mu}(x'_0)f(y')$$

whenever x'_0 is in A'_0. We now compute the action of A' explicitly. Since any (x, ζ) in A' can be uniquely decomposed as

$$(x, \zeta) = (x_0, \zeta(x_0, \tau^{i(x)})_3)(\tau^{i(x)}, 1) \tag{2.1}$$

where x_0 is in A_0 and $0 \leq i(x) \leq 2$, $\{(\tau^i, 1)\ i = 0, 1, 2\}$ is a set of representatives for A'/A'_0. We have

$$(x, \zeta_x)(a, \zeta) = (a, \zeta(a, x^2)_3)(x, \zeta_x),$$

(2.2)
$$\sigma_{\omega,\mu}(a, \zeta)f(\tau^i, 1) = \begin{cases} \mu(a_0)\omega((a_0, \tau^{2i+i(a)})_3\zeta)f(\tau^{i+i(a)}, 1) \\ \mu(\tau^3 a_0)\omega((a_0, \tau^{2i+i(a)})_3\zeta)f(\tau^{i+i(a)-3}, 1) \end{cases}$$

according as $i + i(a) \leq 2$ or not.

We extend $\sigma_{\omega,\mu}$ to B' which is the semi-direct product of A' and N, and then induce to G', and thus obtain the principal series representations of G'. We denote such a representation by $\rho_{\omega,\mu}$. It acts by right translations on the space $\phi_{\omega,\mu}$ of locally constant functions ϕ on $G' \times A'$ satisfying

(i) $\phi(g', a'_0 a') = L_{\omega,\mu}(a'_0)\phi(g', a')$ if $a'_0 \in A'_0$
(ii) $\phi(b'g', a') = \delta(b')\phi(g', a'b')$ if $b' \in A'$

where $\delta(b')$ denotes the modulus of b if $b' = b(1, \zeta)$.

In the rest of this paper we shall restrict ourselves to the case of unramified characters of $K^{\times}$, so that $\mu(x) = |x|^s$ for a complex number s. Furthermore, if $\mu(x) = |x|^s$ and $\mu'(x) = |x|^{s'}$ where s and s' differ by an integer multiple of $2\pi i/3lnq$, then $L_{\omega,\mu}$ and $L_{\omega,\mu'}$ are equal on A'_0. We shall therefore restrict ourselves to the strip $-\pi/3lnq \leq \operatorname{Im} s \leq \pi/3lnq$. *Throughout this paper, we shall be referring to this strip when we say all complex numbers s.* Finally, we shall always assume ω to be nontrivial, and thereby consider only the genuine representations of G'.

An analogue of the Bruhat decomposition holds in G'; we have

$$G' = B' \cup B'(w, 1)N$$

where $w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. We note that, we write g for the element $(g, 1)$ of G' when the meaning is clear from the context.

It follows from the above decomposition that all ϕ in $\phi_{\omega,\mu}$ are determined by their values on N and w. Hence, putting $f(x, a')$ equal to $\phi(w^{-1}n(x), a')$ with $n(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ gives rise to a realization of $\rho_{\omega,\mu}$ on the space $F_{\omega,\mu}$ of locally constant functions on $K \times A'$ satisfying

(2.3)
(i) $f(x, a'_0 a') = L_{\omega,\mu}(a'_0)f(x, a')$ if $a'_0 \in A'_0$
(ii) $|x|\sigma_{\omega,\mu}(x, 1)f(x, a')$ is constant for large $|x|$.

We fix a character χ of K once and for all. We assume, for convenience, that the conductor of χ is $\mathcal{O}$. For a function f in $F_{\omega,\mu}$ we define

https://doi.org/10.1017/S0027763000018754 Published online by Cambridge University Press

$$\mathscr{F} f(x, a') = \sum_{n \in Z} \int_{v(y)=n} f(y, a')\chi(yx)dy$$

where dy is a fixed Haar measure on K normalized so that $\mathcal{O}$ has measure 1. This series converges uniformly on compact subsets of $K^\times$. (cf. [5], Lemma 9; essentially the same proof works here as we have $|f(x, a')| \sim |x|^{-1}|\mu(x)|^{-1}$ for large $|x|$.) $\mathscr{F} f$ will be called the Fourier transform of f, and sometimes be denoted by f^*. Moreover, for each fixed $a', f(x, a')$ is a square integrable function when $\operatorname{Re} s > -1/2$; in this case the Fourier transform of f in the L^2 sense coincides with $\mathscr{F}$.

From distribution theory, it can be seen that the kernel of the map $\mathscr{F}$ contains only functions which are constant on $K \times \{a'\}$ for each a' in A'. However, the only such function satisfying condition (ii) of (2.3) is zero. Hence, $\mathscr{F}$ maps $F_{\omega,\mu}$ injectively onto a space $\mathscr{H}_{\omega,\mu}$. We shall characterize this space only for certain μ and this will be done in § 5. We shall denote the realization of $\rho_{\omega,\mu}$ on $\mathscr{H}_{\omega,\mu}$ by $\rho^*_{\omega,\mu}$.

3. Intertwining operators

We shall fix some notation first: Let K_i denote the set of elements of K whose order is equal to i modulo 3, and ψ_i the characteristic function of K_i. $\mathscr{S}(K)$ (resp. $\mathscr{S}(K^\times)$) will denote the Schwartz-Bruhat space of K (resp. $K^\times$); i.e., the space of locally constant functions whose support is compact in K (resp. $K^\times$). We let $d^\times x$ be the Haar measure on K^* given by $\dfrac{dx}{|x|}$.

3.1. Lemma. *For any Φ in $\mathscr{S}(K)$, complex number s with $0 < \operatorname{Re} s < 1$, and $j = 1, 2$ we have*

$$\int_{K_i} \Phi(x)|x|^s \omega((x, \tau^j)_3)d^\times x$$
$$= c_j q^{s-1/2} \int_{K_{2i-1}} \Phi^*(x)|x|^{1-s}\omega((x, \tau^{-j})_3)d^\times x$$

where Φ^* is the Fourier transform of Φ, and c_j are complex numbers of modulus 1 with $c_1c_2 = 1$.

Proof. We fix a unit D in K so that $(D, \tau)_3$ is a primitive cube root of 1. Then $(D, x)_3$ is a primitive root unless $v(x) \equiv 0 \bmod 3$. We can therefore write the characteristic function of K_i as

$$\psi_i(x) = 1/3 \sum_{l=0}^{2} (D, x\tau^{-i})_3^l .$$

Furthermore, since the character $(D, x)_3$ is unitary and unramified, it is of the form $|x|^d$ for some complex number d with $\operatorname{Re} d = 0$. We can now write the left hand side of the equality in the proposition as

$$(1/3) \sum_{l=0}^{2} q^{ldi} \int_K \Phi(x) |x|^{s+ld} \omega((x, \tau^j)_3) d^\times x .$$

Applying Tate's functional equation to each term and recalling that we have

$$\Gamma(| \cdot |^s \omega((\cdot , \tau^j)_3)) = c_j q^{s-1/2}$$

where Γ is the p-adic gamma function and c_j are complex numbers of modulus 1 such that $c_1 c_2 = 1$ (cf. [9], Theorem 1), the sum becomes

$$(1/3) c_j q^{s-1/2} \sum_{l=0}^{2} q^{ldi} \cdot q^{ld} \int_K \Phi^*(x) |x|^{1-s-ld} \omega((x, \tau^{-j})_3) d^\times x.$$

Observing that we have

$$(1/3) \sum_{l=0}^{2} q^{ldi} \cdot q^{ld} |x|^{-ld} = (1/3) \sum_{l=0}^{2} (D, x^{-1}\tau^{-i-1})_3 = \psi_{2i-1}(x)$$

we prove the proposition.

For the case $j = 0$ we have the following.

3.2. Lemma. *For any Φ in $\mathscr{S}(K)$, complex number s with $0 < \operatorname{Re} s < 1$, we have*

$$\begin{aligned}\int_{K_i} \Phi(x) |x|^s d^\times x = \frac{1 - q^{-1}}{1 - q^{-3s}} \int_{K_{2i}} \Phi^*(x) |x|^{1-s} d^\times x \\ + \frac{q^s(q^{-3s} - q^{-1})}{1 - q^{-3s}} \int_{K_{2i-1}} \Phi^*(x) |x|^{1-s} d^\times x \\ + \frac{q^{-s}(1 - q^{-1})}{1 - q^{-3s}} \int_{K_{2i-2}} \Phi^*(x) |x|^{1-s} d^\times x .\end{aligned}$$

Proof. The left hand side is equal to

$$(1/3) \sum_{l=0}^{2} q^{ldi} \int_K \Phi(x) |x|^{s+ld} d^\times x .$$

By Theorem 1 of [9], $\Gamma(| \cdot |^s) = (1 - q^{s-1})/(1 - q^{-s})$. Applying the functional equation of Tate, we see that the above expression is equal to

$$(1/3)\int_K \Phi^*(x)|x|^{1-s}\left[\frac{1-q^{s-1}}{1-q^{-s}} + (D, x^2\tau^{-i})_3\left(\frac{1-q^{s+d-1}}{1-q^{-d-s}}\right)\right.$$
$$\left. + (D, x\tau^{-2i})_3\left(\frac{1-q^{s+2d-1}}{1-q^{-2d-s}}\right)\right]d^\times x\,.$$

We compute the expression in brackets. We factor out $1-q^{-3s}$, the product of the three denominators; this leaves an expression with a q^0 term coefficient of $3\psi_{2i}(x)$, a q^{-s} term coefficient of $3\psi_{2i-2}(x)$, a q^{-2s} term coefficient of $3\psi_{2i-1}(x)$, a q^{s-1} term coefficient of $-3\psi_{2i-1}(x)$, a q^{-1} term coefficient of $-3\psi_{2i-2}(x)$. Therefore, the integral is

$$\frac{1}{1-q^{-3s}}\int_K \Phi^*(x)|x|^{1-s}[(1-q^{-1})\psi_{2i}(x) + q^s(q^{-3s}-q^{-1})\psi_{2i-1}(x)$$
$$+ q^{-s}(1-q^{-1})\psi_{2i-2}(x)]d^\times x\,.$$

This completes the proof.

For an element ϕ of $\phi_{\omega,\mu}$ we put

$$I\phi(g', a') = \int_K \phi\left(w\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix} g', wa'\,w^{-1}\right)dx\,.$$

The integral converges for $\operatorname{Re} s > 0$ since

$$|\phi(wn(x)g', wa'w^{-1})| \approx |\mu(x)|^{-1}|x|^{-1}\,.$$

It is easy to see that I_ϕ is in $\phi_{\omega,\mu^{-1}}$ and that I commutes with right translations; I intertwines $\rho_{\omega,\mu}$ and $\rho_{\omega,\mu^{-1}}$. Furthermore, if $\phi\in\phi_{\omega,\mu}$ and $\phi'\in\phi_{\bar\omega,\mu^{-1}}$ then $\phi\cdot\phi'$ is invariant under left translations of the second variable by elements of A'_0. The function

$$g' \mapsto \int_{A'_0\backslash A'} \phi\cdot\phi'(g', a')da'$$

is in the space $L(G', B')$ of locally constant functions Φ satisfying

$$\Phi\left(\left(\begin{bmatrix}a & *\\ 0 & a^{-1}\end{bmatrix}, \zeta\right)g'\right) = |a|^2\,\Phi(g')\,.$$

If we denote the essentially unique positive linear form on $L(G', B')$ by

$$\Phi \mapsto \int_{B'\backslash G'} \Phi(g')dg'$$

then

$$\langle\phi, \phi'\rangle = \int_{B'\backslash G'}\int_{A'_0\backslash A'} \phi(g', a')\phi'(g', a')da'dg'$$

https://doi.org/10.1017/S0027763000018754 Published online by Cambridge University Press

gives a non-degenerate bilinear form on $\phi_{\omega,\mu} \times \phi_{\bar{\omega},\mu^{-1}}$. Thus it follows that $\rho_{\bar{\omega},\mu^{-1}}$ is the contragradient representation of $\rho_{\omega,\mu}$. (cf. [5], p. 1.18). By well-known techniques, the above integral can be written as

$$\langle \phi, \phi' \rangle = \int_K \int_{A'_0\backslash A'} \phi(w^{-1}n(x), a')\phi'(w^{-1}n(x), a')da'dx .$$

We shall now restrict ourselves to the case of real s with $0 < s < 1$. In this case the complex conjugate of $I\phi$ is in $\phi_{\bar{\omega},\mu^{-1}}$ if ϕ is in $\phi_{\omega,\mu}$. Thus, the following is an invariant bilinear form on $\phi_{\omega,\mu} \times \phi_{\omega,\mu}$.

$$\begin{aligned}\int_K \int_{A'_0\backslash A'} & \phi_1(w^{-1}n(x), a')\overline{I\phi_2(w^{-1}n(x), a')}da'dx \\ &= \int_K \int_{A'_0\backslash A'} f_1(x, a') \int_K \overline{\phi_2(wn(y)w^{-1}n(x), wa'w^{-1})}dyda'dx \\ &= \int_K \int_{A'_0\backslash A'} f_1(x, a') \int_K \bar{\phi}_2\left(\begin{bmatrix} y^{-1} & 1 \\ 0 & y \end{bmatrix} \overline{w^{-1}n(x + y^{-1}), wa'w^{-1}}\right)dyda'dx .\end{aligned}$$

We note that the arguments of ϕ_2 in the last two expressions are only equal up to a central element of G'; the difference is absorbed by the integration over $A'_0\backslash A'$. We write the integral in the following form:

$$\int_K \int_{A'_0\backslash A'} f_1(x, a') \int_K \overline{\sigma_{\omega,\mu}(y^{-1}, 1), f_2(x + y^{-1}, wa'w^{-1})}d^\times y\, da'dx .$$

By using the set of representatives $\{\tau^i : i = 0, 1, 2\}$ of $A'_0\backslash A'$, this invariant bilinear form becomes

$$\begin{aligned}\int f_1(x, 1) & \int \overline{\sigma_{\omega,\mu}(y, 1)f_2(x + y, 1)}d^\times y\, dx \\ &+ \int f_1(x, \tau) \int \overline{\sigma_{\omega,\mu}(y, 1)f_2(x + y, \tau^{-1})}d^\times ydx \\ &+ \int f_1(x, \tau^2) \int \overline{\sigma_{\omega,\mu}(y, 1)f_2(x + y, \tau^{-2})}d^\times ydx.\end{aligned}$$

By (2.2) this expression is equal to

$$\begin{aligned}\int f_1(x, 1) & \int \psi_0(y)\,|y|^s\, \overline{f_2(x + y, 1)}d^\times ydx \\ &+ \int f_1(x, 1) \int \psi_1(y)\,|\tau^{-1}y|^s\, \overline{\omega((y, \tau)_3)f_2(x + y, \tau)}\, d^\times ydx \\ &+ \int f_1(x, 1) \int \psi_2(y)\,|\tau^{-2}y|^s\, \overline{\omega((y, \tau^2)_3)f_2(x + y, \tau^2)}d^\times y\, dx \\ &+ \int f_1(x, \tau) \int \psi_0(y)\,|\tau^{-3}y|^s\, \overline{\omega((y, \tau)_3)f_2(x + y, \tau^2)}d^\times y\, dx\end{aligned}$$

https://doi.org/10.1017/S0027763000018754 Published online by Cambridge University Press

$$+\int f_1(x,\tau)\int \psi_1(y)|\tau^{-1}y|^s\,\overline{\omega((y,\tau^2)_3)f_2(x+y,1)}\,d^\times y dx$$
$$+\int f_1(x,\tau)\int \psi_2(y)|\tau^{-2}y|^s\,\overline{f_2(x+y,\tau)}\,d^\times y dx$$
$$+\int f_1(x,\tau^2)\int \psi_0(y)|\tau^{-3}y|^s\,\overline{\omega((y,\tau^2)_3)f_2(x+y,\tau)}\,d^\times y dx$$
$$+\int f_1(x,\tau^2)\int \psi_1(y)|\tau^{-4}y|^s\,\overline{f_2(x+y,\tau^2)}\,d^\times y dx$$
$$+\int f_1(x,\tau^2)\int \psi_2(y)|\tau^{-2}y|^s\,\overline{\omega((y,\tau)_3)f_2(x+y,1)}\,d^\times y dx\,.$$

We now assume that f has compact support as a function of x for each a'. Then each term of the above sum can be thought of (by Fubini's theorem) as having the form of the expressions in Lemmas 3.1 and 3.2 where Φ is the convolution of f_1^v and f_2. (f_1^v is the translate by -1 of f_1). By these lemmas, we therefore write the invariant bilinear form as follows, if we write $P(r, t, v)$ for the sum of $(1-q^{-1})r$, $q^{-s}(1-q^{-1})t$ and $q^s(q^{-3s}-q^{-1})v$:

$$\int f_1^*(y,1)\overline{f_2^*(y,1)}\,|y|^{1-s}(1/(1-q^{-3s}))P(\psi_0(y),\psi_1(y),\psi_2(y))\,d^\times y$$
$$+\int f_1^*(y,1)\overline{f_2^*(y,\tau)\,|y|^{1-s}c_1q^{2s-1/2}\omega((y,\tau^2)_3)\psi_1(y)}\,d^\times y$$
$$+\int f_1^*(y,1)\overline{f_2^*(y,\tau^2)\,|y|^{1-s}c_2q^{3s-1/2}\omega((y,\tau)_3)\psi_0(y)}\,d^\times y$$
$$+\int f_1^*(y,\tau)\overline{f_2^*(y,\tau^2)\,|y|^{1-s}c_1q^{4s-1/2}\omega((y,\tau^2)_3)\psi_2(y)}\,d^\times y$$
$$(3.1)\qquad +\int f_1^*(y,\tau)\overline{f_2^*(y,1)\,|y|^{1-s}c_2q^{2s-1/2}\omega((y,\tau)_3)\psi_1(y)}\,d^\times y$$
$$+\int f_1^*(y,\tau)\overline{f_2^*(y,\tau)\,|y|^{1-s}q^{2s}(1-q^{-3s})^{-1}P(\psi_1(y),\psi_2(y),\psi_0(y))}\,d^\times y$$
$$+\int f_1^*(y,\tau^2)\overline{f_2^*(y,\tau)\,|y|^{1-s}c_2q^{4s-1/2}\omega((y,\tau)_3)\psi_2(y)}\,d^\times y.$$
$$+\int f_1^*(y,\tau^2)\overline{f_2^*(y,\tau^2)\,|y|^{1-s}q^{4s}(1-q^{-3s})^{-1}P(\psi_2(y),\psi_0(y),\psi_1(y)}\,d^\times y$$
$$+\int f_1^*(y,\tau^2)\overline{f_2^*(y,1)\,|y|^{1-s}c_1q^{3s-1/2}\omega((y,\tau^2)_3)\psi_0(y)}\,d^\times y\,.$$

By taking a suitable sequence of functions in $F_{\omega,\mu}$ which are compactly supported in their first variable for each a' we can easily see that the above is valid for any f_1 in $F_{\omega,\mu}$.

We can think of the expression (3.1) in the form

https://doi.org/10.1017/S0027763000018754 Published online by Cambridge University Press

$$\int_K \int_{A'_0\backslash A'} f_1^*(y, a')\overline{Jf_2^*(y, a')}\, d^\times y da' \tag{3.2}$$

for some linear map J defined on $\mathscr{H}_{\omega,\mu}$. For any operator T let us denote by T_c the operator $f \mapsto \overline{T\bar{f}}$. Then (3.2) gives an invariant non-degenerate bilinear form on $\mathscr{H}_{\omega,\mu} \times$ Image of J_c. (J is not 0). Thus the image of J_c can be identified with a subspace of the contragradient of $\mathscr{H}_{\omega,\mu}$ i.e.,. $\mathscr{H}_{\bar{\omega},\mu^{-1}}$. If we denote by I^* the intertwining operator obtained by carrying I from the $\phi_{\omega,\mu}$ model to the $\mathscr{H}_{\omega,\mu}$ model, then it is clear that

$$\langle f^*, J_c g^* \rangle = \langle f^*, I_c^* g^* \rangle .$$

Thus $J = I^*$ for $0 < s < 1$.

We now write J in the matrix form by considering f^* to be a vector valued function on $K^\times$; we put $f^*(x)$ equal to

$$(f^*(x, 1), f^*(x, \tau), f^*(x, \tau^2))$$

in C^3—this vector determines $f^*(x, a')$ for all a'. We then have

$$J(x) = |x|^{1-s} \begin{bmatrix} (1 - q^{-3s})^{-1} P(\psi_0, \psi_1, \psi_2) & c_1 q^{2s-1/2} \omega^2 \psi_1 & c_2 q^{3s-1/2} \omega \psi_0 \\ c_2 q^{2s-1/2} \omega \psi_1 & (1 - q^{-3s})^{-1} q^{2s} P(\psi_1, \psi_2, \psi_0) & c_1 q^{4s-1/2} \omega^2 \psi_2 \\ c_1 q^{3s-1/2} \omega^2 \psi_0 & c_2 q^{4s-1/2} \omega \psi_2 & q^{4s}(1 - q^{-3s})^{-1} P(\psi_2, \psi_0, \psi_1) \end{bmatrix}$$

where we write ψ_i (resp. ω^j) instead of $\psi_i(x)$ (resp. $\omega((x, \tau^j)_3)$). We shall sometimes write $J_{\omega,s}$, to emphasize dependence on ω and s.

3.1. PROPOSITION. *The operator J is defined and is equal to I^* on the whole right half-plane* $\{s\colon \mathrm{Re}\,(s) > 0\}$.

Proof. For $i = 0, 1, 2$, we let F_i be a function from $\mathscr{S}(K^\times)$, and put $f^*(x, \tau^i) = F_i(x)$, and $f(x, \tau^i) = F_i^*(x)$. For each μ we can extend f to a function f_μ so that f_μ is in $F_{\omega,\mu}$. Then

$$\mathscr{F} f_\mu(x, \tau^i) = f^*(x, \tau^i)$$

for $i = 0, 1, 2$. Since $J = I^*$ on the interval $(0, 1)$ we have

$$J_{\omega,s} f^*(x, \tau^i) = I_{\omega,s}^* f^*(x, \tau^i)$$

for $i = 0, 1, 2$ when $0 < s < 1$. (Note that the values of f^* in question are independent of s.) Thus, from the principal of analytic continuation and the fact that every function in $F_{\omega,\mu}$ is the pointwise limit of a sequence of functions of the form f_μ, the proposition follows.

4. Composition series of $\rho^*_{\omega,\mu}$ for $\operatorname{Re} s > 0$

We start with an analogue of a theorem for p-adic reductive groups. A simple proof of this theorem for the semi-simple rank 1 case is in [2], pp. 3–4; this proof works verbatim in the case of G'. We therefore omit the proof.

4.1. THEOREM. *The length of* $\rho^*_{\omega,\mu}$ *is at most* 2.

Consequently, to determine the composition series of $\rho^*_{\omega,\mu}$, we only need the following theorem.

4.2. THEOREM. *The image of* $J_{\omega,s}$ *is irreducible for all* s *with* $\operatorname{Re} s > 0$.

This is a theorem of Langlands whose proof for the case of real reductive groups is contained in [8]. We include here a slight modification of Langlands' proof for the sake of completeness. We first need the following.

4.3. LEMMA. *Let* x *be in* $K^\times$, ϕ *in* $\phi_{\omega,\mu}$ *and* ϕ' *in* $\phi_{\bar{\omega},\mu^{-1}}$ *with* s *a real number. If we put*

$$F(x) = \langle \rho_{\omega,\mu}(x^3, 1)\phi, \phi' \rangle$$

then as $|x|$ *approaches* ∞, *we have*

$$F(x) \sim |x|^{3(s-1)} \int_{A'_0\backslash A'} I\phi(w, wa'w^{-1})\phi'(e, a')da'$$

where e *is the identity element of* G'.

Proof. We write $F(x)$ in the form

$$F(x) = \int_{A'_0\backslash A'} \int_{N_1} \rho_{\omega,\mu}(x^3, 1)\phi'(n_1, a')\phi'(n_1, a')dn_1 da'$$

where $N_1 = w^{-1}Nw$. By the "Iwasawa decomposition", $G' = B'M$, we can write n_1 as $n(t, \zeta)k$. We also put

$$(x^{-3}, 1)n_1(x^3, 1) = n_x(t_x, \zeta_x)k_x$$

so that

$$kx^3 = (t, \zeta)^{-1}n^{-1}x^3 n_x(t_x, \zeta_x)k_x \, .$$

Substituting in $F(x)$ first the expression for n_1, and then the one for kx^3, we get

https://doi.org/10.1017/S0027763000018754 Published online by Cambridge University Press

$$\rho_{\omega,\mu}(x^3, 1)\phi(n_1, a') = |x|^3 |t_x| \sigma_{\omega,\mu}(x^3(t_x, \zeta_x))\phi(k_x, a')$$

and

$$\phi'(n_1, a') = |t| \sigma_{\bar{\omega},\mu^{-1}}(t, \zeta)\phi'(k, a') .$$

Now we change variables by putting $n' = x^{-3}n_1x^3$. Observing that $k = (t, \zeta)^{-1}n^{-1}x^3n'x^{-3}$, and that $x^3n'x^{-3}$ approaches e as $|x|$ approaches ∞, we find that

$$F(x) \sim |x|^{3(s-1)} \int_{A'_0 \backslash A'} \int_{N_1} \phi(n_1, a')dn_1\phi'(e, a')da' .$$

We leave it to the reader to prove that one can interchange the integral and the limit as we just did. (cf. [8]). This completes the proof of the lemma.

Proof of the Theorem. Suppose V_1 is the kernel of I and V_2 is a proper invariant subspace of $\phi_{\omega,\mu}$ containing V_1. It clearly suffices to prove that any such V_2 is contained in V_1.

Pick a non-zero element ϕ'_0 in $\phi_{\bar{\omega},\mu^{-1}}$ such that $\langle \phi, \phi'_0 \rangle = 0$ for all ϕ in V_2. Fix an element ϕ_2 of V_2. We have

$$\langle \rho_{\omega,\mu}(g')\phi_2, \phi'_0 \rangle = 0$$

for all g' in G'. Putting $g' = x^3$ for x in $K^\times$, and using Lemma 4.3, we get

$$\int_{A'_0 \backslash A'} I\phi_2(w, wa'w^{-1})\phi'_0(e, a')da' = 0 .$$

As this equality holds for $\rho_{\omega,\mu}(g')\phi_2$ instead of ϕ_2 for any g', we must have $I\phi_2 = 0$, which proves the theorem.

As a consequence of this, we have the following theorem.

4.4. THEOREM. *The representations $\rho^*_{\omega,\mu}$ are irreducible for* $\mathrm{Re}\, s \neq 0$ *except when $s = \pm 1/3$. If r_ω denotes the representation of G' obtained by restricting $\rho^*_{\omega,-1/3}$ to the image of $J_{\omega,1/3}$, then*

$$0 \subsetneqq r_\omega \subsetneqq \rho^*_{\omega,-1/3}$$

is the composition series of $\rho^*_{\omega,-1/3}$.

Proof. It can be seen from (3.1) that for $\mathrm{Re}\, s > 0$ we have

https://doi.org/10.1017/S0027763000018754 Published online by Cambridge University Press

$$\det J_{\omega,s} = \frac{(1-q^{3s-1})^2(q^{3s}-q^{-1})}{(1-q^{-3s})^3}|x|^{3(1-s)} .$$

The kernel of $J_{\omega,s}$ is therefore trivial for $\operatorname{Re} s > 0$ except at $s = 1/3$. The theorem now follows from Theorems 4.1, 4.2 and the equivalence of $\rho^*_{\omega,\mu}$ and $\rho^*_{\omega,\mu^{-1}}$.

Let us denote by π_ω the representation obtained by restricting $\rho^*_{\omega,1/3}$ to the kernel of $J_{\omega,1/3}$. We shall devote the rest of this section to proving that r_ω and π_ω are inequivalent representations, neither of which is equivalent to an irreducible $\rho^*_{\omega,\mu}$.

4.5. PROPOSITION. *The representations $\rho^*_{\omega,\mu}$ and r_ω are spherical; i.e., they contain a nontrivial subspace fixed by M. π_ω is not spherical.*

Proof. We shall consider the $\rho_{\omega,\mu}$ realization. By the Iwasawa decomposition, there exists an element ϕ_0 in $\phi_{\omega,\mu}$ fixed by M if and only if there is a function Φ_0 on A' with the properties

(i) $\Phi_0(a'_0a') = L_{\omega,\mu}(a'_0)\phi_0(a')$ for $a'_0 \in A'_0$
(ii) $\Phi_0(a'b') = \Phi_0(a')$ for $b' \in A' \cap M$.

If $a' = (a, \zeta)$, $b' = (u, 1)$ with a unit element u, then $a'b' = (u, (u, a^2)_3)(a, \zeta)$. Therefore, the second condition is met if and only if $\omega((u, a^2)_3) = 1$ for all units u, whenever $\Phi_0(a')$ is nonzero. Thus, it is necessary that we have $\Phi_0(\tau) = \Phi_0(\tau^2) = 0$. Any such Φ_0 that also satisfies (i) will give a function ϕ_0 in $\phi_{\omega,\mu}$ which is fixed by M by putting $\phi_0(a'k, b')$ equal to $\Phi_0(a'b')$.

As the subspace fixed by M is thus shown to be one-dimensional, to complete the proof of the proposition it suffices to prove that the function ϕ_0 in $\phi_{\omega,1/3}$ is not in the kernel of I. But

$$\begin{aligned} I\phi_0(1,1) &= \int_K \phi_0(wn(x),1)dx \\ &= \phi_0(1,1)\int_{|x|\le 1} dx + \int_{|x|>1} \phi_0(wn(x),1)dx, \end{aligned}$$

and for $|x| > 1$ we have

$$wn(x) = \begin{bmatrix} x^{-1} & 0 \\ 0 & x \end{bmatrix} n(y)k$$

for some element y and element k of M. Hence the second integral is

https://doi.org/10.1017/S0027763000018754 Published online by Cambridge University Press

$$\int_{|x|>1} \phi_0(1, x^{-1}) d^\times x = \int_{|x|>1} \Phi_0(x^{-1}) d^\times x \,.$$

However, since Φ_0 vanishes outside A'_0, this integral becomes

$$\phi_0(1, 1) \int_{|x|>1} |x^{-1}|^{1/3} \psi_0(x) d^\times x = \phi_0(1, 1)(1 - q^{-1}) \sum_{n=1}^{\infty} q^{-n} = q^{-1}\phi_0(1, 1) \,.$$

So $I\phi_0$ takes on the value $\phi_0(1, 1)(1 + q^{-1})$, and therefore is not zero.

This proposition already proves that no irreducible $\rho^*_{\omega,\mu}$ or r_ω is equivalent to π_ω. We now want to show that r_ω is not equivalent to any irreducible $\rho^*_{\omega,\mu}$.

We consider the Iwahori subgroup

$$B_0 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M \colon c \equiv 0 \bmod \mathscr{P} \right\}$$

and compute the subspace $V_{\omega,\mu}(B_0)$ of $\phi_{\omega,\mu}$ fixed under B_0. G' can clearly be written as the disjoint union of $B'B_0$ and $B'wB_0$. The elements of $V_{\omega,\mu}(B_0)$ vanishing on $B'wB_0$ are of the form

$$\phi(b'b_0, a') = \delta(b')\phi(1, a'b') \tag{4.1}$$

where $\phi(1, a')$ is a function on A' satisfying

$$\phi(1, a'_0 a') = L_{\omega,\mu}(a'_0)\phi(1, a'_0 a') \qquad \text{if } a'_0 \in A'_0 \tag{4.2}$$

(4.1) and (4.2) give a well defined function if and only if

$$\delta(b')\phi(1, a'b') = \phi(1, a')$$

for all b' in $B' \cap B_0$. As in the proof of the last proposition, we see that $\phi(1, \tau) = \phi(1, \tau^2) = 0$. Therefore, the subspace of functions in $V_{\omega,\mu}(B_0)$ vanishing on $B'wB_0$ is one dimensional.

We proceed similarly to study the elements of $V_{\omega,\mu}(B_0)$ vanishing on $B'B_0$. They must be given by (4.2) and

$$\phi(b'wb_0, a') = \delta(b')\phi(1, a'b') \,. \tag{4.3}$$

It is then necessary that

$$\delta(b')\phi(1, a'b') = \phi(1, a')$$

whenever b' is in $wB_0w^{-1} \cap B'$; i.e., for $b' = (u, \zeta)$ with a unit u. Hence, by (4.2) $\phi(1, \tau) = \phi(1, \tau^2) = 0$.

We have proved that $V_{\omega,\mu}(B_0)$ is a two-dimensional subspace with a basis consisting of the two functions ϕ_1, ϕ_2 given as follows: ϕ_1 vanishes on $B'wB_0$ and

$$\phi_1(b'b_0, a') = \begin{cases} \delta(b')L_{\omega,\mu}(a'b') \\ 0 \end{cases}$$

according as $a'b'$ is in A'_0 or not; ϕ_2 vanishes on $B'B_0$ and

$$\phi_2(b'wb_0, a') = \begin{cases} \delta(b')L_{\omega,\mu}(a'b') \\ 0 \end{cases}$$

according as $a'b'$ is in A'_0 or not.

We shall now consider the B_0 fixed elements of π_ω and r_ω. We shall, therefore, first compute $I_{\omega,1/3}\phi_1$ and $I_{\omega,1/3}\phi_2$. It suffices to compute their values at $(1, 1)$ and $(w, 1)$ by B_0 invariance.

$$I\phi_1(1, 1) = \int_{|x|\le 1} \phi_1(wn(x), 1)dx + \int_{|x|>1} \phi_1(wn(x), 1)dx .$$

The first integrand is 0. In the second integral we write

$$wn(x) = \begin{bmatrix} x^{-1} & -1 \\ 0 & x \end{bmatrix}\begin{bmatrix} -1 & 0 \\ -x^{-1} & 1 \end{bmatrix}.$$

Thus,

$$I\phi_1(1, 1) = \int_{|x|>1} |x|^{-1}\phi_1(1, x^{-1})dx$$

where the integrand is $|x|^{-4/3}\psi_0(x)$; we get q^{-1}.

Also,

$$I\phi_1(w, 1) = \int_{|x|<1} \phi_1(wn(x)w, 1)dx + \int_{|x|\ge 1} \phi_1(wn(x)w, 1)dx .$$

In the first integral we have $\phi_1\left(\begin{bmatrix} -1 & 0 \\ x & -1 \end{bmatrix}, 1\right)$ which is 1. The second integrand is 0 since

$$wn(x)w = \begin{bmatrix} -x^{-1} & 1 \\ 0 & -x \end{bmatrix} wn(-x^{-1}) .$$

Therefore $I\phi_1(w, 1) = q^{-1}$.

In exactly the same manner we compute $I\phi_2$ and get $I\phi_2(1, 1) = I\phi_2(w, 1) = 1$. We thus see that π_ω contains a one-dimensional subspace fixed under B_0; it is generated by $\phi_2 - q\phi_1$. Therefore the B_0-fixed subspace of r_ω is also one-dimensional. This, along with Proposition 4.5 proves the following theorem.

4.6. THEOREM. *No two representations in the collection consisting of irreducible* $\rho^*_{\omega,\mu}$, r_ω *and* π_ω *are equivalent.*

5. The representation r_ω

In this section we shall study the irreducible representation r_ω more closely, and obtain a more explicit description.

We start by computing $\mathscr{H}_{\omega,\mu}$ for $\mu(x) = |x|^{1/3}$. We recall that this space consists of Fourier transforms of functions in $F_{\omega,\mu}$. $F_{\omega,\mu}$ is the direct sum of $\mathscr{S}_{\omega,\mu}$, which is the subspace of functions vanishing for large $|x|$, and the subspace generated by the function $g(x, a')$ given by

$$g(x, a') = \begin{cases} |x|^{-1}\sigma_{\omega,\mu}(x^{-1}, 1)G(a') \\ 0 \end{cases}$$

according as $|x| \geq 1$ or not, where G is a function on A' satisfying

$$G(a'_0 a') = L_{\omega,\mu}(a'_0)G(a') . \tag{5.1}$$

Thus $\mathscr{H}_{\omega,\mu}$ is the direct sum of $\mathscr{S}_{\omega,\mu}$ and the space generated by g^*. We shall now compute g^*; it suffices to compute its values when a' is 1, τ and τ^2. We have

$$g^*(y, 1) = \sum_{n=0}^{\infty} \int_{v(x)=n} G(x)\chi(x^{-1}y)d^\times x .$$

We break the sum into three parts, $\Sigma^0, \Sigma^1, \Sigma^2$ where Σ^i indicates that summation is to be carried out over those nonnegative integers which are equal to i modulo 3. We observe that by (5.1), $G(x)$ is nothing but $\mu(x)G(1)$ when x is in K_0. When x is in K_1 we write $(x, 1)$ in the form $(x\tau^{-1}, (x, \tau)_3)(\tau, 1)$ so that $G(x) = \mu(x\tau^{-1})\omega((x, \tau)_3)G(\tau)$; when x is in K_2, we find similarly that

$$G(x) = \mu(x\tau^{-2})\omega((x, \tau^2)_3)G(\tau^2) .$$

We thus have

$$\begin{aligned} g^*(y, 1) &= \Sigma^0 G(1) \int_{v(x)=n} \mu(x)\chi(x^{-1}y)d^\times x \\ &= \Sigma^1 G(\tau) \int_{v(x)=n} \mu(x\tau^{-1})\omega((x, \tau)_3)\chi(x^{-1}y)d^\times x \end{aligned}$$

$$= \Sigma^2 G(\tau^2) \int_{v(x)=n} \mu(x\tau^{-2})\omega((x, \tau^2)_3)\chi(x^{-1}y)d^\times x \,.$$

We have for $i = 1, 2$

$$\int_{v(x)=n} \mu(x)\omega((x, \tau^i)_3)\chi(x^{-1}y)d^\times x = \begin{cases} \mu(y)\omega((y, \tau^i)_3)q^{-s-1/2}c_{-i} \\ 0 \end{cases} \tag{5.2}$$

according as $v(y) = n - 1$ or not, where the c_i are the constants that arise as in Lemma 3.1 from the gamma function. (We put $c_i = c_{i+3m}$ for all integers m.)

We now compute Σ^0. We have

$$\int_{v(x)=n} \mu(x)\chi(x^{-1}y)d^\times x = q^{-ns} \int_{\mathcal{O}^\times} \chi(\tau^{-n}yu)du = q(h(\tau^{-n}y) - q^{-1}h(\tau^{-n+1}y))$$

in which $h(y)$ is 1 or 0 according as $v(y) \geq 0$ or not. Therefore,

$$\Sigma^0 \int_{v(x)=n} \mu(x)\chi(x^{-1}y)d^\times x = F_s^0(y) - q^{-1}F_s^0(\tau y)$$

where $F_s^0(y) = \Sigma^0 q^{-ns}h(\tau^{-n}y)$. Changing variables by putting $n = 3m$ in this summation, we easily find that

$$F_s^0(y) = \frac{1 - q^{-3s[v(y)/3]-3s}}{1 - q^{-3s}}$$

where [ ] is the Gauss symbol. We thus get

$$\Sigma^0 = \begin{cases} \dfrac{1}{1 - q^{-3s}}(1 - q^{-1} - q^{-3s[v(y)/3]-3s}(1 - q^{-1-3s})) \\ \dfrac{1}{1 - q^{-3s}}(1 - q^{-1} - q^{-3s[v(y)/3]-3s}(1 - q^{-1})) \end{cases}$$

according as $v(y) \equiv 2$ or $v(y) \not\equiv 2 \bmod 3$. Taking $s = 1/3$, putting the above together with Σ^1, Σ^2 and using (5.2) we find that

$$g^*(y, 1) = G(1) + |y|^{1/3} \begin{cases} G(\tau)c_2 q^{-1/2}\omega((y, \tau)_3) - G(1)q^{-1} \\ G(\tau^2)c_1 q^{-1/6}\omega((y, \tau^2)_3) - G(1)q^{-2/3} \\ -G(1)q^{-1/3}(1 + q^{-1}) \end{cases}$$

according as $v(y) \equiv 0$, $v(y) \equiv 1$ or $v(y) \equiv 2 \bmod 3$, if $|y|$ is sufficiently small—$g^*(y, 1)$ is 0 for large $|y|$. The computations of $g^*(y, \tau)$ and $g^*(y, \tau^2)$ are quite similar; we omit them and collect the results in the following proposition.

5.1. PROPOSITION. *$\mathscr{H}_{\omega,1/3}$ consists of functions f on $K^\times \times A'$ with*

$$f(x, a_0'a') = L_{\omega,1/3}(a_0')f(x, a')$$

which for any fixed a' are locally constant functions on $K^\times$ vanishing outside some compact subset of K and which behave in a neighborhood of 0 as $\eta(x, a') + \nu(x, a')$ for some functions η and ν where $\eta(x, a')$ is constant for a fixed a', and

$$\nu(x, 1) = |x|^{1/3}\begin{cases} -Aq^{-1} + Bc_2q^{-1/2}\omega((x, \tau)_3) \\ -Aq^{-2/3} + Cc_1q^{-1/6}\omega((x, \tau^2)_3) \\ -Aq^{-1/3}(1 + q^{-1}) \end{cases}$$

$$\nu(x, \tau) = |x|^{1/3}\begin{cases} -C(1 + q^{-1}) \\ Ac_2q^{-7/6}\omega((x, \tau)_3) - Cq^{-2/3} \\ Bc_1q^{-5/6}\omega((x, \tau^2)_3) - Cq^{-1/3} \end{cases}$$

$$\nu(x, \tau^2) = |x|^{1/3}\begin{cases} Ac_1q^{-3/2}\omega((x, \tau^2)_3) - Bq^{-1} \\ -Bq^{-2/3}(1 + q^{-1}) \\ Ccq^{-5/6}\omega((x, \tau)_3) - Bq^{-4/3} \end{cases}$$

according as $v(x) \equiv 0$, $v(x) \equiv 1$ or $v(x) \equiv 2 \bmod 3$, for some constants A, B, and C.

We now consider $J_{\omega,1/3}$ as given by (3.2). The following lemma is easily proved.

5.2. LEMMA. *The kernel of $J_{\omega,1/3}$ consists of functions f in $\mathscr{H}_{\omega,1/3}$ which satisfy the following:*

$$\begin{aligned} f(x, 1) &= -c_2q^{1/2}\omega((x, \tau)_3)f(x, \tau^2) && \text{if } v(x) \equiv 0 \bmod 3 \\ f(x, 1) &= -c_1q^{1/2}\omega((x, \tau^2)_3)f(x, \tau) && \text{if } v(x) \equiv 1 \bmod 3 \\ f(x, \tau) &= -c_1q^{1/2}\omega((x, \tau^2)_3)f(x, \tau^2) && \text{if } v(x) \equiv 2 \bmod 3\,. \end{aligned}$$

Consequently, the functions which behave as $\nu(x, a')$ around 0 are in the kernel. Thus to characterize the image it suffices to consider the subspace $\mathscr{S}_{\omega,1/3}$ of $\mathscr{H}_{\omega,1/3}$. We obtain the following easily.

5.3. LEMMA. *The image of $J_{\omega,1/3}$ consists of locally constant functions on $K^\times \times A'$ which satisfy*

(i) $f(x, a_0'a') = L_{\omega,-1/3}(a_0')f(x, a')$,

(ii) *one of the following according as $v(x) \equiv 0$, $v(x) \equiv 1$ or $v(x) \equiv 2 \bmod 3$.*

https://doi.org/10.1017/S0027763000018754 Published online by Cambridge University Press

$$f(x,1)=c_2q^{-1/2}\omega((x,\tau)_3)f(x,\tau^2),\qquad f(x,\tau)=0$$
$$f(x,\tau)=c_2q^{1/2}\omega((x,\tau)_3)f(x,1),\qquad f(x,\tau^2)=0$$
$$f(x,\tau^2)=c_2q^{1/2}\omega((x,\tau)_3)f(x,\tau),\qquad f(x,1)=0$$

and which behave as $\psi(x,a')$ *around* 0, *where*

$$\psi(x,1)=|x|^{-1/3}\begin{cases}A+Bc_2\omega((x,\tau)_3)\\ Aq^{-1/3}+Cc_1q^{-1/2}\omega((x,\tau^2)_3)\\ 0\end{cases}$$
$$\psi(x,\tau)=|x|^{-1/3}\begin{cases}0\\ C+Ac_2q^{1/6}\omega((x,\tau)_3)\\ Cq^{1/3}+Bc_1\omega((x,\tau^2)_3)\end{cases}$$
$$\psi(x,\tau^2)=|x|^{-1/3}\begin{cases}Bq^{1/2}+Ac_1q^{1/2}\omega((x,\tau^2)_3)\\ 0\\ Bq^{5/6}+Cc_2q^{1/6}\omega((x,\tau)_3)\end{cases}$$

according as $v(x)\equiv 0$, $v(x)\equiv 1$ *or* $v(x)\equiv 2 \bmod 3$, *for some constants* A, B, C.

Given any function f on $K^\times$, we define a function ιf on $K^\times \times A'$ by putting

$$\iota f(x,1)=\begin{cases}f(x)\\ c_1q^{-1/2}\omega((x,\tau^2)_3)f(x)\\ 0\end{cases}$$
$$\iota f(x,\tau)=\begin{cases}0\\ f(x)\\ c_1q^{1/2}\omega((x,\tau^2)_3)f(x)\end{cases}$$
$$\iota f(x,\tau^2)=\begin{cases}c_1q^{1/2}\omega((x,\tau^2)_3)f(x)\\ 0\\ f(x)\end{cases}$$

according as $v(x)\equiv 0$, $v(x)\equiv 1$ or $v(x)\equiv 2 \bmod 3$, and requiring that

$$\iota f(x,a_0'a')=L_{\omega,-1/3}(a_0')\iota f(x,a').$$

5.4. THEOREM. *The representation* r_ω *has a realization on the space of locally constant functions on* $K^\times$, *which have compact support in* K, *and which behave around* 0 *as*

https://doi.org/10.1017/S0027763000018754 Published online by Cambridge University Press

$$\psi(x) = |x|^{-1/3}\begin{cases} A + Bc_2\omega((x,\tau)_3) \\ Ac_2q^{1/6}\omega((x,\tau)_3) + C \\ Bq^{5/6} + Cc_2q^{1/6}\omega((x,\tau)_3) \end{cases}$$

according as $v(x) \equiv 0$, $v(x) \equiv 1$ *or* $v(x) \equiv 2 \bmod 3$. *The action of* G' *is given by*

$$r_\omega(g')f = (\iota^{-1}\rho_{\omega,-1/3}(g')\iota)f\,.$$

Moreover, r_ω *is a pre-unitary representation with the inner product*

$$(f_1, f_2) = -\int_K \int_{A'_0\backslash A'} \iota f_1(y, a')\overline{J_{\omega,-1/3}\iota f_2(y, a')}da'\, d^\times y\,.$$

Proof. It only remains to prove that (,) is positive definite. $J_{\omega,-1/3}$ does not vanish on the image of $J_{\omega,1/3}$—in fact $J_{\omega,-1/3}\circ J_{\omega,1/3}$ is a scalar. Furthermore, for each y, $-J_{\omega,-1/3}(y)$ is a Hermitian matrix with positive diagonal elements whose principal minors have nonnegative determinants. Thus at each y, $-J_{\omega,-1/3}(y)$ can be written as B^*B for some matrix B (which does not vanish on the image of $J_{\omega,1/3}$). This completes the proof.

References

[1] E. Artin and J. Tate, Class Field Theory, W.A. Benjamin, New York, 1968.

[2] W. Casselman, Some general results in the theory of admissable representations of P-adic reductive groups, to appear.

[3] S. Gelbart, Weil's representation and the spectrum of the metaplectic groups, Lecture Notes in Mathematics, No. 530, Springer-Verlag, 1976.

[4] S. Gelbart and P. J. Sally, Intertwining operators and automorphic forms on the metaplectic group, Proc. Nat. Acad. Sci., USA, **72** (1975), 1406–1410.

[5] R. Godement, Notes on Jacquet-Langlands Theory, Institute for Advanced Study, Princeton, 1970.

[6] H. Jacquet and R. P. Langlands, Automorphic forms on $GL(2)$, Lecture Notes in Mathematics, No. 114, Springer-Verlag, 1970.

[7] T. Kubota, Automorphic functions and the reciprocity law in a number field, Kyoto University, 1969.

[8] R. P. Langlands, On the classification of irreducible representations of real reductive groups, Mimeographed notes, Institute for Advanced Study, 1973.

[9] P. J. Sally and M. H. Taibleson, Special functions on locally compact fields, Acta Mathematica, **116** (1966), 279–309.

Bogazici University
P.K. 2, Bebek
Istanbul, Turkey