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ALMOST ALL CONVEX POLYHEDRA ARE ASYMMETRIC

EDWARD A. BENDER AND NICHOLAS C. WORMALD

1. Introduction. Although many types of rooted planar maps have been enumerated (see **[8]** for example), not much has been done on enumeration of entirely unrooted planar maps. Yet in virtually all cases of interest, it has appeared that comparatively very few of the maps are symmetric (have non-trivial automorphisms). This suggests that an asymptotic formula for the numbers of unrooted maps of a particular type on n edges can be obtained by dividing the numbers of rooted maps of that type on n edges by $4n$, where $4n$ is the number of potentially distinct rootings of an asymmetric n-edged map. The assertion that almost all maps of a given type are asymmetric has previously been proved in only two non-trivial cases: for 3-connected planar triangulations by Tutte **[9]** and for all n-edged 3-connected planar maps in **[5]**. We prove here that it is also true for 3-connected planar maps with a given number of vertices and faces, uniformly as either parameter approaches infinity. The results of **[9]** and **[5]** follow from this result, as does an asymptotic formula for the number of 3-connected planar maps with v vertices and f faces, or with v vertices. All of these results carry over to asymptotic enumeration of the combinatorial types of convex polyhedra, via a theorem of Steinitz to the effect that a convex polyhedron is combinatorially equivalent to a 3-connected planar map. Federico **[2]** gives a history of this problem. In particular, Steiner asked in 1832 for the number of n-faced convex polyhedra, which we answer asymptotically in Corollary 4.5.

In what follows we refer to a planar map simply as a map. In Section 2 some of the constructions to be used are introduced. The purpose of these constructions is to take from a symmetric map a chunk which determines the whole map. The main theorem of **[1]**, giving an asymptotic formula for the number of rooted 3-connected maps by faces and vertices, is a basic tool in our method. It is stated in Section 3, where useful bounds are also obtained on the number of rooted 3-connected maps by vertices, faces and valence of the root face. The main result and some of its consequences are given in Section 4.

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2. Automorphisms and constructions. Mani [3] showed that for each 3-connected map M there is a convex polyhedron whose 1-skeleton is isomorphic to M such that every automorphism of M is induced by an isometry of the convex polyhedron. We do not use this result, but it helps in the visualisation of some of the results in this section.

Throughout this paper, M denotes a 3-connected map with a non-trivial automorphism α of minimal order, which will be a prime. As usual we regard M as a dissection of a sphere Π into the vertices, edges and faces of M (called the *cells* of M). We say α is *sense-preserving* if it preserves the orientation of Π, and *sense-reversing* otherwise. (This concept can alternatively be defined combinatorially by studying the action of α on the cells of M.) An *invariant* cell of α is a cell of M fixed by α.

If α is sense-reversing, then it has even order and hence order 2. So altogether, α must fall into one of the following three classes. Firstly, we say α is *plane-reflective* if it is sense-reversing of order 2 and has at least one invariant cell. (By Mani's result this would imply that α corresponds to the reflection of some convex polyhedron in a plane.) This is the most troublesome type of automorphism to deal with in regard to proving that hardly any polyhedra have one. Secondly, we say α is *antipodal* if it is sense-reversing and has no invariant cell (corresponding to a reflection of a polyhedron in some point). Thirdly, we say α is *rotative* if it is sense-preserving. (This corresponds to a rotation of a polyhedron.) An edge is *flipped* by α if it is fixed by α but its incident vertices are interchanged. All other invariant edges are *transfixed* by α.

A *cell-cycle* of M is a set S of cells such that the graph G, whose vertex set is S and whose edge set consists of pairs of incident cells of M, is a cycle of length at least 4. In particular, each cell is adjacent to two others in G. Clearly no edge of S can be incident with both a face and a vertex of S. We use v or $v(G)$ to denote the number of vertices of a graph or map G, n for edges, and f or $f(M)$ for the number of faces of a map M. We often use Euler's formula $v + f - n = 2$ and its consequence for triangulations, that $f = 2v - 4$ and $n = 3v - 6$. The following structural result provides a convenient bound on one of the variables defined in the proof of Theorem 4.1.

LEMMA 2.1. *The diameter of a k-connected graph G does not exceed* $1 + (v - 2)/k$.

Proof. By Menger's Theorem there are at least k internally-disjoint paths between any two vertices of G. At least one of these paths therefore has length at most $1 + (v - 2)/k$.

From [5] we have the following three results.

LEMMA 2.2. *If α is plane-reflective, then the set of invariant cells of α is a cell-cycle.*

LEMMA 2.3. *If α is antipodal, then there is a cell-cycle of M which is fixed setwise by α, contains no faces, and contains at most* $2(v + 1)/3$ *vertices.*

Actually, to obtain Lemma 2.3 from [**5**, Lemma 2.2] one must take the dual result, and also observe that since the cell-cycle of that lemma is obtained as the union of two shortest paths between two vertices, its length is at most $2(v + 1)/3$ by Lemma 2.1.

LEMMA 2.4. *If α is rotative, then it has precisely two invariant cells.*

A cell-cycle S of M partitions the remaining cells of M into two connected subsets (by the Jordan Curve Theorem) which we call the *caps* of S. Given M, either cap determines S and the other cap. Given a cap D, we can perform the following constructions. They are each begun by taking a simple closed curve C, contained in the union of the cells in S and intersecting them in consecutive cyclic order, such that C intersects each edge of S incident with faces of S at a unique point. The cells of the *slice* of M with respect to D are all the cells in D, together with the intersections of C with each cell of M, and the portions of the faces and edges of S lying on the same side of C as D. The other side of C is also a face of the slice, called the *base* of the slice. The slice corresponds to cutting a polyhedron by a plane of reflection. The *join* or *1-join* of M with respect to D is obtained from M by removing all edges and vertices lying on the other side of C from D, introducing a new vertex there called the *apex*, joining the apex to all vertices on C, and extending all edges which cross C so that they meet the apex. For $r \geqq 3$, an *r-join* of M with respect to D is obtained from the join by replacing the apex with an r-cycle, called the *apex cycle*, in such a way that each edge incident with the apex becomes incident with a vertex of the r-cycle, and so that each such vertex gets degree at least 3.

The next result is immediate.

LEMMA 2.5. *The slice and join are dual constructions, in the sense that the dual map of the join of the dual of M (with respect to the dual of D) is the slice of M with respect to D.*

We next give situations in which the slice and join preserve the 3-connectedness of M.

LEMMA 2.6. *If α is plane-reflective and S is the set of invariant cells of α, then S is a cell-cycle and the slice, the join and every r-join ($r \geqq 3$) of M with respect to either cap of S are all* 3*-connected.*

Proof. For the slice this comes from [**5**, Lemma 2.4]. For the join it then follows from Lemma 2.5 and the fact that the dual of a 3-connected map is 3-connected. This in turn implies that every r-join is 3-connected by simple graph theoretic arguments.

LEMMA 2.7. *If S contains no faces, then the join of M with respect to either cap of S is 3-connected.*

Proof. This is immediate from [**5**, Lemma 2.5], in view of Lemma 2.5.

Given a cell-cycle S, we use $a = a(S)$ to denote the number of edges in S whose ends are in S, $b = b(S)$ to denote the number of other edges in S (whose two incident faces are also in S), $k = k(S)$ to denote the number of faces in S, and set $d = a + k$. We use v and f to denote the numbers of vertices and faces of M respectively, and v^* and f^* for a given map M^*, and set $\delta = d/v$.

LEMMA 2.8. *The valence of the apex of the join, and the valence of the base of the slice, of M with respect to a cap of S, are both d.*

Proof. For the join this is immediate. For the slice it then follows from the duality mentioned in Lemma 2.5.

LEMMA 2.9. *If S is such that its two caps have equal numbers of vertices, faces and edges, then*

(i) *for the slice M^* of M with respect to a cap of S, we have*

$$v^* = (v + a + k + b)/2,$$

and

$$f^* = 1 + (f + k)/2.$$

(ii) *for an r-join M^* of M with respect to a cap of S, we have*

$$v^* = r + (v + a + k - b)/2,$$

and

$$f^* = 1 - \delta_{1,r} + (f + 2a + k)/2$$

where $\delta_{1,r}$ is the Kronecker delta.

Proof. The number of vertices in S is $d - b = a + k - b$, so the number of vertices in each cap is $(v - a - k + b)/2$. The number of faces in each cap is $(f - k)/2$. To these must be added the contributions due to the cells arising from S in each construction. The lemma follows.

LEMMA 2.10. *If α is plane-reflective and S is the set of invariant cells of α, then*

(i) $f \geqq 2a + k$,
(ii) $v \geqq \max\{a + k - b, a + 3k - 3b\}$,
(iii) $f/v \geqq 4\delta - 2$,
(iv) $2v - 4 \geqq f + 2b - k$.

Proof. From Lemma 2.6, S is a cell-cycle and so neither of the two faces incident with any of the a edges transfixed by α is one of the k faces in S, which gives (i). To obtain (ii) we first distinguish the caps D_1 and D_2 of S, and give S a cyclic orientation so that each of its elements has a successor. Each of the a edges transfixed by α is associated with the vertex which is its successor. Each of the k faces in S is associated with its successor if that is a vertex, and otherwise (when the successor is a flipped edge) with the end of its successor which is in D_1. Each of the b edges flipped by α is associated with its end in D_2. Each of the faces in S not incident with a flipped edge (of such faces there are at least $k - 2b$) is associated in addition with all its incident vertices not fixed by α, of which there are at least two. In this way no vertex of M is associated with more than one cell in S, and at least $\max\{a + k + b, a + k + b + 2(k - 2b)\}$ such associations have been defined, so (ii) follows.

Set $\beta = b/v$ and $\kappa = k/v$. From (ii) we get

$$\beta \leqq 1 - \delta \quad \text{and} \quad 2\kappa \leqq 3\beta + 1 - \delta \leqq 4 - 4\delta.$$

Now from (i)

$$\begin{aligned} f/v &\geqq 2(a + k)/v - \kappa \\ &\geqq 2\delta - (2 - 2\delta), \end{aligned}$$

which is (iii).

Note that at least $2b - k$ faces in S are incident with two flipped edges. Thus we must add at least $2b - k$ edges if we wish to triangulate M. A 3-connected triangulation has $2v - 4$ faces, and (iv) follows.

LEMMA 2.11. *Suppose M^* is an r-join of M with respect to a cap D, with its apex or apex cycle distinguished. There are at most $\binom{d}{b}$ possibilities for $M - D$ given M^*, and for $r \geqq 3$ there are at least $\frac{1}{d}\binom{d}{r}$ possibilities for M^* given $M - D$.*

Proof. M^* defines the join of M uniquely. To choose an $M - D$ given the join, choose which b of the d edges incident with the apex are extensions of edges in the cell-cycle S in the definition of the join. This identifies the faces in S incident with edges in S. The other faces in S are determined by the non-triangular faces incident with the apex of the join of M. Thus $M - D$ is determined. Conversely, given $M - D$ the d edges can be hooked up to an r-cycle in $\binom{d}{r}$ ways if one of the d edges is distinguished to remove any rotational symmetries of $M - D$. The binomial results from choosing which of the d edges is to be the first (in the clockwise direction say) incident with each vertex in the r-cycle. Since an edge can be distinguished in at most d ways, the stated lower bound follows.

3. Enumeration results for rooted maps. In this section we state and derive formulae which are used in the next section to bound the numbers of polyhedra with non-trivial automorphisms.

Henceforth in this paper, $B(\mu)$ denotes

$$\left(\frac{4\mu(2 - \mu)}{(2\mu - 1)^2}\right)^{\mu} \frac{4(2\mu - 1)}{(2 - \mu)^2},$$

$A(\sigma, \lambda)$ denotes

$$\frac{(2 - \sigma)^{4(2-\sigma)}}{(1 - \sigma)^{2(1-\sigma)}\lambda^{\lambda}(3 - 2\sigma - \lambda)^{3-2\sigma-\lambda}},$$

$q_{v,f}$ denotes the number of rooted 3-connected maps with v vertices and f faces, and $\sim$ and $o(\ )$ refer to the passage of v to infinity unless otherwise specified.

We take the following result from [**1**, Theorem 1].

LEMMA 3.1. *Suppose* $\epsilon > 0$. *Then*

$$q_{v,f} \sim \frac{1}{\pi v^3 \mu^{3/2}}\left(\frac{(2\mu - 1)(2 - \mu)}{9\mu}\right)^{5/2} B(\mu)^{v-1}$$

uniformly as $v, f \to \infty$ *provided*

$$\frac{1}{2} + \epsilon < \mu < 2 - \epsilon \quad \text{where } \mu = (f - 1)/(v - 1).$$

The next lemma is the result of simple calculations.

LEMMA 3.2. *For* $\frac{1}{2} < \mu < 2$,

(i) $$\frac{d \log B(\mu)}{d\mu} = \log \frac{4\mu(2 - \mu)}{(2\mu - 1)^2},$$

(ii) $$\frac{d^2 \log B(\mu)}{d\mu^2} = -\frac{2(\mu + 1)}{\mu(2 - \mu)(2\mu - 1)} < 0,$$

(iii) *the maximum of* $B(\mu)$ *occurs at*

$$\mu = \frac{3 + \sqrt{7}}{4}$$

and has a value of

$$\frac{16}{27}(17 + 7\sqrt{7}) < 21.05,$$

(iv) $$B(1) = 16,\ 16 < B(1.8),\ \lim_{\mu \to 2^-} B(\mu) = \frac{256}{27} > 9.48.$$

For most of our purposes the following rough bound on numbers of maps will suffice.

LEMMA 3.3. *Let* $\mu = f/v$. *Then*

$$\lim_{v\to\infty} q_{v,f}^{1/v} = B(\mu)$$

uniformly for $1 \leqq \mu < 2$.

Proof. Since $B(\mu)$ is bounded this follows from Lemma 3.1 provided $1 \leqq \mu < 2 - \epsilon$ for any fixed $\epsilon > 0$. To close the gap near $\mu = 2$, we first observe that from [**6**, equation (5.11)] the number of rooted 3-connected triangulations with $v \geqq 4$ vertices is

$$(3.1) \qquad T(v) = \frac{2(4v - 11)!}{(3v - 7)!(v - 2)!} \leqq \left(\frac{256}{27} + o(1)\right)^v.$$

The latter inequality comes from Stirling's formula. A rooted 3-connected map with v vertices and μv faces can be triangulated by adding $(2 - \mu)v - 4$ edges. Thus each such map can be obtained from a 3-connected triangulation with v vertices by removing some set of $(2 - \mu)v - 4$ of its $3v - 6$ edges. Hence

$$q_{v,f} \leqq \binom{3v - 6}{(2 - \mu)v - 4} T(v),$$

and so for $\mu > 2 - \epsilon$ we have

$$(3.2) \qquad q_{v,f}^{1/v} \leqq \left(\frac{3^3}{\epsilon^\epsilon(3 - \epsilon)^{3-\epsilon}} + o(1)\right)\left(\frac{256}{27} + o(1)\right)$$
$$\leqq \frac{256}{27} + \epsilon_1 + o(1)$$

where $\epsilon_1 \to 0$ as $\epsilon \to 0$.

Similarly, a rooted 3-connected map with $(2 - \epsilon)v$ faces can be turned into a rooted 3-connected map with μv faces, for $\mu > 2 - \epsilon$, by adding $(\mu - (2 - \epsilon))v$ edges. Thus each of the former can be obtained from at least one of the latter by removing some set of $(\mu - 2 + \epsilon)v$ of its edges. Hence, as before,

$$q_{v,(2-\epsilon)v} < \binom{3v}{(\mu - 2 + \epsilon)v} q_{v,f}$$

for $\mu > 2 - \epsilon$. Since $\mu - 2 + \epsilon < \epsilon$, this gives

$$q_{v,f}^{1/v} > \left(\frac{3^3}{\epsilon^\epsilon(3 - \epsilon)^{3-\epsilon}} + o(1)\right)^{-1}(B(2 - \epsilon) + o(1))$$

for fixed ϵ by Lemma 3.1. Thus

$$q_{v,f}^{1/v} > \frac{256}{27} - \epsilon_2 + o(1)$$

where $\epsilon_2 \to 0$ as $\epsilon \to 0$ by Lemma 3.2 (iv). Together with (3.2) this closes the gap.

A *rooted near-triangulation* is a rooted map in which every non-root face has valence 3.

LEMMA 3.4. *Let $T(v, \sigma)$ be the number of 3-connected rooted near-triangulations with v vertices and root face valence σv, and take $\epsilon > 0$. Then*

$$T(v, \sigma)^{1/v} \leqq \frac{(2 - \sigma)^{4(2-\sigma)}}{(1 - \sigma)^{2(1-\sigma)}(3 - 2\sigma)^{3-2\sigma}} + o(1)$$

uniformly for $0 \leqq \sigma \leqq 1 - \epsilon$.

Proof. Let $m = \sigma v$. From [**6**, equation (5.10)],

$$T(v, \sigma) = \frac{3(m - 1)!(m - 4)!}{(3v - 6)!} \sum \frac{(4v - m - j - 8)!}{j!(j + 1)!(m - j - 3)!} \times \frac{(m + j - 1)(m - 3j - 3)}{(m - j - 1)!(v - m - j - 1)!}$$

where the summation ranges over $0 \leqq j \leqq \min\{m - 3, v - m - 1\}$.
Thus

$$T(v, \sigma)^{1/v} \leqq \left[\frac{(m!)^2}{(3v)!} \max_j \frac{(4v - m - j)!}{(j!)^2(m - j)!^2(v - m - j)!}\right]^{1/v} \times (1+o(1))$$

uniformly for $0 \leqq \sigma \leqq 1$ as $v \to \infty$. The ratio of consecutive terms in the function to be maximised is

$$(m - j)^2(v - m - j)/(j^2(4v - m - j)) + o(1)$$

when all factors are sufficiently large. Setting this equal to 1 gives

$$j \sim (v - m)m/(3v - 2m).$$

This clearly determines the maximum of the function in question when $\epsilon < \sigma < 1 - \epsilon$. Thus, from Stirling's formula, for $\epsilon < \sigma < 1 - \epsilon$

$$T(v, \sigma)^{1/v} \leqq [m^{2m}(3(2v - m)^2)^{4v-m-j}]^{1/v}/[(3v)^{3v}(m(v - m))^{2j} \times (m(2v - m))^{2(m-j)}(3(v - m)^2)^{v-m-j}(3v - 2m)^{3v-2m}]^{1/v}(1 + o(1))$$

$$= \frac{(2 - \sigma)^{4(2-\sigma)}}{(1 - \sigma)^{2(1-\sigma)}(3 - 2\sigma)^{3-2\sigma}} + o(1).$$

In the near-triangulations under consideration, we have $f = (2 - \sigma)v - 1$, so

$$f \geqq (2 - \epsilon)v - 1 \quad \text{for } \sigma \leqq \epsilon.$$

Lemmas 3.2 (iv) and 3.3 complete the proof since

$$\lim_{\sigma \to 0} \frac{(2 - \sigma)^{4(2-\sigma)}}{(1 - \sigma)^{2(1-\sigma)}(3 - 2\sigma)^{3-2\sigma}} = 256/27.$$

LEMMA 3.5. *Let $T(v, \sigma, \lambda)$ be the number of rooted 3-connected maps with root face of valence σv and with $f/v = 2 - \sigma - \lambda$. Then*

$$T(v, \sigma, \lambda)^{1/v} \leqq A(\sigma, \lambda) + o(1)$$

uniformly for $0 < \sigma < 1$ and $0 < \lambda < 2 - \sigma$.

Proof. A near-triangulation with root face of valence σv has $v(2 - \sigma) - 1$ faces (counting the root face) and $v(3 - 2\sigma) - 3$ edges which are *internal*, that is, non-incident with the root face. Hence each of the maps counted by $T(v, \sigma, \lambda)$ can be formed by removing $\lambda v - 1$ of the internal edges of some near-triangulation with v vertices and root face of valence σv. Therefore

$$T(v, \sigma, \lambda) \leqq T(v, \sigma)\binom{3v - 2\sigma v - 3}{\lambda v - 1}.$$

Thus the lemma follows for $\sigma < 1 - \epsilon$ from Lemma 3.4 and Stirling's formula. For $\sigma \geqq 1 - \epsilon$, each map M counted by $T(v, \sigma, \lambda)$ can be modified to a new 3-connected map M' with root face valence $\lfloor (1 - \epsilon)v \rfloor$ by the addition of a new edge in the root face of M, which becomes the root edge of M'. It follows that

$$T(v, \sigma, \lambda) \leqq T(v, \sigma', \lambda')$$

where

$$\begin{aligned} \sigma' &= \lfloor (1 - \epsilon)v \rfloor \quad \text{and} \quad \lambda' = 2 - \sigma' - (f + 1)/v \\ &= \lambda + O(\epsilon). \end{aligned}$$

Hence, from what we have already shown,

$$\begin{aligned} T(v, \sigma, \lambda)^{1/v} &\leqq A(\sigma', \lambda') + o(1) \\ &\leqq A(\sigma, \lambda) + o(1) + \epsilon_2 \end{aligned}$$

where $\epsilon_2 \to 0$ as $\epsilon \to 0$. This completes the proof.

4. Bounds on the numbers of symmetric polyhedra. Throughout this section c denotes a positive constant strictly less than 1, perhaps different at each occurrence, and similarly ϵ denotes some positive constant at each occurrence.

THEOREM 4.1. *The number of symmetric unrooted* 3-*connected maps with* v *vertices and* f *faces is* $o(c^v q_{v,f})$ *uniformly for* $1/2 < f/v < 2$.

Proof. Antipodal and rotative automorphisms are the easiest to deal with, so we do these before plane-reflective. In each case, bounds on the constant c may readily be obtained from our proof, but we do not do so since the results will not be best possible. In each case, we verify the theorem with the implicit assumption that $1 \leqq f/v < 2$. The reciprocal range is covered by applying the established result to the dual maps in each case.

The numbers of unrooted 3-connected maps with v vertices and f faces and with an antipodal, rotative or plane-reflective automorphism α are denoted by $p_1(v, f)$, $p_2(v, f)$ and $p_3(v, f)$ respectively.

First of all suppose α is antipodal. Let S denote the cell-cycle of M whose existence is asserted by Lemma 2.3 and let D be a cap of S. The caps of S are interchanged by α and at the same time each element of S is mapped to its opposite in the cycle. Hence D determines M, and also the hypotheses of Lemma 2.9 are satisfied. Let M^* be the join of M rooted so that the apex is the root vertex. Then by Lemma 2.11, M^* determines M uniquely. By Lemma 2.7, M^* is 3-connected. We have $k = b = 0$ and $a \leqq (2v + 2)/3$. Thus by Lemma 2.9.

$$v^* = 1 + (v + a)/2 < 2 + 5v/6 \tag{4.1}$$

and

$$f^* = (f + 2a)/2. \tag{4.2}$$

By Lemmas 3.2 (iii) and 3.3, the number $p_1(v, f)$ of possibilities for M^* is bounded by 21.05^{v^*}, which is at most 12.7^v by (4.1). By Lemmas 3.2 and 3.3,

$$q_{v,f} > (16 - o(1))^v$$

uniformly for $1 \leqq f/v \leqq 1.8$, which establishes the theorem in this range. From (4.1), (4.2) and $f \leqq 2v$ we have

$$f^*/v^* > f/v - o(1).$$

Thus for $f/v > 1.8$, Lemma 3.2 gives

$$B(f^*/v^*) < B(f/v) + o(1),$$

so by Lemma 3.3.

$$\begin{aligned} p_1(v, f)^{1/v} &< (B(f^*/v^*) + o(1))^{v^*/v} \\ &< (B(f/v) + o(1))^{5/6} \qquad \text{(by (4.1))} \\ &< (B(f/v) + (1))^{-1/6} q_{v,f}^{1/v} \end{aligned}$$

and so

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$$(4.3) \qquad p_1(v, f) = o(c^v q_{v,f}).$$

Next suppose α is rotative, of order $l \geqq 2$. By Lemma 2.4, α has just two invariant cells, which we assume at first to be vertices. Suppose $p_2{}^*(v, f)$ is the number of such M. Let P be a shortest path joining the two invariant vertices. Since P is shortest, any vertex in $P \cap \alpha^k(P)$ for some $1 \leqq k \leqq l - 1$ must be invariant under α^k. Since $\alpha^k \neq 1$, such a vertex must be invariant under α by Lemma 2.4 applied to α^k. Hence the images of P under $\alpha^0, \alpha^1, \dots, \alpha^{l-1}$ are internally disjoint. The union of two adjacent such images is a cell-cycle S of vertices and faces, such that one cap D of S contains precisely $1/l$ of the faces of M. S contains at most $2(v - 2)/l + 2$ vertices, and also at most $2(v + 1)/3$ vertices by Lemma 2.1 applied to P. Let M^* be the join of M with respect to D, rooted so that its apex is the root vertex. Then

$$a < \min\{2v/l + 2, 2v/3 + 1\}, \quad b = 0 \text{ and } k = 0,$$

and the number of possibilities for M given M^* is at most $a \leqq v$. Also

$$v^* = (v - 2)/l + a/2 + 2,$$
$$f^* = f/l + a.$$

Again, M^* is 3-connected by Lemma 2.7. If $l = 2$, we get (4.1) and (4.2), and hence the number of possibilities for M^* is bounded by $o(c^v q_{v,f})$ by (4.3). If $l \geqq 3$, $v^* < 2v/3 + 3$, and so the number of possibilities for M^* is bounded by

$$21.05^{v^*} < (8 + o(1))^v,$$

which is again $(c^v q_{v,f})$. Hence

$$p_2{}^*(v, f) < v \sum_l o(c^v q_{v,f}) = o(c^v q_{v,f}).$$

Now suppose the invariant cells of α are not both vertices. If an edge, w say, is invariant, introduce a new vertex, u say, in that edge and regard the two portions of w separated by u as edges. Two more new edges incident with u can be introduced, contained in the faces incident with w, so that the resulting map is still 3-connected and fixed by α and u is invariant. Similarly, if a face is invariant, we can introduce a new vertex u and l (if $l \geqq 3$) or 4 (if $l = 2$) new incident edges in the face, to achieve the same result. Hence we get a map M' with vertices invariant under α, with $v' = v + 1$ or $v + 2$ vertices and $f' = f + 2$ or $f + l - 1$ or $f + 2l - 1$ faces. By the previous result and Lemma 3.3, the number of such M' is

$$o(c^{v'} q_{v',f'}) = o(c^v q_{v,f}) \quad \text{if } l = 2,$$

and is at most

$$(8 + o(1))^{v'} = o(c^v q_{v,f}) \quad \text{if } l \geqq 3.$$

As each M' corresponds to a bounded number of M, we now have

$$(4.4) \qquad p_2(v, f) = o(c^v q_{v,f}).$$

Finally suppose α is plane-reflective. Let S denote the cell-cycle of invariant cells of α, whose existence is asserted by Lemma 2.2, and let D be a cap of S. Let M^* be an r-join of M with respect to D, rooted so that the root face is the one bounded by the apex cycle. By Lemma 2.6, M^* is 3-connected. At some points in our argument we will demonstrate that for v sufficiently large (recalling $\delta = d/v$)

$$(4.5) \qquad B(\mu) > B(f^*/v^*)^{v^*/v}\left[\binom{\delta v}{b}\Big/\binom{\delta v}{r}\right]^{1/v} + \epsilon.$$

By Lemmas 2.11 and 3.3 this implies that the number of possibilities for M in such cases is $o(c^v q_{v,f})$. Set

$$\mu = f/v, \quad H = (3 + \sqrt{7})/4 \quad \text{and} \quad \hat{B} = B(H).$$

We distinguish two ranges of μ.

Case 1. $1 \leqq \mu \leqq H$. Here let $r = \max\{\lfloor b/2 \rfloor, 3\}$. Note that

$$\left[\binom{\delta v}{b}\Big/\binom{\delta v}{r}\right]^{1/v} < (\delta v e/(2b))^{b/(2v)} + o(1)$$

by Stirling's formula. This is at most $e^{\delta/4} + o(1)$ since $w^{1/w} \leqq e^{1/e}$ for $w > 0$. Also by Lemma 2.9 (ii)

$$v^* = v/\lambda + O(1) \quad \text{where } \lambda = 2/(1 + \delta).$$

Thus for v large (4.5) is implied by

$$B(\mu) > e^{\delta/4}B(f^*/v^*)^{1/\lambda} + \epsilon$$

or

$$\lambda \log B(\mu) - \log B(f^*/v^*) - (2 - \lambda)/4 \geqq \epsilon.$$

By Lemmas 2.9 (ii) and 2.10 (i), $f^* \leqq f + 1$, so

$$f^*/v^* \leqq \lambda\mu + o(1).$$

Thus by Lemma 3.2, it suffices to find $\epsilon > 0$ such that ϵ is a lower bound on each of the following quantities for the stated ranges of $\lambda\mu$:

$$(4.6) \qquad \lambda \log B(\mu) - \log B(\lambda\mu) - (2 - \lambda)/4$$

when $\lambda\mu \leqq H$; and

$$(4.7) \qquad \lambda \log B(\mu) - \log \hat{B} - (2 - \lambda)/4$$

when $\lambda\mu \geqq H$.

We have by Lemma 2.10 (iii)

(4.8) $\lambda \geqq 8/(\mu + 6) \geqq 32/(27 + \sqrt{7})$.

Differentiating (4.6) with respect to μ gives

(4.9) $$\lambda \frac{d \log B(\mu)}{d\mu} - \lambda \frac{d \log B(\lambda\mu)}{d(\lambda\mu)}.$$

By Lemma 3.2, $d \log B(x)/dx$ is decreasing and so (4.9) is positive as $\lambda > 1$. Hence (4.6) is increasing in μ. By (4.8), $\mu \geqq 8/\lambda - 6$. So for (4.6) we need only consider

(4.10) $\lambda \log B(8/\lambda - 6) - \log B(8 - 6\lambda) - (2 - \lambda)/4$

for $32/(27 + \sqrt{7}) \leqq \lambda \leqq 8/7$ and

(4.11) $\lambda \log B(1) - \log B(\lambda) - (2 - \lambda)/4$

for $8/7 \leqq \lambda \leqq H$. The second derivative of (4.10) is

$$1/\lambda - 2/(\lambda - 1) + 48/(5 - 4\lambda) - 169/(16 - 13\lambda).$$

Since $4(16 - 13\lambda)/13 < 5 - 4\lambda$ we have

$$169/(16 - 13\lambda) > 52/(5 - 4\lambda),$$

and so this second derivative is negative for the appropriate λ. Thus (4.10) is minimised for extremal λ. It is at least .032 at $\lambda = 32/(27 + \sqrt{7})$ and at least .022 at $\lambda = 8/7$. The derivative of (4.11) is

$$\log(2\lambda - 1)^2 - \log(\lambda(2 - \lambda)) + \log 4 + 1/4,$$

which is positive since $\lambda(2 - \lambda) \leqq (2\lambda - 1)^2$ for $\lambda \geqq 1$. Hence (4.11) is minimised for $\lambda = 8/7$, at which point it has the same value as (4.10).

By Lemma 3.2, (4.7) has positive derivative with respect to μ, so we only consider the minimum value of μ for each λ, subject to

$$\mu \geqq 8/\lambda - 6, \mu \geqq 1 \quad \text{and} \quad \mu \geqq H/\lambda.$$

This gives

(4.12) $\lambda \log B(8/\lambda - 6) - \log \hat{B} - (2 - \lambda)/4$

for $32/(27 + \sqrt{7}) \leqq \lambda \leqq (29 - \sqrt{7})/24$,

(4.13) $\lambda \log B(H/\lambda) - \log \hat{B} - (2 - \lambda)/4$

for $(29 - \sqrt{7})/24 \leqq \lambda \leqq H$, and

(4.14) $\lambda \log B(1) - \log \hat{B} - (2 - \lambda)/4$

for $H \leqq \lambda \leqq 2$. The second derivative of (4.12) is

$$\frac{8}{\lambda}\left(\frac{6\lambda - 7}{(\lambda - 1)(4 - 3\lambda)} - \frac{26}{16 - 13\lambda}\right)$$

which is negative for this range of λ. At the extreme values of λ, (4.12) is at least .011 and .045. The derivative of (4.13) is

$$\log 4 - 2\log(2 - H/\lambda) + \log(2H/\lambda - 1) + 1/4$$

which is positive since $H/\lambda \geqq 1$ here. Hence in (4.13) we may take λ at its minimum, which is covered by (4.12). Finally, (4.14) has positive derivative, so again take λ at its minimum, H, which is covered by (4.13). Thus $\epsilon = .011$ is a lower bound on (4.6) and (4.7), and we have (4.5).

Case 2. $H \leqq \mu < 2$. This time let

$$\begin{aligned} r &= \max\{\lfloor (a + k)/(1 + \hat{B}) \rfloor, 3\} \\ &= \delta v/(1 + \hat{B}) + O(1). \end{aligned}$$

Using Lemma 2.9 (ii) for v^*, and $B(f^*/v^*) \leqq \hat{B}$, (4.5) follows from

$$B(\mu) > \hat{B}^{(1+\delta-b/v)/2}\binom{\delta v}{b}^{1/v} \Big/ \left[\binom{\delta v}{r}\hat{B}^{-r}\right]^{1/v} + \epsilon.$$

Note that

$$x^{-t}\binom{\delta v}{t} = \left(\frac{x+1}{x}\right)^{\delta v}\left[\binom{\delta v}{t}(x+1)^{-t}(x/(x+1))^{\delta v - t}\right]$$

where the bracketed quantity is a probability whose maximum over t is attained at

$$t = \delta v/(x + 1) + O(1).$$

Hence

$$\left[\binom{\delta v}{r}\hat{B}^{-r}\right]^{1/v} = (1 + 1/\hat{B})^{\delta} + o(1)$$

and

$$\binom{\delta v}{b}^{1/v}\hat{B}^{-b/(2v)} < (1 + \hat{B}^{-1/2})^{\delta} + o(1).$$

Thus it suffices to have

(4.15) $\quad B(\mu) > E(\delta) + \epsilon$

where

$$E(\delta) = \hat{B}^{1/2}((1 + \hat{B}^{1/2})\hat{B}/(1 + \hat{B}))^{\delta}.$$

Now let M^* be the slice of M with respect to D, rooted so that the base of the slice is the root face. By Lemma 2.6, M^* is 3-connected. Given M^*, there are at most $\binom{\delta v}{a} \leqq 2^{\delta v}$ possibilities for M, since then M is determined by the a edges in S transfixed by α. Set

$$\sigma = d/v^* \quad \text{and} \quad \lambda = 2 - \sigma - f^*/v^*.$$

By Lemmas 2.9 (i), 2.10 (ii) and 2.10 (iv) we have $v^* \leqq v$ and so $\sigma \geqq \delta$,

$$2\lambda v^* = 2v - f + 2b - k - 2 < 4v - 2f$$

$$\text{and} \quad 2\lambda v^* \geqq \lambda(v + \delta v).$$

Thus

(4.16) $\quad 0 < \lambda < 2(2 - \mu)/(1 + \delta).$

Also $1 \geqq v^*/v$ implies $\sigma \geqq \delta$. So by Lemmas 2.8, 3.3 and 3.5 it suffices to have

(4.17) $\quad B(\mu) > 2^{\delta} A(\sigma, \lambda) + \epsilon.$

Note that for $0 < \mu < 1$,

(4.18) $\quad (1 - \sigma)(3 - \sigma) < (2 - \sigma)^2.$

Hence

$$\frac{\partial \log A(\sigma, \lambda)}{\partial \sigma} = 2 \log[\,(1 - \sigma)(3 - 2\sigma - \lambda)/(2 - \sigma)^2] < 0.$$

Thus (4.17) is implied by

(4.19) $\quad B(\mu) > 2^{\delta} A(\delta, \lambda) + \epsilon.$

To complete the proof we will demonstrate the existence of $\epsilon > 0$ such that for $H \leqq \mu \leqq 2$ and $0 < \delta < 1$, either (4.15) holds or (4.19) holds for all λ satisfying (4.16).

Observe that $A(\delta, \lambda) \leqq A(\delta, (3 - 2\delta)/2)$ and that

$$\frac{d \log[2^{\delta} A(\delta, (3 - 2\delta)/2)\,]}{d\delta}$$

$$= 2 \log[\,(1 - \delta)(3 - 2\delta)(2 - \delta)^{-2}/\sqrt{2}],$$

which is negative by (4.18). Hence $2^{\delta} A(\delta, (3 - 2\delta)/2)$ is a decreasing function of δ. At $\delta = .8$, it is less than 13.2, and thus so is $2^{\delta} A(\delta, \lambda)$ for $\delta \geqq .8$ and all λ. Also, $E(\delta)$ increases with δ and is less than 17.6 when $\delta = .8$. Since $B(1.7) > 18$, we now have (4.15) or (4.19) for $H \leqq \mu \leqq 1.7$.

We may now assume $1.7 \leqq \mu < 2$. Then

$$4(2 - \mu)/(1 + \delta) < 2/(1 + \delta) < (3 - \delta)/(1 + \delta) < 3 - 2\delta,$$

and so $A(\delta, \lambda)$ is maximised at $\lambda = 2(2 - \mu)/(1 + \delta) = \bar{\lambda}$ say. Therefore (4.19) is implied by

(4.20) $\quad B(\mu) > F(\mu, \delta) + \epsilon$

where

$$F(\mu, \delta) = 2^{\delta} A(\delta, 2(2 - \mu)/(1 + \delta)).$$

We may also assume $\delta \geqq .3$, since $E(.3) < 8$. We have

$$\partial\bar{\lambda}/\partial\delta < 0 \quad \text{and} \quad \partial \log A(\delta, \bar{\lambda})/\partial\bar{\lambda} > 0$$

as $\bar{\lambda} < (3 - 2\delta)/2$. Hence

$$\frac{\partial \log F(\mu, \delta)}{\partial\delta} = \log 2 + \frac{\partial \log A(\delta, \bar{\lambda})}{\partial\delta} + \frac{\partial \log A(\delta, \bar{\lambda})}{\partial\bar{\lambda}} \frac{\partial\bar{\lambda}}{\partial\delta}$$

$$\leqq \log 2 + 2 \log[(1 - \delta)(3 - 2\delta)/(2 - \delta)^2].$$

The latter is negative at $\delta = .3$, and its derivative is less than

$$2/(2 - \delta) - 1/(1 - \delta) < 0,$$

so it stays negative. Thus $F(\mu, \delta)$ is decreasing in δ for fixed μ. It is also decreasing in μ for fixed δ as

$$\partial \log A(\delta, \bar{\lambda})/\partial\bar{\lambda} > 0 \quad \text{and} \quad \partial\bar{\lambda}/\partial\mu < 0.$$

Recall that $E(\delta)$ is increasing in δ. The following table completes the proof that

$$(4.21) \quad p_3(v, f) = o(c^v q_{v,f}),$$

where, for instance, the implication of the third line is that for $1.941 \leqq \mu \leqq 1.968$, (4.15) holds whenever $\delta \leqq .52$ and (4.20) holds whenever $\delta \geqq .52$. The theorem follows from (4.3), (4.4) and (4.21).

μ	δ	$B(\mu)$	$E(\delta)$	$\bar{\mu}$	$F(\delta, \bar{\mu})$
2.0	.43	> 9.48	< 9.43	1.987	< 9.48
1.987	.475	> 10.17	< 10.17	1.968	< 10.17
1.968	.52	> 10.96	< 10.96	1.941	< 10.96
1.941	.57	> 11.95	< 11.92	1.9	< 11.91
1.9	.635	> 13.29	< 13.29	1.82	< 13.28
1.82	.729	> 15.56	< 15.55	1.6	< 15.0

A map with n edges has at most $4n$ automorphisms. Thus, once it has been shown that the symmetric maps comprise at most $o(n^{-1})$ of the maps of a given type with n edges, we can obtain an asymptotic formula for the number of such unrooted maps by dividing the number of rooted maps by $4n$.

COROLLARY 4.2. *The number of combinatorially distinct* 3-*connected maps, or convex polyhedra, with v vertices and f faces, is asymptotic to* $q_{v,f}/(4(v + f - 2))$ *as* $v \to \infty$ *uniformly for* $1/2 < f/v < 2$.

Lemma 3.1 gives $q_{v,f}$ asymptotically, and Mullin and Schellenberg [**4**] give it exactly. Putting $f = 2v - 4$ gives the following.

COROLLARY 4.3. ([**9**]). *At most $o(c^v T(v))$ of the convex polyhedra with v vertices and all faces triangular are symmetric. Hence the number with v vertices is given asymptotically by $T(v)/(4(3v - 6))$, where $T(v)$ is given explicitly by* (3.1).

Since $2v \leqq n \leqq 3v - 6$ we obtain the following.

COROLLARY 4.4. ([**5**]). *The fraction of the n-edged convex polyhedra which are symmetric is at most $o(c^n)$.*

This implies that Tutte's asymptotic formula for the number of rooted 3-connected n-edged maps [**7**] can be divided by $4n$ to give an asymptotic formula for the unrooted ones.

Let

$$\bar{q}_v = \sum_f q_{v,f}$$

be the number of rooted 3-connected maps with v vertices. Then [**1**, Theorems 1 and 2] gives $\bar{q}_v \sim K(v)$ where

$$K(v) = \left(\frac{4 + \sqrt{7}}{8\sqrt{7}}\right)^{1/2} \frac{3 + \sqrt{7}}{(8\pi v^5)^{1/2}} \left(\frac{38\sqrt{7} - 100}{9}\right)^{5/2} \times \left(\frac{16}{27}(17 + 7\sqrt{7})\right)^{v-1}.$$

By Lemmas 3.2 and 3.3 almost all 3-connected maps satisfy

$$f = v(3 + \sqrt{7})/4 + o(1).$$

Hence for almost all, $n \sim v(7 + \sqrt{7})/4$, so we have the following.

COROLLARY 4.5. *At most $o(c^v K(v))$ convex polyhedra with v vertices are symmetric. Hence the number of convex polyhedra with v vertices is asymptotic to $K(v)/[v(7 + \sqrt{7})]$ as $v \to \infty$. Duality gives the same result with vertices replaced by faces.*

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University of California at San Diego,
La Jolla, California;
University of Auckland,
Auckland, New Zealand

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