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ABSTRACT

We prove that any hyper-Kähler sixfold K of generalized Kummer type has a naturally associated manifold Y_K of K3[3] type. It is obtained as crepant resolution of the quotient of K by a group of symplectic involutions acting trivially on its second cohomology. When K is projective, the variety Y_K is birational to a moduli space of stable sheaves on a uniquely determined projective K3 surface S_K. As an application of this construction we show that the Kuga–Satake correspondence is algebraic for the K3 surfaces S_K, producing infinitely many new families of K3 surfaces of general Picard rank 16 satisfying the Kuga–Satake Hodge conjecture.

1. Introduction

Together with manifolds of K3[n] type, deformations of generalized Kummer varieties constitute the most well-studied hyper-Kähler manifolds. We refer to this deformation type in dimension $2n$ as to the Kum^n type. After Beauville [Bea83] gave the first examples of such hyper-Kähler manifolds, many more have been constructed from moduli spaces of stable sheaves on abelian surfaces, see [Yos01]. However, the varieties so obtained always have Picard rank at least 2, and our understanding of a general projective variety of Kum^n type remains poor. In fact, for the time being, no construction of such variety is known (but see [O'Gr22] for some recent ideas).

In the present article we partially remedy this for hyper-Kähler sixfolds of generalized Kummer type, by associating to any K of Kum^3 type a hyper-Kähler manifold Y_K of K3[3] type. They are related by a dominant rational map

$$K \dashrightarrow Y_K$$

of degree 2^5, described as follows.

Any K of Kum^3 type admits an action of the group $(\mathbb{Z}/4\mathbb{Z})^4 \rtimes \mathbb{Z}/2\mathbb{Z}$ by symplectic automorphisms, where $\mathbb{Z}/2\mathbb{Z}$ acts on the first factor as -1. In fact, this is the group $\mathrm{Aut}_0(K)$ of automorphisms of K which act trivially on its second cohomology, which is deformation invariant by [HT13]. It has been computed in [BNS11] for the generalized Kummer variety associated to an abelian surface. We let $G \subset \mathrm{Aut}_0(K)$ be the subgroup generated by the automorphisms whose fixed locus contains a 4-dimensional component. We will show in Lemma 2.4 that

$$G \cong (\mathbb{Z}/2\mathbb{Z})^5.$$

THEOREM 1.1. *Let K be a manifold of* Kum^3 *type. The quotient K/G admits a resolution $Y_K \to K/G$ with Y_K a manifold of* K3$^{[3]}$ *type.*

The resolution $Y_K \to K/G$ is obtained via a single blow-up of the reduced singular locus. It can also be described as the quotient by G of the blow-up of K at the union of the fixed loci of all non-trivial automorphisms in G. We show that there is an isometry of transcendental Hodge structures

$$H^2_{\mathrm{tr}}(Y_K, \mathbb{Q}) \xrightarrow{\sim} H^2_{\mathrm{tr}}(K, \mathbb{Q})(2),$$

where on the right-hand side the form is multiplied by 2. This parallels the classical construction of the Kummer K3 surface $\mathrm{Km}(A)$ associated to an abelian surface A, where $H^2_{\mathrm{tr}}(\mathrm{Km}(A), \mathbb{Z})$ is Hodge isometric to $H^2_{\mathrm{tr}}(A, \mathbb{Z})(2)$. The study of the integral transcendental lattices of the manifolds Y_K will be the subject of future work.

As a consequence, we obtain a well-defined K3 surface associated to any projective sixfold of generalized Kummer type.

THEOREM 1.2. *Let K be a projective variety of* Kum^3 *type. There exists a unique (up to isomorphism) projective K3 surface S_K such that the variety Y_K of* K3$^{[3]}$ *type given by Theorem 1.1 is birational to a moduli space $M_{S_K, H}(v)$ of stable sheaves on S_K, for some primitive Mukai vector v and a v-generic polarization H.*

We call S_K the K3 surface associated to the sixfold K. It is characterized by the existence of a Hodge isometry $H^2_{\mathrm{tr}}(S_K, \mathbb{Z}) \xrightarrow{\sim} H^2_{\mathrm{tr}}(Y_K, \mathbb{Z})$. These surfaces come in countably many 4-dimensional families of general Picard rank 16; up to isogeny, they are the K3 surfaces S admitting an isometric embedding $H^2_{\mathrm{tr}}(S, \mathbb{Q}) \hookrightarrow \Lambda_{\mathrm{Kum}^3}(2) \otimes_{\mathbb{Z}} \mathbb{Q}$ of rational quadratic spaces. Here, Λ_{Kum^3} is the lattice which is the second cohomology of manifolds of Kum^3 type; it was computed in [Bea83] that $\Lambda_{\mathrm{Kum}^3} = U^{\oplus 3} \oplus \langle -8 \rangle$, where we denote by U the hyperbolic plane.

Applications

In a series of papers [Mar20, Mar22, Mar23], Markman has proven striking results on the Hodge conjecture for hyper-Kähler varieties of Kum^n and K3$^{[n]}$ type. He uses Verbitsky's theory of hyperholomorphic sheaves [Ver96] to produce very interesting algebraic cycles on general projective hyper-Kähler varieties via deformation. Through our construction, we are able to deduce more cases of the Hodge conjecture from his results.

Our main application is to the Kuga–Satake Hodge conjecture for K3 surfaces [vGe00]. The Kuga–Satake construction [Del71] associates via Hodge theory an abelian variety $\mathrm{KS}(S)$ to any projective K3 surface S, and the conjecture predicts the existence of an algebraic cycle inducing an embedding of the transcendental Hodge structure of the surface into the second cohomology of $\mathrm{KS}(S) \times \mathrm{KS}(S)$. It is known to hold in many cases for K3 surfaces of Picard number at least 17 (see [Mor85]), but it is wide open otherwise. There are a couple of families of K3 surfaces of general Picard rank 16 for which the conjecture is known, namely, the family of double covers of the plane branched at six lines [Par88] and the family of K3 surfaces with 15 nodes in $\mathbb{P}^4$ (see [ILP22]).

Theorem 1.3 gives infinitely many new families of K3 surfaces of general Picard rank 16 for which the Kuga–Satake Hodge conjecture holds true.

THEOREM 1.3. *Let S be a projective K3 surface such that there exists an isometric embedding of $H^2_{\mathrm{tr}}(S, \mathbb{Q})$ into $\Lambda_{\mathrm{Kum}^3}(2) \otimes_{\mathbb{Z}} \mathbb{Q}$, where (2) indicates that the form is multiplied by 2. Then, there exists an algebraic cycle γ on $S \times \mathrm{KS}(S) \times \mathrm{KS}(S)$ inducing an embedding $\gamma_* : H^2_{\mathrm{tr}}(S, \mathbb{Q}) \hookrightarrow H^2(\mathrm{KS}(S) \times \mathrm{KS}(S), \mathbb{Q})$ of rational Hodge structures.*

Via Theorem 1.2, we deduce this statement from the validity of the Kuga–Satake Hodge conjecture for varieties of Kum^n type, established by Voisin [Voi22] as a consequence of results of O'Grady [O'Gr21] and Markman [Mar23]. Once Theorem 1.3 is proven, a result of Varesco [Var22] implies that the Hodge conjecture holds for all powers of any K3 surface satisfying the assumption of the above theorem, see Corollary 5.8.

In the recent preprint [Mar22], Markman proves that any rational Hodge isometry $H^2(X, \mathbb{Q}) \xrightarrow{\sim} H^2(X', \mathbb{Q})$ between varieties of $\mathrm{K3}^{[n]}$ type is algebraic. He expects that an extension of his argument will lead to the analogous result for varieties of Kum^n type. In dimension 6, we can obtain it from the $\mathrm{K3}^{[3]}$ case via Theorem 1.1.

THEOREM 1.4. *Let K, K' be projective varieties of deformation type* Kum^3. *Let*

$$f \colon H^2(K, \mathbb{Q}) \xrightarrow{\sim} H^2(K', \mathbb{Q})$$

be a Hodge isometry. Then f is induced by an algebraic correspondence.

Overview of the contents

We sketch the proof of Theorem 1.1. First, we calculate the fixed loci K^g of all the automorphisms $g \in G$: we show that $\bigcup_{g \neq 1 \in G} K^g$ is the union of 16 hyper-Kähler manifolds of $\mathrm{K3}^{[2]}$ type, and we determine the various intersections of these components. Thanks to the deformation invariance of G, it is in fact enough to calculate these loci in the special case of the generalized Kummer sixfold on an abelian surface.

We then describe explicitly the singularities of K/G. It turns out that these are of a particular simple nature, modeled on products of ordinary double points on surfaces. A resolution Y_K of K/G is obtained via a single blow-up of the singular locus. The quotient K/G is a primitive symplectic orbifold as studied by Fujiki [Fuj83] and Menet [Men20], and we use a criterion due to Fujiki to prove that Y_K is a hyper-Kähler manifold. Moreover, the resolution can be performed in families, so that the Y_K are deformation equivalent to each other. To complete the proof of Theorem 1.1 it is therefore sufficient to find a single K of Kum^3 type such that Y_K is of $\mathrm{K3}^{[3]}$ type.

The specific example we study is a Beauville–Mukai system $K_J(v_3)$ on a general principally polarized abelian surface $\Theta \subset J$. It admits a Lagrangian fibration to the linear system $|2\Theta| = \mathbb{P}^3$, whose general fibres parametrize certain degree-3 line bundles supported on curves in the linear system. We show that the norm map for line bundles induces a dominant rational map of degree 2^5 from $K_J(v_3)$ onto a Beauville–Mukai system $M_{\mathrm{Km}(J)}(w_3)$, a moduli space of sheaves on the Kummer K3 surface $\mathrm{Km}(J)$ associated to J; the hyper-Kähler variety $M_{\mathrm{Km}(J)}(w_3)$ is birational to $\mathrm{Km}(J)^{[3]}$. We next prove that the norm map descends to a birational map

$$K_J(v_3)/G \dashrightarrow M_{\mathrm{Km}(J)}(w_3).$$

It follows that the hyper-Kähler manifold $Y_{K_J(v_3)}$ is birational to $\mathrm{Km}(J)^{[3]}$, and hence $Y_{K_J(v_3)}$ is of $\mathrm{K3}^{[3]}$ type by Huybrechts' theorem [Huy99] that birational hyper-Kähler manifolds are deformation equivalent.

Once Theorem 1.1 is proven, the other results follow from it. When K is projective, we define the associated K3 surface S_K as the unique (up to isomorphism) K3 surface whose transcendental lattice is Hodge isometric to $H^2_{\mathrm{tr}}(Y_K, \mathbb{Z})$. This is justified by the fact that $H^2_{\mathrm{tr}}(Y_K, \mathbb{Z})$ has rank at most 6 and, hence, it appears as transcendental lattice of some K3 surface of Picard rank at least 16. Moreover, for such Picard numbers, two K3 surfaces with Hodge isometric transcendental lattices are isomorphic; therefore, S_K is determined up to isomorphism. To prove our applications, we exploit the algebraic cycle in $K \times Y_K$ given by the rational map $K \dashrightarrow Y_K$.

The organization of this text is as follows. In §2 we calculate the fixed loci of the automorphisms of G. In §3 we study in detail the case of $K_J(v_3)$ as outlined above. In §4 we prove Theorems 1.1 and 1.2. Finally, in §5 we use our construction to prove Theorems 1.3 and 1.4.

2. Automorphisms trivial on the second cohomology

Let K be a manifold of Kum^3 type and let $\mathrm{Aut}_0(K)$ be the group of automorphisms of K which act trivially on $H^2(K, \mathbb{Z})$. We let K^h denote the fixed locus of an automorphism h of K.

DEFINITION 2.1. We denote by G the subgroup of $\mathrm{Aut}_0(K)$ generated by the automorphisms $g \in \mathrm{Aut}_0(K)$ such that K^g has a 4-dimensional component.

We will see in Lemma 2.4 that $G \cong (\mathbb{Z}/2\mathbb{Z})^5$. The main result proven in this section is then the following.

THEOREM 2.2. Let $Z \subset K$ be defined as $Z := \bigcup_{g \neq 1 \in G} K^g$. Then Z is the union of 16 fourfolds Z_j, each of which is a smooth hyper-Kähler manifold of K3$^{[2]}$ type. Moreover, two distinct components intersect in a K3 surface, three distinct components intersect in four points, and four or more distinct components do not intersect.

In order to prove this, we will compute the fixed loci of all automorphisms in G. Thanks to the deformation invariance of automorphisms trivial on the second cohomology it will be sufficient to treat the case of the generalized Kummer variety associated to an abelian surface A, i.e. K is the fibre over 0 of the composition

$$A^{[4]} \xrightarrow{\nu} A^{(4)} \xrightarrow{\sum} A$$

of the Hilbert–Chow morphism with the summation map. We denote by $A_0^{(4)} \subset A^{(4)}$ the fibre over 0 of $\sum$. The restriction $\nu \colon K \to A_0^{(4)}$ is a crepant resolution.

In this case the group of automorphisms of K acting trivially on the second cohomology has the natural description $\mathrm{Aut}_0(K) = A_4 \rtimes \langle -1 \rangle$, and the action on K and $A^{[4]}$ is induced by that on A, see [BNS11].

2.3 Notation

We let A_n be the group of points of order n of A. Via the isomorphism above, we write the elements of $\mathrm{Aut}_0(K^3(A))$ as $(\epsilon, \pm 1)$ for $\epsilon \in A_4$. For $\tau \in A_2$, we denote by $A_{2,\tau}$ the set $\{a \in A \mid 2a = \tau\}$, which consists of 16 points. The quotient surface $A/\langle(\tau, -1)\rangle$ has 16 nodes corresponding to points in $A_{2,\tau}$; its minimal resolution $\mathrm{Km}^\tau(A)$ is isomorphic to the Kummer K3 surface associated to A.

LEMMA 2.4. Let K be any manifold of Kum^3 type. Then $G \cong (\mathbb{Z}/2\mathbb{Z})^5$.

Proof. Since automorphisms in $\mathrm{Aut}_0(K)$ deform with K, their fixed loci deform as well. Therefore, it suffices to prove the lemma for the generalized Kummer variety $K = K^3(A)$ on an abelian surface A. In this case we show that

$$G = A_2 \times \langle -1 \rangle \cong (\mathbb{Z}/2\mathbb{Z})^5.$$

We have $\mathrm{Aut}_0(K^3(A)) = A_4 \rtimes \langle -1 \rangle$. If $h \in \mathrm{Aut}_0(K^3(A))$ is induced by translation by $0 \neq \epsilon \in A_4$, then, by [Ogu20, Lemma 3.5], the fixed locus $(K^3(A))^h$ is a union of K3 surfaces if $2\epsilon = 0$ and consists of isolated point if $2\epsilon \neq 0$. Consider now $h = (\epsilon, -1)$. The fixed locus of $h = (\epsilon, -1)$

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on $A^{(4)}$ consists of

$$W_\epsilon = \{(a, -a - \epsilon, b, -b - \epsilon), \text{ for } a, b \in A\} \subset A^{(4)},$$

and subvarieties of lower dimension. If $\epsilon \in A_2$, then W_ϵ is contained in $A_0^{(4)}$ and, hence, $(\epsilon, -1) \in G$. If $\epsilon \notin A_2$, then W_ϵ has empty intersection with $A_0^{(4)}$; the action of h on $A_0^{(4)}$ fixes isolated points and the surfaces $\{(\alpha, b, -b - \epsilon, -\alpha + \epsilon), b \in A\}$, where $\alpha \in A$ satisfies $\alpha = -\alpha - \epsilon$. The general point of such a surface is supported on four distinct points. Therefore, the fixed locus $(K^3(A))^h$ consists of surfaces and isolated points, because it is a union of symplectic varieties and the fibres of ν have dimension at most 3. We conclude that G is generated by the 16 involutions $(\tau, -1)$ with $\tau \in A_2$ and, hence, $G = A_2 \times \langle -1 \rangle$. $\square$

2.5 Relevant subvarieties

In what follows we let $K = K^3(A)$ be the generalized Kummer variety on an abelian surface A. We will calculate the fixed loci of automorphisms in $G = A_2 \times \langle -1 \rangle$ acting on K. The next definition introduces the relevant subvarieties. We denote by $\xi = (a, b, c, d)$ a point of $A_0^{(4)}$, and refer to $\{a, b, c, d\}$ as the support of ξ.

DEFINITION 2.6. We define the following subvarieties of $A_0^{(4)}$:

– for any $\tau \in A_2$, we let

$$W_\tau := \{(a, b, -a + \tau, -b + \tau), \text{ for } a, b \in A\};$$

– for a pair $(\tau, \theta) \in A_2 \times A_2$ with $\tau \neq 0$, we let

$$V_{\tau,\theta} := \{(a, a + \tau, -a + \theta, -a + \tau + \theta), \text{ for } a \in A\}.$$

We denote by $\overline{W_\tau}$ and $\overline{V_{\tau,\theta}}$ the subvarieties of K obtained as strict transform of W_τ and $V_{\tau,\theta}$ under the Hilbert–Chow morphism, respectively.

Note that none among the W_τ and $V_{\tau,\theta}$ is contained in the exceptional locus of $\nu \colon K \to A_0^{(4)}$, so that $\overline{W_\tau}$ and $\overline{V_{\tau,\theta}}$ are well-defined. We will show in Lemma 2.9 that the $\overline{W_\tau}$ are hyper-Kähler varieties of K3$^{[2]}$ type and that the $\overline{V_{\tau,\theta}}$ are K3 surfaces.

Remark 2.7. (i) For $\tau \in A_2$, the G-equivariant morphism $\psi_\tau \colon A \times A \to A^{(4)}$ such that $(a, b) \mapsto (a, b, -a + \tau, -b + \tau)$ has degree 8 and induces an isomorphism

$$(A/\langle (\tau, -1) \rangle)^{(2)} \xrightarrow{\sim} W_\tau.$$

Moreover, $W_\tau \neq W_{\tau'}$ unless $\tau = \tau'$, as can be seen considering a point $(\epsilon, \epsilon, \epsilon, \epsilon)$ for some $\epsilon \in A_{2,\tau}$.

(ii) Given $0 \neq \tau \in A_2$ and $\theta \in A_2$, the G-equivariant morphism $\psi_{\tau,\theta} \colon A \to A^{(4)}$ defined by $a \mapsto (a, a + \tau, -a + \theta, -a + \tau + \theta)$ induces an isomorphism

$$A/\langle \tau, (\theta, -1) \rangle \xrightarrow{\sim} V_{\tau,\theta}.$$

For a given $\tau \in A_2$, the union $\bigcup_{\theta \in A_2} V_{\tau,\theta}$ has eight irreducible components. In fact, $V_{\tau,\theta} = V_{\tau,\theta'}$ if and only if $\theta' \in \{\theta, \theta + \tau\}$, which can be seen considering a point $(a, a + \tau, a, a + \tau)$: it belongs to $V_{\tau,\theta}$ if and only if $a \in A_{2,\theta}$ or $a \in A_{2,\theta+\tau}$. Moreover, it can be easily checked that $V_{\tau,\theta} \neq V_{\tau',\theta'}$ for $\tau \neq \tau'$. In total, we therefore have 120 distinct surfaces $V_{\tau,\theta}$.

We recall a result of Kamenova, Mongardi and Oblomonkov which will be used in the sequel.

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Proposition 2.8 [KMO22, Lemma B.1]. *Let $\iota \colon \mathbb{C}^2 \to \mathbb{C}^2$ be the involution given by $(x, y) \mapsto (-x, -y)$. Consider the rational map*

$$\Phi \colon (\mathrm{Bl}_0(\mathbb{C}^2/\iota))^{[n]} \dashrightarrow (\mathbb{C}^2)^{[2n]}$$

which maps a general $\zeta = (P_1, \ldots, P_n)$ to $\xi = (P_1, \iota(P_1), \ldots, P_n, \iota(P_n))$. Then Φ extends to a closed embedding, with image the fixed locus for the action of ι on $(\mathbb{C}^2)^{[2n]}$.

The next lemma describes the varieties $\overline{V_{\tau, \theta}}$ and $\overline{W_\tau}$ introduced above.

Lemma 2.9.

(i) *For $\tau \neq 0$, $\theta \in A_2$, the rational map $\phi_{\tau, \theta}$ defined by the commutative diagram*

$$
\begin{array}{ccc}
\mathrm{Km}^\theta(A/\langle\tau\rangle) & \overset{\phi_{\tau,\theta}}{\dashrightarrow} & K \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \nu} \\
A/\langle\tau, (\theta,-1)\rangle & \overset{\psi_{\tau,\theta}}{\longrightarrow} & A_0^{(4)}
\end{array}
$$

where q is the natural map, extends to a regular morphism, which is an isomorphism onto its image $\overline{V_{\tau,\theta}}$.

(ii) *For $\tau \in A_2$, the rational map ϕ_τ defined by the commutative diagram*

$$
\begin{array}{ccc}
(\mathrm{Km}^\tau(A))^{[2]} & \overset{\phi_\tau}{\dashrightarrow} & K \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \nu} \\
(A/\langle(\tau,-1)\rangle)^{(2)} & \overset{\psi_\tau}{\longrightarrow} & A_0^{(4)}
\end{array}
$$

where q is the composition of the Hilbert–Chow morphism with the natural map $(\mathrm{Km}^\tau(A))^{(2)} \to (A/\langle(\tau,-1)\rangle)^{(2)}$, extends to a regular morphism, which is an isomorphism onto its image $\overline{W_\tau}$.

Proof. We prove part (i). As translations are fixed point free, for any $\alpha \in A/\langle\tau, (\theta,-1)\rangle$ the support of $\psi_{\tau,\theta}(\alpha) \in V_{\tau,\theta}$ consists of either two or four distinct points, according to whether α is a node or not. If α is not a node $q^{-1}(\alpha)$ is a single point at which $\phi_{\tau,\theta}$ is well-defined. Otherwise, $\psi_{\tau,\theta}(\alpha) = (a, a + \tau, a, a + \tau)$ for some a in $A_{2,\theta}$ or $A_{2,\tau+\theta}$. Then, there is a canonical identification $\nu^{-1}(\psi_{\tau,\theta}(\alpha)) = \mathbb{P}(T_a A) \times \mathbb{P}(T_{a+\tau} A)$. The exceptional divisor in $\mathrm{Km}^\theta(A/\langle\tau\rangle)$ corresponding to the node α is identified with $\mathbb{P}(T_a A)$. Translation by τ gives an isomorphism $\mathbb{P}(T_a A) \overset{\sim}{\longrightarrow} \mathbb{P}(T_{a+\tau} A)$, and the morphism $\phi_{\tau,\theta}$ is extended via $\phi_{\tau,\theta}(t) := (t, \tau(t))$ for $t \in \mathbb{P}(T_a A)$.

The proof of part (ii) is similar. If $\psi_\tau(\alpha, \beta)$ consists of four distinct points, then $q^{-1}(\alpha, \beta)$ is a single point at which ϕ_τ is well-defined. If the support of $\psi_\tau(\alpha, \beta)$ consists of three distinct points, then exactly one between α and β is a node of $A/\langle(\tau,-1)\rangle$, and $\psi_\tau(\alpha, \beta) = (a, \epsilon, -a + \tau, \epsilon)$ for some $a \in A \setminus A_{2,\tau}$ and $\epsilon \in A_{2,\tau}$; we extend ϕ_τ via the canonical identifications $q^{-1}(\alpha, \beta) = \mathbb{P}(T_\epsilon A) = \nu^{-1}(\psi_\tau(\alpha, \beta))$. If the support of $\psi_\tau(\alpha, \beta)$ consists of two distinct points then either $\alpha = \beta$ for some smooth point or $\alpha \neq \beta$ with both α and β nodes of $A/\langle(\tau,-1)\rangle$. In the first case there exists $a \in A \setminus A_{2,\tau}$ such that $\psi_\tau(\alpha, \beta) = (a, a, -a + \tau, -a + \tau)$. Then $q^{-1}(\alpha, \beta) = \mathbb{P}(T_a A)$ and $\nu^{-1}(\psi_\tau(\alpha, \beta)) = \mathbb{P}(T_a A) \times \mathbb{P}(T_{-a+\tau} A)$, and we define ϕ_τ by $t \mapsto (t, (\tau,-1)(t))$. In the second case $\psi_\tau(\alpha, \beta) = (\epsilon_1, \epsilon_2, \epsilon_1, \epsilon_2)$ for some $\epsilon_1 \neq \epsilon_2$ both in $A_{2,\tau}$, and we extend ϕ_τ via the canonical isomorphisms $q^{-1}(\alpha, \beta) = \mathbb{P}(T_{\epsilon_1} A) \times \mathbb{P}(T_{\epsilon_2} A) = \nu^{-1}(\psi_\tau(\alpha, \beta))$. Finally, consider the case when $\alpha = \beta$ and α is a node. Locally analytically around the node α of $A/\langle(\tau,-1)\rangle$, the morphism q is isomorphic to the composition

$$(\mathrm{Bl}_0(\mathbb{C}^2/\iota))^{[2]} \to (\mathrm{Bl}_0(\mathbb{C}^2/\iota))^{(2)} \to (\mathbb{C}^2/\iota)^{(2)},$$

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where $\iota(x, y) = (-x, -y)$ is the restriction of the involution $(\tau, -1)$. The rational map ϕ_τ is identified with $\Phi \colon (\mathrm{Bl}_0(\mathbb{C}^2/\iota))^{[2]} \dashrightarrow (\mathbb{C}^2)^{[4]}$ which maps a general $\zeta = (\alpha, \beta)$ to the subscheme $\xi = (\alpha, \iota(\alpha), \beta, \iota(\beta))$. Then Proposition 2.8 implies that ϕ_τ extends to a closed embedding $(\mathrm{Bl}_0(\mathbb{C}^2/\iota))^{[2]} \hookrightarrow (\mathbb{C}^2)^{[4]}$, completing the proof. $\qquad\square$

The various intersections of the submanifolds $\overline{W_\tau} \subset K$ are as follows.

LEMMA 2.10.

(i) For $\tau_1 \neq \tau_2$ in A_2 we have $\overline{W_{\tau_1}} \cap \overline{W_{\tau_2}} = \overline{V_{\tau_1 + \tau_2, \tau_1}}$.
(ii) For pairwise distinct τ_1, τ_2, τ_3 in A_2, the intersection $\overline{W_{\tau_1}} \cap \overline{W_{\tau_2}} \cap \overline{W_{\tau_3}}$ consists of the four distinct points of the set

$$\nu^{-1}(\{(a, a + \tau_1 + \tau_2, -a + \tau_1, -a + \tau_2) \in A_0^{(4)}, \ a \in A_{2, \tau_1 + \tau_2 + \tau_3}\}) \subset K.$$

(iii) The intersection of four or more distinct submanifolds $\overline{W_\tau}$ is empty.

Proof. (i) A point $\xi \in A_0^{(4)}$ belongs to $W_{\tau_1} \cap W_{\tau_2}$ if and only if it can be written as $\xi = (a, b, -a + \tau_1, -b + \tau_1)$ as well as $\xi = (a', b', -a' + \tau_2, -b' + \tau_2)$, for some a, b, a', b' in A. Then, we may assume that $a' = a$; since $\tau_1 \neq \tau_2$, it follows that $-a + \tau_1 \neq -a' + \tau_2$. This forces either $b' = -a + \tau_1$ or $-b' + \tau_2 = -a + \tau_1$; in both cases, we can write

$$\xi = (a, -a + \tau_1, -a + \tau_2, a + \tau_1 + \tau_2),$$

i.e. $\xi \in V_{\tau_1 + \tau_2, \tau_1}$. Thus, $W_{\tau_1} \cap W_{\tau_2} = V_{\tau_1 + \tau_2, \tau_1}$ and, hence, for their strict transforms we have $\overline{W_{\tau_1}} \cap \overline{W_{\tau_2}} = \overline{V_{\tau_1 + \tau_2, \tau_1}}$.

(ii) By part (i), a point ξ lies in the intersection $W_{\tau_1} \cap W_{\tau_2} \cap W_{\tau_2}$ if and only if it can be written as $\xi = (a, a + \tau_1 + \tau_2, -a + \tau_1, -a + \tau_2)$ as well as $\xi = (a'', b'', -a'' + \tau_3, -b'' + \tau_3)$. Again, we may assume $a'' = a$. Then we must have $-a + \tau_3 = a + \tau_1 + \tau_2$; equivalently, $a \in A_{2, \tau_1 + \tau_2 + \tau_3}$. Moreover, either we have $b'' = -a + \tau_1$ or $b'' = -a + \tau_2$. Note that, since $a \in A_{2, \tau_1 + \tau_2 + \tau_3}$, in the first case $-b'' + \tau_3 = -a + \tau_2$, while in the second case $-b'' + \tau_3 = -a + \tau_1$. This shows that

$$W_{\tau_1} \cap W_{\tau_2} \cap W_{\tau_3} = \{(a, a + \tau_1 + \tau_2, -a + \tau_1, -a + \tau_2) \in A_0^{(4)}, \ a \in A_{2, \tau_1 + \tau_2 + \tau_3}\}.$$

This set consists of four distinct points. Indeed, G acts transitively on it and the subgroup of G generated by the $(\tau_i, -1)$, $i = 1, 2, 3$ acts trivially. Moreover, it is easy to see that if $g \in G$ is not an element of this subgroup, then it has no fixed points in the above set. Hence, the intersection $W_{\tau_1} \cap W_{\tau_2} \cap W_{\tau_3}$ is in bijection with $(\mathbb{Z}/2\mathbb{Z})^2$. No point of this set belongs to the exceptional locus of the Hilbert–Chow morphism, so that $\overline{W_{\tau_1}} \cap \overline{W_{\tau_2}} \cap \overline{W_{\tau_3}}$ consists of four distinct points.

(iii) Note that $W_{\tau_1} \cap W_{\tau_2} \cap W_{\tau_3} \cap W_{\tau_1 + \tau_2 + \tau_3}$ is empty. We have seen above that if $\theta \in A_2 \setminus \{\tau_1, \tau_2, \tau_3, \tau_1 + \tau_2 + \tau_3\}$, the action of $(\theta, -1)$ on $W_{\tau_1} \cap W_{\tau_2} \cap W_{\tau_3}$ is free; as W_θ is clearly fixed by $(\theta, -1)$, it follows that $W_{\tau_1} \cap W_{\tau_2} \cap W_{\tau_3} \cap W_\theta = \emptyset$. $\qquad\square$

2.11 Fixed loci

We calculate the fixed locus of the automorphisms $g \in G$. Our results confirm those of Oguiso [Ogu20] and Kamenova, Mongardi and Oblomkov [KMO22].

PROPOSITION 2.12. *Let $g \neq 1 \in G$. Then:*

– *if $g = (\tau, 1)$, with $\tau \neq 0 \in A_2$, the fixed locus K^g is the disjoint union of the eight K3 surfaces $\overline{V_{\tau, \theta}}$, for θ varying in A_2;*

– if $g = (\tau, -1)$ with $\tau \in A_2$, the fixed locus K^g is the union of the $K3^{[2]}$ type variety $\overline{W_\tau}$ and 140 isolated points; the isolated points are given by the set

$$\left\{ (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4), \ \epsilon_i \in A_{2,\tau} \text{ pairwise distinct and such that } \sum_{i=1}^4 \epsilon_i = 0 \in A \right\}.$$

Proof. Let $g = (\tau, 1)$ with $\tau \neq 0 \in A_2$. The action of g on $\xi = (a, b, c, d) \in A^{(4)}$ is given by $\xi \mapsto (a + \tau, b + \tau, c + \tau, d + \tau)$. We see immediately that the surfaces $V_{\tau,\theta}$ are fixed by g. Conversely, if ξ is fixed and a belongs to its support, then also $a + \tau$ must be in the support. Therefore, $\xi = (a, a + \tau, b, b + \tau)$; if $\xi \in A_0^{(4)}$, then $2a + 2b = 0$, so that $\theta := a + b$ belongs to A_2 and $\xi \in V_{\tau,\theta}$. By the description of $\overline{V_{\tau,\theta}}$ given in Lemma 2.9(i), the fixed locus of g in $\nu^{-1}(V_{\tau,\theta})$ is precisely $\overline{V_{\tau,\theta}}$. Since the fixed locus is smooth, K^g is the disjoint union of the eight K3 surfaces $\overline{V_{\tau,\theta}}$, for $\theta \in A_2$.

Assume now that $g = (\tau, -1)$. If $\xi = (a, b, c, d) \in A^{(4)}$, the action of g is given by $\xi \mapsto (-a + \tau, -b + \tau, -c + \tau, -d + \tau)$. Note that W_τ is contained in the fixed locus of g on $A_0^{(4)}$. Moreover any ξ entirely supported at points of $A_{2,\tau}$ is fixed by g. Conversely, let $\xi \in A_0^{(4)}$ be fixed. Then we have the following possibilities. If the support of ξ does not intersect $A_{2,\tau}$, we have $\xi = (a, -a + \tau, b, -b + \tau)$ for some $a, b \in A \setminus A_{2,\tau}$ and ξ belongs to W_τ. If the support of ξ contains some $\epsilon \in A_{2,\tau}$ and some $a \notin A_{2,\tau}$, then $\xi = (a, -a + \tau, \epsilon, b)$ for some $b \in A$; as $\xi \in A_0^{(4)}$, we must have $b = \epsilon$, and hence ξ belongs to W_τ. Finally, assume that $\xi = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ is entirely supported at $A_{2,\tau}$. As ξ belongs to $A_0^{(4)}$ we have $\sum_i \epsilon_i = 0$ in A. If the support of ξ consists of four distinct points, then $\xi \notin W_\tau$. Otherwise, $\xi = (\epsilon_1, \epsilon_1, \epsilon_2, \epsilon_2)$ lies in W_τ.

We conclude that the fixed locus of g acting on $A_0^{(4)}$ consists of W_τ and the isolated points in the set

$$\left\{ (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4), \ \epsilon_i \in A_{2,\tau} \text{ pairwise distinct and such that } \sum_{i=1}^4 \epsilon_i = 0 \in A \right\}.$$

There are 140 isolated fixed points. Indeed, the total number of ordered sequences $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ such that $\sum_i \epsilon_i = 0$ is 16^3. There are 16 such sequences with support a single point, and $6\binom{16}{2} = 16 \cdot 45$ with support two distinct points. The number of ordered sequences as above supported on four distinct points is $16^3 - 16 - 16 \cdot 45 = 16 \cdot 210$; under the symmetric group, each of them has an orbit of cardinality 24. Hence, the number of isolated fixed points is $\frac{210 \cdot 16}{24} = 140$.

Therefore, K^g consists of 140 isolated fixed points and the fixed locus of g acting on $\nu^{-1}(W_\tau)$. To conclude we have to check that the latter coincides with $\overline{W_\tau}$. Using the description of $\overline{W_\tau}$ given in the proof of Lemma 2.9(ii) and the second assertion of Proposition 2.8 it is readily seen that $\nu^{-1}(\xi) \cap K^g = \nu^{-1}(\xi) \cap \overline{W_\tau}$ for any $\xi \in W_\tau$. $\square$

Remark 2.13. Let $h = (\epsilon, \pm 1)$ be an automorphism in $\mathrm{Aut}_0(K^3(A)) = A_4 \rtimes \langle -1 \rangle$. One can see directly from Definition 2.6 that h maps $\overline{W_\tau}$ to $\overline{W_{\tau+2\epsilon}}$, for any $\tau \in A_2$.

We can finally give the proof of the main result of this section.

Proof of Theorem 2.2. Let us first assume that K is the generalized Kummer variety associated to an abelian surface A. Let $g = (\tau, 1) \in G$, with $\tau \neq 0 \in A_2$. Consider a component $\overline{V_{\tau,\theta}}$ of K^g. The surface $V_{\tau,\theta}$ is clearly contained in W_θ and, hence, the same holds for their strict transforms. Now let $g = (\tau, -1) \in G$. Let P be an isolated fixed point in K^g. Then we have $P = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ for some pairwise distinct $\epsilon_i \in A_{2,\tau}$ summing up to 0 in A. The last condition

implies that we can write $P = (\epsilon_1, \epsilon_2, \epsilon_1 + \theta, \epsilon_2 + \theta)$ for some $0 \neq \theta \in A_2$. This is the same as

$$P = (\epsilon_1, \epsilon_2, -\epsilon_1 + \theta + \tau, -\epsilon_2 + \theta + \tau),$$

which lies in $\overline{W_{\tau+\theta}}$. By Lemma 2.12 we conclude that for a generalized Kummer variety the union of the fixed loci K^g consists of the 16 varieties $\overline{W_\tau}$ of K3$^{[2]}$ type. The intersections of these components were computed in Lemma 2.10.

Now let $f: \mathcal{K} \to B$ be a smooth and proper family of Kum^3 manifolds over a connected base B, with a fibre $\mathcal{K}_0$ isomorphic to the generalized Kummer variety on an abelian surface A. By the result of Hassett and Tschinkel [HT13, Theorem 2.1], up to a finite étale base change, there is a fibrewise action of $G = A_2 \times \langle -1 \rangle$ on $\mathcal{K}$. By [Fuj83, Lemma 3.10], this action is locally trivial with respect to f, i.e. any $x \in \mathcal{K}$ has a neighborhood $U_x = (f^{-1}f(U_x) \cap \mathcal{K}_{f(x)}) \times f(U_x)$ such that the action on U_x of the stabilizer subgroup of x is induced by that on the fibre $\mathcal{K}_{f(x)}$. Therefore, for any $\tau \in A_2$ the fixed locus $\mathcal{K}^g$ of $g = (\tau, -1)$ contains a smooth family $\overline{\mathcal{W}_\tau} \to B$ of manifolds of K3$^{[2]}$ type as the unique component of $\mathcal{K}^g$ with positive-dimensional intersection with the fibres of f. We conclude that for any $b \in B$ we have

$$\bigcup_{1 \neq g \in G} (\mathcal{K}_b)^g = \bigcup_{\tau \in A_2} \overline{\mathcal{W}_\tau}|_{\mathcal{K}_b},$$

and that the components $\overline{\mathcal{W}_\tau}|_{\mathcal{K}_b}$ intersect as claimed. $\qquad\qquad\square$

3. A Lagrangian fibration

Throughout this section, J denotes a general principally polarized abelian surface. We fix a symmetric theta divisor $\Theta \subset J$, which is unique up to translation by a point of J_2. For any integer d, we consider the Mukai vector $v_d := (0, 2\Theta, d - 4)$ on J and the moduli space $M_J(v_d)$ of Θ-semistable sheaves with Mukai vector v_d ([HL10]). The Albanese fibration

$$M_J(v_d) \xrightarrow{\mathrm{alb}} \mathrm{Pic}^0(J) \times J$$

is isotrivial [Yos01]. We denote the fibre of alb over $(\mathcal{O}_J, 0)$ by $K_J(v_d)$. For d even, $K_J(v_d)$ is singular and admits a crepant resolution of OG6 type, by [O'Gr03] and [LS06]. When d is odd, $K_J(v_d)$ is smooth, and it is a hyper-Kähler sixfold of Kum^3 type [Yos01]. In this case, we consider the group $G \subset \mathrm{Aut}_0(K_J(v_d))$ introduced in Definition 2.1.

The main result of this section is the following.

THEOREM 3.1. *The quotient $K_J(v_3)/G$ is birational to $\mathrm{Km}(J)^{[3]}$.*

We will, in fact, show that for any odd d the quotient $K_J(v_d)/G$ is birational to a variety of K3$^{[3]}$ type. Before proceeding with the proof we need to fix some notation.

3.2 Preliminaries

Let C be a smooth projective curve of genus g. Its Picard variety $\mathrm{Pic}^\bullet(C) = \bigsqcup_d \mathrm{Pic}^d(C)$ is the group of line bundles modulo isomorphism. Tensor product gives multiplication maps $m: \mathrm{Pic}^a(C) \times \mathrm{Pic}^b(C) \to \mathrm{Pic}^{a+b}(C)$. We denote by $[n]: \mathrm{Pic}^d(C) \to \mathrm{Pic}^{nd}(C)$ the nth power map. The Jacobian of C is the abelian variety $\mathrm{Pic}^0(C)$; for $d \neq 0$, the variety $\mathrm{Pic}^d(C)$ is a torsor under $\mathrm{Pic}^0(C)$.

Assume now that $\pi\colon \tilde{C} \to C$ is an étale double cover, so that $\tilde{C}$ is of genus $2g-1$. In this situation, for each integer d, we have:

– the pull-back map $\pi^*\colon \mathrm{Pic}^d(C) \to \mathrm{Pic}^{2d}(\tilde{C})$;
– the norm map $\mathrm{Nm}_\pi\colon \mathrm{Pic}^d(\tilde{C}) \to \mathrm{Pic}^d(C)$;
– the covering involution $\sigma\colon \tilde{C} \to \tilde{C}$, inducing $\sigma^*\colon \mathrm{Pic}^d(\tilde{C}) \to \mathrm{Pic}^d(\tilde{C})$.

The composition $\mathrm{Nm}_\pi \circ \pi^*\colon \mathrm{Pic}^d(C) \to \mathrm{Pic}^{2d}(C)$ coincides with multiplication by 2, while $\pi^* \circ \mathrm{Nm}_\pi\colon \mathrm{Pic}^d(\tilde{C}) \to \mathrm{Pic}^{2d}(\tilde{C})$ is given by $L \mapsto L \otimes \sigma^*(L)$.

We refer to Mumford's paper [Mum74] for the following. The étale double cover π is uniquely determined by a 2-torsion line bundle η on C, which is the only non-trivial element in $\ker(\pi^*\colon \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(\tilde{C}))$. Moreover, the image of the pull-back map $\pi^*\colon \mathrm{Pic}^d(C) \to \mathrm{Pic}^{2d}(\tilde{C})$ is precisely the fixed locus of σ^*. The involution σ^* acts as the inverse $L \mapsto L^\vee$ on the kernel of the norm map $\mathrm{Nm}_\pi\colon \mathrm{Pic}^0(\tilde{C}) \to \mathrm{Pic}^0(C)$. This kernel has two connected components; the one containing the neutral element is the Prym variety $P(\pi)$ of the cover, an abelian variety of dimension $g-1$. The fibres of $\mathrm{Nm}_\pi\colon \mathrm{Pic}^d(\tilde{C}) \to \mathrm{Pic}^d(C)$ are torsors under $P(\pi) \times \mathbb{Z}/2\mathbb{Z}$.

3.3 Geometric set-up

Let $\Theta \subset J$ be the general principally polarized abelian surface of Picard rank 1, with Θ a symmetric theta divisor. By Riemann–Roch $H^0(J, \mathcal{O}_J(2\Theta)) = 4$; we will identify the complete linear system $|2\Theta|$ with $\mathbb{P}^3$. It is classically known that this linear system is base-point free and induces an embedding of $J/\pm 1$ into $\mathbb{P}^{3,\vee}$ as a quartic surface with 16 nodes. Blowing-up the nodes of this quartic surface one obtains the Kummer K3 surface $\mathrm{Km}(J)$ associated to J. We denote by H the divisor on $\mathrm{Km}(J)$ obtained as pull-back of a hyperplane section of $J/\pm 1 \subset \mathbb{P}^{3,\vee}$. The linear system $|H|$ is naturally identified with $|2\Theta| = \mathbb{P}^3$.

Consider the universal family of genus-5 curves $\tilde{\mathcal{C}} \to |2\Theta|$. The involution -1 of J acts trivially on $|2\Theta|$, and we consider the quotient family $\mathcal{C} \to \mathbb{P}^3$, which is identified with the hyperplane linear system on $J/\pm 1 \subset \mathbb{P}^{3,\vee}$. There is a degree 2 morphism $\pi\colon \tilde{\mathcal{C}} \to \mathcal{C}$ over $\mathbb{P}^3$. We denote by $\mathcal{D} \to |H|$ the universal family of curves over the linear system $|H|$ on $\mathrm{Km}(J)$; clearly, $\mathcal{C}_b = \mathcal{D}_b$ if and only if $\mathcal{C}_b$ does not pass through any of the nodes of $J/\pm 1$.

A detailed study of the linear system $|2\Theta|$ and the map $\pi\colon \tilde{\mathcal{C}} \to \mathcal{C}$ can be found in Verra's article [Ver87]. In particular, he shows that, for any $b \in \mathbb{P}^3$, the curve $\tilde{\mathcal{C}}_b$ is smooth if and only if $\mathcal{C}_b$ is so, and in this case the Prym variety of the étale double cover $\pi_b\colon \tilde{\mathcal{C}}_b \to \mathcal{C}_b$ is the abelian surface J. Moreover, $\mathcal{C}_b$ is smooth if and only if it does not contain any node of $J/\pm 1$ and it does not lie on a tangent hyperplane.

3.4 Beauville–Mukai systems

Taking the Jacobian of the smooth curves in the families introduced above yields families of abelian varieties over a Zariski open subvariety of $\mathbb{P}^3$. The total space of these families can be compactified considering suitable moduli spaces of stable sheaves on J and $\mathrm{Km}(J)$. The construction leads to certain hyper-Kähler varieties equipped with Lagrangians fibrations, called Beauville–Mukai systems [Bea91, Muk84].

Consider the Mukai vector $v_d = (0, 2\Theta, d-4)$ on J, for an integer d. The moduli space $M_J(v_d)$ parametrizes pure dimension 1 sheaves on J which are push-forward of semistable and torsion-free sheaves of rank 1 and degree d supported on curves algebraically equivalent to 2Θ. Mapping a sheaf to its support [LeP93] gives a morphism

$$M_J(v_d) \xrightarrow{\ \mathrm{supp}\ } \mathrm{Pic}^0(J) \times \mathbb{P}^3.$$

We define the degree-d relative compactified Jacobian as

$$\overline{\mathrm{Pic}}^d(\tilde{\mathcal{C}}/\mathbb{P}^3) \coloneqq \mathrm{supp}^{-1}(\{\mathcal{O}_J\} \times \mathbb{P}^3).$$

If $b \in \mathbb{P}^3$ corresponds to a smooth curve in $|2\Theta|$, the fibre over b of the support morphism $\mathrm{supp} \colon \overline{\mathrm{Pic}}^d(\tilde{\mathcal{C}}/\mathbb{P}^3) \to \mathbb{P}^3$ is the degree-d Picard variety of the curve. The Albanese morphism $\mathrm{alb} \colon \overline{\mathrm{Pic}}^d(\tilde{\mathcal{C}}/\mathbb{P}^3) \to J$ is an isotrivial fibration with fibre $K_J(v_d)$, and it maps a sheaf $F \in K_J(v_d)$ to the sum $\sum c_2(F) \in J$, see [Wie18, §6].

A similar construction can be done for the family of curves $\mathcal{D}$ on $\mathrm{Km}(J)$. Given an integer d, we consider the Mukai vector $w_d = (0, H, d-2)$ on $\mathrm{Km}(J)$. The degree d relative compactified Jacobian of curves in $|H|$ is the moduli space

$$\overline{\mathrm{Pic}}^d(\mathcal{D}/\mathbb{P}^3) \coloneqq M_{\mathrm{Km}(J)}(w_d),$$

of semistable sheaves with Mukai vector w_d, with respect to a fixed w_d-generic polarization. It is a hyper-Kähler variety of K3$^{[3]}$ type by [O'Gr97] and [Yos99]. The moduli space $M_{\mathrm{Km}(J)}(w_d)$ parametrizes pure dimension-1 sheaves on $\mathrm{Km}(J)$ which are push-forward of torsion-free sheaves of rank 1 and degree d on curves in the linear system $|H|$. There is a morphism $\mathrm{supp} \colon \overline{\mathrm{Pic}}^d(\mathcal{D}/\mathbb{P}^3) \to \mathbb{P}^3$ which is a Lagrangian fibration, whose general fibres are the degree-d Picard varieties of the smooth curves in $|H|$.

3.5 The relative norm map

We denote by $B \subset \mathbb{P}^3$ the locus parametrizing smooth curves. The maps introduced in §3.2 give morphisms over B. Hence, we obtain rational maps

$$\mathrm{Nm}_\pi \colon \overline{\mathrm{Pic}}^d(\tilde{\mathcal{C}}/\mathbb{P}^3) \dashrightarrow \overline{\mathrm{Pic}}^d(\mathcal{D}/\mathbb{P}^3);$$
$$\pi^* \colon \overline{\mathrm{Pic}}^d(\mathcal{D}/\mathbb{P}^3) \dashrightarrow \overline{\mathrm{Pic}}^{2d}(\tilde{\mathcal{C}}/\mathbb{P}^3).$$

The pull-back π^* has been studied by Rapagnetta [Rap07] and by Mongardi, Rapagnetta and Saccà [MRS18]. They show that it gives a degree-2 dominant rational map

$$\overline{\mathrm{Pic}}^d(\mathcal{D}/\mathbb{P}^3) \dashrightarrow K_J(v_{2d}),$$

exhibiting a variety of OG6 type as the quotient of a variety of K3$^{[3]}$ type by a birational symplectic involution. We consider instead the norm map Nm_π.

LEMMA 3.6. *For each d, the restriction $\mathrm{Nm}_\pi \colon K_J(v_d) \dashrightarrow \overline{\mathrm{Pic}}^d(\mathcal{D}/\mathbb{P}^3)$ is dominant and generically finite of degree 2^5.*

Proof. For each k, the kth power map $[k] \colon \overline{\mathrm{Pic}}^d(\tilde{\mathcal{C}}/\mathbb{P}^3) \dashrightarrow \overline{\mathrm{Pic}}^{kd}(\tilde{\mathcal{C}}/\mathbb{P}^3)$ restricts to a rational map $[k] \colon K_J(v_d) \dashrightarrow K_J(v_{kd})$. This can be easily checked at smooth curves $\tilde{\mathcal{C}}_b$ in the family using the description of the Albanese map $L \mapsto \sum c_2(L)$.

Consider the following commutative diagram.

$$
\begin{array}{ccc}
K_J(v_d) & \xdashrightarrow{[2]} & K_J(v_{2d}) \\
\big\downarrow{\scriptstyle \mathrm{Nm}_\pi} & & \big\downarrow{\scriptstyle \mathrm{Nm}_\pi} \\
\overline{\mathrm{Pic}}^d(\mathcal{D}/\mathbb{P}^3) & \xdashrightarrow{[2]} & \overline{\mathrm{Pic}}^{2d}(\mathcal{D}/\mathbb{P}^3)
\end{array}
$$

Both the horizontal maps are dominant and generically finite of the same degree 2^6. Indeed, they preserve the fibres of the respective support morphisms, which, generically, are torsors

under abelian threefolds. Hence, the degree of [2] is the number of points of order 2 on an abelian threefold.

It thus suffices to prove the lemma for $\mathrm{Nm}_\pi\colon K_J(v_{2d}) \dashrightarrow \overline{\mathrm{Pic}}^{2d}(\mathcal{D}/\mathbb{P}^3)$. Since $K_J(v_{2d})$ is the closure of the image of π^*, the involution σ^* is the identity on $K_J(v_{2d})$. Hence, the rational map

$$K_J(v_{2d}) \xdashrightarrow{\mathrm{Nm}_\pi} \overline{\mathrm{Pic}}^{2d}(\mathcal{D}/\mathbb{P}^3) \xdashrightarrow{\pi^*} K_J(v_{4d})$$

coincides with multiplication by [2]; this composition is therefore dominant and generically finite of degree 2^6. Since $\pi^*\colon \overline{\mathrm{Pic}}^{2d}(\mathcal{D}/\mathbb{P}^3) \dashrightarrow K_J(v_{4d})$ is generically finite of degree 2, it follows that Nm_π is dominant and it has degree 2^5. □

3.7 The action of G

Assume now that d is odd, so that $K_J(v_d)$ is a smooth variety of Kum^3 type and the support map gives a Lagrangian fibration on it. In this case the group $\mathrm{Aut}_0(K_J(v_d))$ has been explicitly identified by Kim in [Kim21].

PROPOSITION 3.8. *Let d be odd. Then $G = \mathrm{Pic}^0(J)_2 \times \langle -1 \rangle$, where -1 acts on $K_J(v_d)$ via the pull-back $F \mapsto (-1)^*(F)$ of sheaves, and the action of $L \in \mathrm{Pic}^0(J)_2$ is given by $F \mapsto F \otimes L$. Any element of G preserves the fibres of the support fibration.*

Proof. We first show that $\mathrm{Pic}^0(J)_2 \times \langle -1 \rangle$ acts on $K_J(v_d)$ via automorphisms trivial on the second cohomology and which preserve the Lagrangian fibration given by the support morphism.

Since $-1\colon J \to J$ acts trivially on the cohomology of J in even degrees, the pull-back of sheaves defines an automorphism $(-1)^*$ of $K_J(v_d)$; this automorphism is trivial for d even but not for d odd. By [MW15, Lemma 2.34], the action of $(-1)^*$ is symplectic, i.e. the induced action on the transcendental cohomology $H^2_{\mathrm{tr}}(K_J(v_d), \mathbb{Z})$ is the identity. Since all curves in $|2\Theta|$ are stable under -1, the automorphism $(-1)^*$ preserves all fibres of the support fibration. The Picard rank of $K_J(v_d)$ is 2, because J is a general abelian surface by assumption (see [Yos01]). Hence, by Lemma 3.10, the induced action of $(-1)^*$ on $\mathrm{NS}(K_J(v_d))$ is also the identity. Since $H^2(K_J(v_d), \mathbb{Z})$ is torsion free, we conclude that $(-1)^*$ acts trivially on the second cohomology.

Now let $L \in \mathrm{Pic}^0(J)_2$. Let $B \subset |2\Theta|$ be the locus of smooth curves, and consider the universal curve $j\colon \tilde{\mathcal{C}}_B \hookrightarrow J \times B$. By [Wie18, Lemma 6.9], the corresponding pull-back $j^*\colon \mathrm{Pic}^0(J) \times B \to \overline{\mathrm{Pic}}^0(\tilde{\mathcal{C}}/\mathbb{P}^3)_B$ is an injection of abelian schemes. Note that the covering involution σ for the cover $\pi\colon \tilde{\mathcal{C}}_B \to \mathcal{C}_B$ is the restriction of $-1 \times \mathrm{id}_B$. Hence, the induced involution σ^* on $\overline{\mathrm{Pic}}^0(\tilde{\mathcal{C}}/\mathbb{P}^3)_B$ is the inverse map on $j_b^*(\mathrm{Pic}^0(J))$, for each $b \in B$. As $K_J(v_0)$ is the fixed locus of this involution, the 2-torsion line bundle L on J corresponds uniquely to a 2-torsion section s_L of $K_J(v_0)_B \to B$. For any d, the variety $K_J(v_d)_B$ is a torsor under $K_J(v_0)_B$ and, hence, $F \mapsto F \otimes L$ induces a birational automorphism g_L of $K_J(v_d)$, which restricts to a translation on the smooth fibres of the support morphism. This implies that g_L is symplectic. We deduce from Lemma 3.10 that g_L extends to a regular automorphism of $K_J(v_d)$, whose induced action is trivial on $\mathrm{NS}(K_J(v_d))$ as well.

According to [Kim21, Theorem 5.1] we have described all automorphisms in $\mathrm{Aut}_0(K_J(v_d))$ preserving the fibration $\mathrm{supp}\colon K_J(v_d) \to \mathbb{P}^3$. We claim now that an automorphism in $\mathrm{Aut}_0(K_J(v_d))$ which does not preserve the fibration cannot fix a component of dimension 4 on $K_J(v_d)$; this will show that $G = \mathrm{Pic}^0(J)_2 \times \langle -1 \rangle$.

Let $D = \mathrm{supp}^*(\mathcal{O}_{\mathbb{P}^n}(1)) \in \mathrm{Pic}(K_J(v_d))$ be the line bundle inducing the fibration. Since $\mathrm{Aut}_0(K_J(v_d))$ acts trivially on $H^2(K_J(v_d), \mathbb{Z})$, it acts on $|D| = \mathbb{P}^3 = |2\Theta|$. We obtain an action of $\mathrm{Aut}_0(K_J(v_d))/(\mathrm{Pic}^0(J)_2 \times \langle -1 \rangle) \cong (\mathbb{Z}/2\mathbb{Z})^4$ on $|2\Theta|$, which, up to conjugation by an

automorphism of $\mathbb{P}^3$, is identified with the action generated by

$$(x,y,z,w) \mapsto (z,w,x,y), \quad (x,y,z,w) \mapsto (y,x,w,z),$$
$$(x,y,z,w) \mapsto (x,y,-z,-w), \quad (x,y,z,w) \mapsto (x,-y,z,-w);$$

see [Gon94, Lemma 1.52, Note 1.4]. Thus, $h \in \mathrm{Aut}_0(K_J(v_d))$ either acts trivially on $|2\Theta|$ or it fixes a pair of skew lines. Assume by contradiction that $h \in \mathrm{Aut}_0(K_J(v_d))$ fixes a fourfold but does not preserve the support fibration. Then $(K_J(v_d))^h$ contains one of the varieties Z_j of K3$^{[2]}$ type from Theorem 2.2, and the image of Z_j via the fibration must be a line $R \subset |2\Theta|$ fixed by h. But, by a theorem of Matsushita [Mat99], the K3$^{[2]}$ variety Z_j does not admit any non-constant morphism to $\mathbb{P}^1$. $\qquad\square$

Remark 3.9. By [Ver87], the relative Prym variety $\mathcal{P} \to B$ of the double cover $\pi\colon \tilde{\mathcal{C}}_B \to \mathcal{C}_B$ is an isotrivial family over B with fibre J. Since the involution σ^* acts as the inverse on the image of the pull-back j^*, the constant abelian scheme $j^*(\mathrm{Pic}^0(J) \times B)$ is contained into, and hence equal to, the relative Prym variety.

LEMMA 3.10. *Let $f\colon X \to \mathbb{P}^n$ be a Lagrangian fibration, where X is a $2n$-dimensional projective hyper-Kähler variety of Picard rank 2. Let $g\colon X \dashrightarrow X$ be a birational automorphism such that $f \circ g = f$. Then g extends to a regular automorphism of X, and $g^*_{|\mathrm{NS}(X)}\colon \mathrm{NS}(X) \to \mathrm{NS}(X)$ is the identity.*

Proof. The Néron–Severi group $\mathrm{NS}(X)$ is a rank 2 lattice of signature $(1,1)$. Any birational automorphism g induces an isometry g^* of $H^2(X,\mathbb{Z})$ which restricts to an isometry of $\mathrm{NS}(X)$. Denote by $D \in \mathrm{NS}(X)$ the class of the line bundle $f^*(\mathcal{O}_{\mathbb{P}^n}(1))$, and pick $E \in \mathrm{NS}(X)$ such that D and E generate $\mathrm{NS}(X) \otimes \mathbb{Q}$. Since $f \circ g = f$, we have $g^*(D) = D$. Using that D is isotropic one sees that this forces $g^*(E) = E$ as well, so that g^* is the identity on $\mathrm{NS}(X)$; by [Fuj81], g extends to an automorphism of X. $\qquad\square$

We can now complete the proof of the main result of this section.

Proof of Theorem 3.1. Using the identification of G given in Proposition 3.8 we will show that the norm map descends to a birational map $K_J(v_d)/G \dashrightarrow \overline{\mathrm{Pic}}^d(\mathcal{D}/\mathbb{P}^3)$, for any odd d. The varieties $\overline{\mathrm{Pic}}^d(\mathcal{D}/\mathbb{P}^3)$ are of K3$^{[3]}$ type, and $\overline{\mathrm{Pic}}^3(\mathcal{D}/\mathbb{P}^3)$ is birational to $\mathrm{Km}(J)^{[3]}$, see [Bea99, Proposition 1.3].

For a smooth curve $\tilde{\mathcal{C}}_b \in |2\Theta|$ the norm map is induced by the map of divisors which sends $\sum_i a_i[P_i]$ to $\sum_i a_i[\pi(P_i)]$. It is then clear that $\mathrm{Nm}_\pi((-1)^*(F)) = \mathrm{Nm}_\pi(F)$ for any $F \in K_J(v_d)_B$. Instead let $L \in \mathrm{Pic}^0(J)_2$, and let $s_L\colon \mathbb{P}^3 \dashrightarrow K_J(v_0)$ be the rational section defined by $s_L(b) = j_b^*(L)$, where $j\colon \tilde{\mathcal{C}}_B \hookrightarrow J \times B$ denotes the natural inclusion. By Remark 3.9, the section s_L is contained in the relative Prym variety; in particular, the composition $\mathrm{Nm}_\pi \circ s_L$ gives the zero section of $\overline{\mathrm{Pic}}^0(\mathcal{D}/\mathbb{P}^3)_B \to \mathbb{P}^3$.

Therefore, for any $g \in G$ and $y \in K_J(v_d)_B$ we have $\mathrm{Nm}_\pi(g(y)) = \mathrm{Nm}_\pi(y)$, which means that $\mathrm{Nm}_\pi\colon K_J(v_d) \dashrightarrow \overline{\mathrm{Pic}}^d(\mathcal{D}/\mathbb{P}^3)$ descends to a rational map

$$\overline{\mathrm{Nm}}_\pi\colon K_J(v_d)/G \dashrightarrow \overline{\mathrm{Pic}}^d(\mathcal{D}/\mathbb{P}^3).$$

This map is, in fact, birational, because $\mathrm{Nm}_\pi\colon K_J(v_d) \dashrightarrow \overline{\mathrm{Pic}}^d(\mathcal{D}/\mathbb{P}^3)$ is generically finite of degree 2^5 by Lemma 3.6, and G has also order 2^5. $\qquad\square$

4. The K3 surface associated to a Kum^3 variety

In this section we give the proof of Theorems 1.1 and 1.2. We let K be a manifold of Kum^3 type and consider the quotient K/G by the group $G \cong (\mathbb{Z}/2\mathbb{Z})^5$ of Definition 2.1. We will show that the blow-up of the singular locus of K/G yields a hyper-Kähler manifold Y_K of K3$^{[3]}$ type.

4.1 The singularities of K/G

By Theorem 2.2, the locus $Z = \bigcup_{1 \neq g \in G} K^g$ is the union of 16 irreducible components Z_i, for $i = 1, \ldots, 16$. Denote by $X_i \subset K/G$, the image of Z_i via the quotient map $q \colon K \to K/G$. We introduce the following stratification of K/G into closed subspaces:

$$X^j := \{x \in K/G \mid x \text{ belongs to at least } j \text{ components } X_i\};$$

clearly, $X^0 = K/G$.

PROPOSITION 4.2. *The subspace X^j is empty for $j \geqslant 4$. For $j < 4$, a point $x \in X^j \setminus X^{j+1}$ has a neighborhood $U_x \subset K/G$ analytically isomorphic to*

$$(\mathbb{C}^2/\iota)^j \times (\mathbb{C}^2)^{3-j},$$

where ι is the involution $(x, y) \mapsto (-x, -y)$.

Proof. Theorem 2.2 implies immediately that X^j is empty for $j \geqslant 4$. The group G acts freely on $q^{-1}(X^0 \setminus X^1)$ and, hence, $X^0 \setminus X^1$ is smooth.

If $\mathcal{K} \to B$ is a smooth proper family of Kum^3 manifolds, the quotient $\mathcal{K}/G \to B$ is a locally trivial family (see [Fuj83, Lemma 3.10]): any point $x \in \mathcal{K}/G$ has a neighborhood U_x of the form to $f(U_x) \times (f^{-1}(f(x)) \cap U_x)$. Therefore, to prove the proposition we may assume that K is the generalized Kummer sixfold on an abelian surface A.

In this case the components Z_i are the explicit $\overline{W_\tau}$, for $\tau \in A_2$ (see Definition 2.6). Recall that $\overline{W_\tau} \subset K$ is the unique positive-dimensional component of the fixed locus of $(\tau, -1) \in G$, and that the induced action of $G/\langle (\tau, -1) \rangle$ on $\overline{W_\tau}$ is faithful. For any $z \in \overline{W_\tau}$ there is a decomposition $T_z(K) = N_{\overline{W_\tau}|K,z} \oplus T_z(\overline{W_\tau})$, where the first factor is the normal space. The action of $(\tau, -1)$ on $T_z(K)$ is $(-1, 1)$. Now let $x \in X^1 \setminus X^2$, and let $z \in K$ be a preimage of x. By definition, z belongs to exactly one component $\overline{W_\tau}$. The stabilizer is $G_z = \langle (\tau, -1) \rangle$. By the above, there exists a neighborhood V_z of $z \in K$ such that $g(V_z) \cap V_z$ is empty for any $g \notin G_z$, and $V_z = (\mathbb{C}^2)^3$ with $(\tau, -1)_{|V_z} = (\iota, \mathrm{id}, \mathrm{id})$. The image of V_z under the quotient map is thus a neighborhood $U_x = (\mathbb{C}^2/\iota) \times (\mathbb{C}^2)^2$ of x in K/G.

In the other cases we proceed similarly. Let $x \in X^2 \setminus X^3$, and let $z \in K$ be one of its preimages. Then z belongs to two distinct components $\overline{W_{\tau_1}}$ and $\overline{W_{\tau_2}}$. The stabilizer $G_z \cong (\mathbb{Z}/2\mathbb{Z})^2$ is the subgroup $\langle (\tau_1, -1), (\tau_2, -1) \rangle$ of G. There is a decomposition $T_z(K) = N_{\overline{W_{\tau_1}}|K,z} \oplus N_{\overline{W_{\tau_2}}|K,z} \oplus T_z(\overline{W_{\tau_1}} \cap \overline{W_{\tau_2}})$, and the action of G_z is generated by $(-1, 1, 1)$ and $(1, -1, 1)$. This implies that the image in K/G of a sufficiently small neighborhood of z in K is isomorphic to $(\mathbb{C}^2/\iota)^2 \times (\mathbb{C}^2)$.

Finally, let $x \in X^3$ and $z \in K$ a preimage of it. Then z belongs to exactly three components $\overline{W_{\tau_i}}$, for $i = 1, 2, 3$. The stabilizer G_z is the subgroup generated by the involutions $(\tau_i, -1)$, for $i = 1, 2, 3$, and there is a decomposition $T_z(K) = \bigoplus_{i=1}^3 N_{\overline{W_{\tau_i}}|K,z}$ of the tangent space. The action of $(\tau_i, -1)$ on $T_z(K)$ is -1 on the ith summand and the identity on the complement. We conclude that there exists a neighborhood V_z of z in K whose image in K/G is isomorphic to $(\mathbb{C}^2/\iota)^3$. $\square$

4.3 The symplectic resolution of K/G

We will now conclude the proof of Theorem 1.1 in several steps. First we show that the blow-up of the singular locus of K/G resolves the singularities.

Proposition 4.4. *Let $Y_K \coloneqq \mathrm{Bl}_{X^1}(K/G)$ be the blow-up of the singular locus with reduced structure. Then Y_K is a smooth manifold. It is identified with the quotient by G of the blow-up of K at $Z = \bigcup_{1 \neq g \in G} K^g$.*

The proposition will be reduced to the following elementary statement.

Lemma 4.5. *Let $\iota \colon \mathbb{C}^2 \to \mathbb{C}^2$ be the involution $(x, y) \mapsto (-x, -y)$, let j, k be positive integers. Then the blow-up of $(\mathbb{C}^2/\iota)^j \times (\mathbb{C}^2)^k$ along its singular locus is isomorphic to $(\mathrm{Bl}_0(\mathbb{C}^2/\iota))^j \times (\mathbb{C}^2)^k$. It is thus smooth, and identified with the quotient $(\mathrm{Bl}_0(\mathbb{C}^2)/\iota)^j \times (\mathbb{C}^2)^k$ of $(\mathrm{Bl}_0(\mathbb{C}^2))^j \times (\mathbb{C}^2)^k$.*

Proof. Given noetherian schemes T_1 and T_2 and closed subschemes $V_1 \subsetneq T_1$ and $V_2 \subsetneq T_2$, we have

$$\mathrm{Bl}_{(V_1 \times T_2) \cup (T_1 \times V_2)}(T_1 \times T_2) = \mathrm{Bl}_{V_1}(T_1) \times \mathrm{Bl}_{V_2}(T_2).$$

It is easy to see this when T_1 and T_2 are affine schemes, using the definition of blowing-up via the Proj construction [Har77, II, § 7]. The singular locus of $(\mathbb{C}^2/\iota)^j \times (\mathbb{C}^2)^k$ is the union of $\mathrm{pr}_i^{-1}(\{0\})$ for i from 1 to j, where $\mathrm{pr}_i \colon (\mathbb{C}^2/\iota)^j \times (\mathbb{C}^2)^k \to (\mathbb{C}^2/\iota)$ is the projection onto the ith factor. Via the above observation, the statement is reduced to the case $j = 1$, $k = 0$, which is the well-known minimal resolution of an isolated A_1-singularity of a surface. $\square$

Proof of Proposition 4.4. By Proposition 4.2, any point $x \in K/G$ has a neighborhood of the form $(\mathbb{C}^2/\iota)^j \times (\mathbb{C}^2)^k$. Therefore, Lemma 4.5 immediately implies that the blow-up Y_K of the singular locus is smooth. It also implies that the natural birational map $\mathrm{Bl}_Z(K)/G \dashrightarrow Y_K$ extends to an isomorphism. $\square$

We will now use a criterion due to Fujiki [Fuj83] to show that Y_K is symplectic, i.e. that it admits a nowhere degenerate holomorphic 2-form.

Proposition 4.6. *The manifold Y_K is hyper-Kähler.*

Proof. The singular space K/G is a primitively symplectic V-manifold in the sense of Fujiki. According to his [Fuj83, Proposition 2.9], to show that Y_K admits a symplectic form it suffices to check that the following two conditions are satisfied:

(i) for each component X_j of the singular locus of K/G, a general point $x \in X_j$ has a neighborhood $U_x = A \times V_x$ in K/G, where A is a surface and $V_x = U_x \cap X_j$ is a smooth neighborhood of x in X_j;

(ii) the restriction of $p \colon Y_K \to K/G$ to the preimage of U_x is the product

$$p' \times \mathrm{id} \colon \tilde{A} \times V_x \to A \times V_x,$$

where p' is the minimal resolution of A.

We have already shown in the course of proof of Propositions 4.2 and 4.4 that these conditions hold, with A a neighborhood of the ordinary node $0 \in \mathbb{C}^2/\iota$, whose minimal resolution is the blow-up of the singular point.

Hence, Y_K is symplectic. From its description as the quotient $\mathrm{Bl}_Z(K)/G$ we obtain $h^{2,0}(Y_K) = 1$ (see also § 4.7). Finally, Y_K is simply connected by [Fuj83, Lemma 1.2]. $\square$

We can now complete the proof of our main result.

Proof of Theorem 1.1. Let $\mathcal{K} \to B$ be a smooth proper family of manifolds of Kum^3 type. Up to a finite étale base-change, G acts fibrewise on $\mathcal{K}$, and the fixed loci of automorphisms in G are smooth families over B. By Theorem 2.2, the union $\mathcal{Z}$ of $\mathcal{K}^g$ for $1 \neq g \in G$ has 16 components

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$\mathcal{Z}_i$, each of which is a smooth family of manifolds of K3[2] type over B. The blow-up of $\mathcal{K}$ along $\mathcal{Z}$ is a smooth family $\widetilde{\mathcal{K}}$ over B, and the action of G extends to a fibrewise action on $\widetilde{\mathcal{K}}$. By Propositions 4.4 and 4.6, the quotient $\mathcal{Y} = \widetilde{\mathcal{K}}/G$ is a smooth proper family of hyper-Kähler manifolds over B, with fibre over the point $b \in B$ the manifold $Y_{\mathcal{K}_b}$. It thus suffices to find a single K of Kum^3 type such that Y_K is hyper-Kähler of K3[3] type.

Consider the sixfold $K_J(v_3)$ introduced in § 3, which is constructed from a Beauville–Mukai system on a principally polarized abelian surface J. By Theorem 3.1, in this case $Y_{K_J(v_3)}$ is birational to the Hilbert scheme $\mathrm{Km}(J)^{[3]}$ on the Kummer K3 surface associated to J. By [Huy99, Theorem 4.6], birational hyper-Kähler manifolds are deformation equivalent, and therefore $Y_{K_J(v_3)}$ is of K3[3] type. $\qquad\square$

4.7 The associated K3 surface

The following computation is entirely analogous to [Flo23, § 3]. If Z is a compact Kähler manifold, we let $H^2_{\mathrm{tr}}(Z, \mathbb{Z})$ denote the smallest sub-Hodge structure of $H^2(Z, \mathbb{Z})$ whose complexification contains $H^{2,0}(Z)$. Let K be a manifold of Kum^3 type, and denote by $\widetilde{K}$ the blow-up of K along $\bigcup_{1 \neq g \in G} K^g$. By Proposition 4.4 we have a commutative diagram
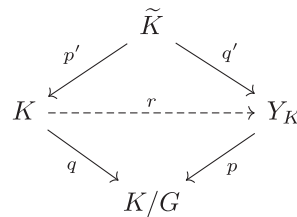
where p, p' are blow-up maps and q, q' are the quotient maps for the action of G.

The pull-back gives an isomorphism $p'^*: H^2_{\mathrm{tr}}(K, \mathbb{Z}) \to H^2_{\mathrm{tr}}(\widetilde{K}, \mathbb{Z})$. Since G acts trivially on $H^2(K, \mathbb{Z})$, the push-forward $q'_*: H^2_{\mathrm{tr}}(\widetilde{K}, \mathbb{Z}) \to H^2_{\mathrm{tr}}(Y_K, \mathbb{Z})$ is injective, and $q'^* q'_*$ is multiplication by 2^5 on $H^2_{\mathrm{tr}}(\widetilde{K}, \mathbb{Z})$. It follows that

$$r_*: H^2_{\mathrm{tr}}(K, \mathbb{Z}) \to H^2_{\mathrm{tr}}(Y_K, \mathbb{Z})$$

becomes an isomorphism of Hodge structures after tensoring with $\mathbb{Q}$. Denote by q_K and q_{Y_K} the Beauville–Bogomolov forms on $H^2(K, \mathbb{Z})$ and $H^2(Y_K, \mathbb{Z})$, respectively.

LEMMA 4.8. *For any $x \in H^2_{\mathrm{tr}}(K, \mathbb{Z})$ we have $q_{Y_K}(r_*(x), r_*(x)) = 2^9 q_K(x, x)$. Therefore,*

$$\frac{1}{16} r_*: H^2_{\mathrm{tr}}(K, \mathbb{Q})(2) \xrightarrow{\sim} H^2_{\mathrm{tr}}(Y_K, \mathbb{Q}),$$

is a rational Hodge isometry, where (2) indicates that the form is multiplied by 2.

Proof. Let c_K and c_Y be the Fujiki constants of K and Y, respectively [Fuj87]. This means that we have $\int_K x^6 = c_K q_K(x, x)^3$ for any $x \in H^2(K, \mathbb{Z})$, and similarly for Y_K. Let $x \in H^2_{\mathrm{tr}}(K, \mathbb{Z})$;

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since x is G-invariant, we have $r^*(r_*x) = 2^5 x$. We compute

$$
\begin{aligned}
q_{Y_K}(r_*x, r_*x)^3 &= \frac{1}{c_{Y_K}} \int_{Y_K} (r_*x)^6 \\
&= \frac{1}{2^5 c_{Y_K}} \int_K (r^* r_*x)^6 \\
&= \frac{1}{2^5 c_{Y_K}} \int_K (2^5 x)^6 \\
&= \frac{2^{25} c_K}{c_{Y_K}} q_K(x,x)^3.
\end{aligned}
$$

By [Rap08], we have $c_{Y_K} = 15$ and $c_K = 60$. Hence, $q_{Y_K}(r_*x, r_*x) = 2^9 q_K(x,x)$, so $r_*/2^4$ multiplies the form by 2 and yields the claimed rational Hodge isometry. $\qquad\square$

We now prove that every projective K of Kum^3 type has a naturally associated K3 surface.

Proof of Theorem 1.2. Let K be a projective variety of Kum^3 type and let Y_K be the crepant resolution of K/G. By the above lemma, the transcendental lattice $H^2_{\mathrm{tr}}(Y_K, \mathbb{Z})$ is an even lattice of signature $(2, k)$, and rank at most 6. By [Mor84, Corollary 2.10], there exists a K3 surface S_K such that $H^2_{\mathrm{tr}}(S_K, \mathbb{Z})$ is Hodge isometric to $H^2_{\mathrm{tr}}(Y_K, \mathbb{Z})$. A criterion independently due to Mongardi and Wandel [MW15] and Addington [Add16, Proposition 4] ensures that Y_K is birational to $M_{S_K, H}(v)$, for some primitive Mukai vector v and a v-generic polarization H on S_K. The surface S_K is uniquely determined up to isomorphism, because two K3 surfaces of Picard rank at least 12 with Hodge isometric transcendental lattices are isomorphic, see [Huy16, Chapter 16, Corollary 3.8] $\qquad\square$

Remark 4.9. The K3 surfaces S_K come in countably many 4-dimensional families. A projective K_0 of Kum^3 type with Picard rank 1 gives such a family, consisting of the K3 surfaces S such that $H^2_{\mathrm{tr}}(S, \mathbb{Z})$ is a sublattice of $H^2_{\mathrm{tr}}(S_{K_0}, \mathbb{Z})$.

Up to isogeny, the K3 surfaces obtained are easily characterized as follows. Recall that $\Lambda_{\mathrm{Kum}^3} = U^{\oplus 3} \oplus \langle -8 \rangle$ is the lattice $H^2(K, \mathbb{Z})$ for K of Kum^3 type.

LEMMA 4.10. *Let S be a projective K3 surface. The following are equivalent:*

– *there exists an isometric embedding of $H^2_{\mathrm{tr}}(S, \mathbb{Q})$ into $\Lambda_{\mathrm{Kum}^3}(2) \otimes_{\mathbb{Z}} \mathbb{Q}$;*
– *there exist a projective variety K of Kum^3 type with associated K3 surface S_K and a rational Hodge isometry $H^2_{\mathrm{tr}}(S, \mathbb{Q}) \xrightarrow{\sim} H^2_{\mathrm{tr}}(S_K, \mathbb{Q})$.*

Proof. The second assertion implies the first thanks to Lemma 4.8. Conversely, let $\Phi \colon H^2_{\mathrm{tr}}(S, \mathbb{Q}) \hookrightarrow \Lambda_{\mathrm{Kum}^3}(2) \otimes_{\mathbb{Z}} \mathbb{Q}$ be an isometric embedding. Choose a primitive sublattice $T \subset \Lambda_{\mathrm{Kum}^3}$ such that $T(2) \otimes_{\mathbb{Z}} \mathbb{Q}$ coincides with the image of Φ. Equip T with the Hodge structure induced by that on $H^2_{\mathrm{tr}}(S, \mathbb{Q})$ via Φ. By the surjectivity of the period map [Huy99], there exists a manifold K of Kum^3 type such that $H^2_{\mathrm{tr}}(K, \mathbb{Z})$ is Hodge isometric to T, and Lemma 4.8 gives a Hodge isometry $H^2_{\mathrm{tr}}(S, \mathbb{Q}) \xrightarrow{\sim} H^2_{\mathrm{tr}}(S_K, \mathbb{Q})$. Since the signature of T is necessarily $(2, k)$ with $k \leqslant 4$, Huybrechts' projectivity criterion [Huy99] implies that K is projective. $\qquad\square$

5. Applications to the Hodge conjecture

In this section we prove Theorems 1.3 and 1.4. Throughout, all cohomology groups are taken with rational coefficients, which are thus suppressed from the notation.

We start with a simple observation. Let X, Y be smooth projective varieties and let $\phi\colon H^2(X) \to H^2(Y)$ be a morphism of Hodge structures.

LEMMA 5.1. *The morphism ϕ is induced by an algebraic correspondence if and only if its restriction to $H^2_{\mathrm{tr}}(X)$ is so.*

Proof. There is a decomposition $H^2(X) = H^2_{\mathrm{tr}}(X) \oplus H^2_{\mathrm{alg}}(X)$, where $H^2_{\mathrm{alg}}(X)$ is spanned by cycle classes of divisors. Similarly, $H^2(Y) = H^2_{\mathrm{tr}}(Y) \oplus H^2_{\mathrm{alg}}(Y)$. Any morphism $H^2_{\mathrm{tr}}(X) \to H^2_{\mathrm{alg}}(Y)$ or $H^2_{\mathrm{alg}}(X) \to H^2_{\mathrm{tr}}(Y)$ of Hodge structures is trivial. Hence, ϕ gives a Hodge class

$$\phi_{\mathrm{tr}} \oplus \phi_{\mathrm{alg}} \ \in \ \left(H^2_{\mathrm{tr}}(X)^\vee \otimes H^2_{\mathrm{tr}}(Y)\right) \oplus \left(H^2_{\mathrm{alg}}(X)^\vee \otimes H^2_{\mathrm{alg}}(Y)\right).$$

The lemma follows because the second summand consists of algebraic classes. $\qquad\square$

Proof of Theorem 1.4. Let K and K' be projective varieties of Kum^3 type and assume that $f\colon H^2(K) \xrightarrow{\sim} H^2(K')$ is a rational Hodge isometry. By the above lemma, it suffices to show that the component $f_{\mathrm{tr}}\colon H^2_{\mathrm{tr}}(K) \xrightarrow{\sim} H^2_{\mathrm{tr}}(K')$ is algebraic.

Let Y, Y' be the varieties of K3$^{[3]}$ type given by Theorem 1.1 applied to K and K', respectively, and let $r\colon K \dashrightarrow Y$ and $r'\colon K' \dashrightarrow Y'$ be the corresponding rational maps. They induce isomorphisms of Hodge structures $r_*\colon H^2_{\mathrm{tr}}(K) \xrightarrow{\sim} H^2_{\mathrm{tr}}(Y)$ and $r'_*\colon H^2_{\mathrm{tr}}(K') \xrightarrow{\sim} H^2_{\mathrm{tr}}(Y')$. Let $\bar{f}_{\mathrm{tr}}$ be defined as

$$\bar{f}_{\mathrm{tr}} \coloneqq r'_* \circ f_{\mathrm{tr}} \circ (r_*)^{-1}\colon H^2_{\mathrm{tr}}(Y) \to H^2_{\mathrm{tr}}(Y').$$

The inverse of r_* is $r^*/2^5$, and similarly for r'. It follows that f_{tr} is algebraic if and only if $\bar{f}_{\mathrm{tr}}$ is so. By Lemma 4.8, the map $\bar{f}_{\mathrm{tr}}$ is a rational Hodge isometry and, hence, it is algebraic by Markman's theorem in [Mar22]. $\qquad\square$

5.2 The Kuga–Satake correspondence

The Kuga–Satake construction associates an abelian variety to any polarized Hodge structure of K3 type. We briefly recall this construction, referring to [vGe00] and [Huy16, Chapter 4] for more details.

Let (V, q) be an effective polarized $\mathbb{Q}$-Hodge structure of weight 2 with $h^{2,0}(V) = 1$. The Clifford algebra $C(V)$ is defined as the quotient

$$C(V) \coloneqq \bigoplus_{k \geqslant 0} V^{\otimes k}/\langle v \otimes v - q(v,v)1\rangle_{v \in V}.$$

As a $\mathbb{Q}$-vector space, $C(V) \cong \bigwedge^\bullet(V)$; hence, $\dim(C(V)) = 2^{\mathrm{rk}(V)}$. The Clifford algebra is $\mathbb{Z}/2\mathbb{Z}$-graded, $C(V) = C^+(V) \oplus C^-(V)$. In [Del71], Deligne shows that the Hodge structure of V induces a Hodge structure of weight 1 on $C^+(V)$ of type $(1,0), (0,1)$, which admits a polarization and, hence, defines an abelian variety $\mathrm{KS}(V)$ up to isogeny. This $2^{\mathrm{rk}(V)-2}$-dimensional abelian variety is called the Kuga–Satake variety of V. Upon fixing some $v_0 \in V$, the action of V on $C(V)$ via left multiplication induces an embedding of weight 0 rational Hodge structures

$$V(1) \hookrightarrow \mathrm{End}(C^+(V)) = H^1(\mathrm{KS}(V))^\vee \otimes H^1(\mathrm{KS}(V)),$$

which maps v to the endomorphism $w \mapsto vwv_0$.

Remark 5.3. If $V' \subset V$ is a sub-Hodge structure such that $V'^\perp$ consists of Hodge classes, then $\mathrm{KS}(V)$ is isogenous to a power of $\mathrm{KS}(V')$. Replacing the form q with a non-zero rational multiple results in isogenous Kuga–Satake varieties. See [Huy16, Chapter 4, Example 2.4 and Proof of Proposition 3.3]

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Let X be a projective hyper-Kähler variety of dimension $2n$. The Kuga–Satake variety $\mathrm{KS}(X)$ of X is the abelian variety obtained from $(H^2_{\mathrm{tr}}(X), q_X)$ via the Kuga–Satake construction, where q_X is the restriction of the Beauville–Bogomolov form. Identifying $H^1(\mathrm{KS}(X))$ with its dual, there exists an embedding of Hodge structures of $H^2_{\mathrm{tr}}(X)$ into $H^2(\mathrm{KS}(X) \times \mathrm{KS}(X))$. According to the Hodge conjecture this embedding should be induced by an algebraic cycle.

CONJECTURE 5.4 (Kuga–Satake Hodge conjecture). Let X be a projective hyper-Kähler variety. There exists an algebraic cycle ζ on $X \times \mathrm{KS}(X) \times \mathrm{KS}(X)$ such that the associated correspondence induces an embedding of Hodge structures

$$\zeta_* \colon H^2_{\mathrm{tr}}(X) \hookrightarrow H^2(\mathrm{KS}(X) \times \mathrm{KS}(X)).$$

Remark 5.5. For a K3 surface X, the above form of the conjecture is equivalent to [vGe00, § 10.2]. To see this, let Mot be the category of Grothendieck motives over $\mathbb{C}$, in which morphisms are given by algebraic cycles modulo homological equivalence, see [And04]. The motive $\mathfrak{h}(X) \in \mathsf{Mot}$ of X decomposes as the sum of its transcendental part $\mathfrak{h}_{\mathrm{tr}}(X)$ and some motives of Hodge–Tate type. Since the standard conjectures hold for X and $\mathrm{KS}(X)$, the tensor subcategory of Mot generated by their motives is abelian and semisimple [Ara06, Theorem 4.1]. Therefore, Conjecture 5.4 and the formulation of van Geemen [vGe00, § 10.2] are both equivalent to $\mathfrak{h}^2_{\mathrm{tr}}(X)$ being a direct summand of $\mathfrak{h}^2(\mathrm{KS}(X) \times \mathrm{KS}(X))$ in the category Mot.

We will use the following easy lemma.

LEMMA 5.6. *Let X and Z be projective hyper-Kähler varieties. Assume that there exists an algebraic cycle γ on $Z \times X$ which induces a rational Hodge isometry*

$$\gamma_* \colon H^2_{\mathrm{tr}}(Z) \xrightarrow{\sim} H^2_{\mathrm{tr}}(X)(k),$$

for some non-zero $k \in \mathbb{Q}$. Then, if Conjecture 5.4 holds for X, it holds for Z as well.

Proof. By Remark 5.3 there exists an isogeny $\phi \colon \mathrm{KS}(X) \to \mathrm{KS}(Z)$ of Kuga–Satake varieties. It induces an isomorphism

$$\phi_* \colon H^2(\mathrm{KS}(X) \times \mathrm{KS}(X)) \xrightarrow{\sim} H^2(\mathrm{KS}(Z) \times \mathrm{KS}(Z))$$

of rational Hodge structures. If Conjecture 5.4 holds for X, there exists an algebraic cycle ζ on $X \times \mathrm{KS}(X) \times \mathrm{KS}(X)$ giving an embedding of Hodge structures

$$\zeta_* \colon H^2_{\mathrm{tr}}(X) \hookrightarrow H^2(\mathrm{KS}(X) \times \mathrm{KS}(X)).$$

It follows that the embedding of Hodge structure given by the composition

$$\phi_* \circ \zeta_* \circ \gamma_* \colon H^2_{\mathrm{tr}}(Z) \hookrightarrow H^2(\mathrm{KS}(Z) \times \mathrm{KS}(Z))$$

is induced by an algebraic cycle on $Z \times \mathrm{KS}(Z) \times \mathrm{KS}(Z)$. $\qquad\square$

For convenience of the reader we restate Theorem 1.3.

THEOREM 5.7. *Let S be a projective K3 surface such that there exists an isometric embedding of $H^2_{\mathrm{tr}}(S)$ into $\Lambda_{\mathrm{Kum}^3}(2) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then Conjecture 5.4 holds for S.*

Proof. By Lemma 4.10, there exists a projective variety K of Kum^3 type with associated K3 surface S_K and a rational Hodge isometry $t_0 \colon H^2_{\mathrm{tr}}(S) \xrightarrow{\sim} H^2_{\mathrm{tr}}(S_K)$. Denote by Y_K the crepant resolution of K/G given by Theorem 1.1 and let v, H be given by Theorem 1.2, so that Y_K is birational to the moduli space $M_{S_K,H}(v)$. Denote by $r \colon K \dashrightarrow Y_K$ the natural rational map of degree 2^5.

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By [Bus19, Huy19], the isometry t_0 is algebraic. Next, by [Muk87], there exists a quasi-tautological sheaf U over $S_K \times M_{S_K,H}(v)$, which means that there exists an integer ρ such that for any $F \in M_{S_K,H}(v)$ the restriction of U to $S_K \times \{F\}$ is $F^{\oplus \rho}$. Consider the algebraic class

$$\gamma := \frac{1}{\rho}\mathrm{ch}(U) \cdot \mathrm{pr}^*\sqrt{\mathrm{td}_{S_K}} \in H^\bullet(S_K \times M_{S_K,H}(v)),$$

where $\mathrm{pr}\colon S_K \times M_{S_K,H}(v) \to S_K$ is the projection. By [O'Gr97], its Künneth component $\gamma_3 \in H^6(S_K \times M_{S_K,H}(v))$ induces a Hodge isometry

$$t_1\colon H^2_{\mathrm{tr}}(S_K) \xrightarrow{\ \sim\ } H^2_{\mathrm{tr}}(M_{S_K,H}(v)).$$

By [CM13], the standard conjectures holds for $M_{S_K,H}(v)$ (this also follows from [Bül20] via the arguments of [Ara06]). Hence, all Künneth components of γ are algebraic. Now let $f\colon M_{S_K,H}(v) \dashrightarrow Y_K$ be a birational map. Then the push-forward $f_*\colon H^2_{\mathrm{tr}}(M_{S_K,H}(v)) \xrightarrow{\ \sim\ } H^2_{\mathrm{tr}}(Y_K)$ is a Hodge isometry. By Lemma 4.8, a multiple of the composition $r^* \circ f_*$ gives a Hodge isometry $t_2\colon H^2_{\mathrm{tr}}(M_{S_K,H}(v)) \xrightarrow{\ \sim\ } H^2_{\mathrm{tr}}(K)(2)$.

It follows that the Hodge isometry

$$t_2 \circ t_1 \circ t_0\colon H^2_{\mathrm{tr}}(S) \xrightarrow{\ \sim\ } H^2_{\mathrm{tr}}(K)(2)$$

is induced by an algebraic cycle on $S \times K$. The Kuga–Satake Hodge conjecture holds for K by [Voi22]. By Lemma 5.6, Conjecture 5.4 holds for the K3 surface S as well. $\qquad\square$

For the general projective variety K of Kum^3 type, $H^2_{\mathrm{tr}}(K,\mathbb{Q})$ is a rank-6 Hodge structure with Hodge numbers $(1,4,1)$ and the explicit knowledge of the quadratic form allows one to show that the Kuga–Satake variety is A^4, where A is an abelian fourfold of Weil type (cf. [vGe00, Theorem 9.2]). O'Grady moreover shows in [O'Gr21] that $H^1(A,\mathbb{Q}) \cong H^3(K,\mathbb{Q})$. This is a crucial ingredient in the subsequent work of Markman [Mar23] and Voisin [Voi22]. O'Grady's theorem further allows us to apply a result of Varesco [Var22] to prove the Hodge conjecture for all powers of the K3 surfaces appearing in Theorem 1.3.

COROLLARY 5.8. *Let S be a projective K3 surface such that there exists an isometric embedding of $H^2_{\mathrm{tr}}(S)$ into $\Lambda_{\mathrm{Kum}^3}(2) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the Hodge conjecture holds for all powers of S.*

Proof. By Remark 4.9, there exists a 4-dimensional family $\mathcal{S} \to B$ of projective K3 surfaces of general Picard rank 16 such that: for each $b \in B$ the fibre $\mathcal{S}_b$ is of the form S_{K_b} for some K_b of Kum^3 type and for some $0 \in B$ there exists a rational Hodge isometry $H^2_{\mathrm{tr}}(S) \xrightarrow{\ \sim\ } H^2_{\mathrm{tr}}(\mathcal{S}_0)$. By [Huy19], it is sufficient to prove the Hodge conjecture for all powers of $\mathcal{S}_0$.

To this end, it will be enough to check that $\mathcal{S} \to B$ satisfies the two assumptions of [Var22, Theorem 0.2]. As explained in [Var22, Theorem 4.1], the first of these assumptions is that the Kuga–Satake variety of a general K3 surface in the family is isogenous to A^4 for an abelian fourfold A of Weil type with trivial discriminant. This holds by O'Grady's theorem in [O'Gr21] as already mentioned. The other assumption is that the Kuga–Satake Hodge conjecture holds for the surfaces $\mathcal{S}_b$ for all $b \in B$, which is the content of Theorem 5.7. $\qquad\square$

Remark 5.9. The conclusions of Theorem 5.7 and Corollary 5.8 were known for the two 4-dimensional families of K3 surfaces studied in [Par88] and [ILP22]. The general fibre has transcendental lattice $U^{\oplus 2} \oplus \langle -2 \rangle^{\oplus 2}$ in the first case and $U^{\oplus 2} \oplus \langle -6 \rangle \oplus \langle -2 \rangle$ in the second. The reader may check that the K3 surfaces studied in [Par88] satisfy the assumption of our Theorem 5.7, while those appearing in [ILP22] do not.

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