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CURVES OF INFINITE LENGTH IN LABYRINTH FRACTALS

LIGIA L. CRISTEA[1] AND BERTRAN STEINSKY[2]

[1] *Technical University of Graz, Steyrergasse 30, 8010 Graz, Austria* (strublistea@gmail.com)
[2] *Department of Knowledge-Based Mathematical Systems, Johannes Kepler University Linz, Altenberger Strasse 69, 4040 Linz, Austria* (bertran.steinsky@jku.at)

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Abstract We define an infinite class of fractals, called horizontally and vertically blocked labyrinth fractals, which are dendrites and special Sierpiński carpets. Between any two points in the fractal there is a unique arc a; the length of a is infinite and the set of points where no tangent to a exists is dense in a.

Keywords: infinite curve length; fractal; dendrite; tree; infinite distance

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1. Introduction

Having discussed 4×4-labyrinth fractals in [**5**], we now devote our attention to $m \times m$-labyrinth fractals for $m \geqslant 5$. These fractal sets are constructed iteratively, as described in § 1.1.1. Additionally, we demand three properties be satisfied, which we formulate in § 1.1.2. An example for the first two steps of the construction is given in Figure 1, where the black squares indicate the points that are not in the fractal.

1.1. Labyrinth fractals

In this section it will be helpful to keep in mind the example in Figure 1, which we have already discussed [**5**].

1.1.1. *Construction*

We note that Figure 1 gives an example of the construction we now describe. Let $Q \subseteq [0,1] \times [0,1]$ be a square. Then, for any point $(z_1, z_2) \in [0,1] \times [0,1]$, we define the function

$$P_Q(z_1, z_2) = (qz_1 + x_1, qz_2 + x_2),$$

where (x_1, x_2) is the lower-left corner of Q and q is the side length of Q.

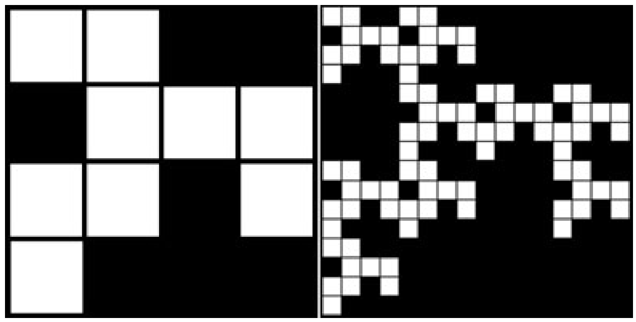

Figure 1. L_1 and L_2.

Let $m \geqslant 1$,

$$S_{i,j,m} = \left\{(z_1, z_2) \;\middle|\; \frac{i}{m} \leqslant z_1 \leqslant \frac{i+1}{m} \text{ and } \frac{j}{m} \leqslant z_2 \leqslant \frac{j+1}{m}\right\},$$

and $\mathcal{S}_m = \{S_{i,j,m} \mid 0 \leqslant i \leqslant m-1 \text{ and } 0 \leqslant j \leqslant m-1\}$. We let $\mathcal{W}_1 \subset \mathcal{S}_m$, and call it the *set of white squares of order* 1. Then we define $\mathcal{B}_1 = \mathcal{S}_m \setminus \mathcal{W}_1$ as the *set of black squares of order* 1. For $n \geqslant 2$ we define the *set of white squares of order* n by

$$\mathcal{W}_n = \bigcup_{W_1 \in \mathcal{W}_1, W_{n-1} \in \mathcal{W}_{n-1}} \{P_{W_{n-1}}(W_1)\}.$$

We note that $\mathcal{W}_n \subset \mathcal{S}_{m^n}$, and we define the *set of black squares of order* n by $\mathcal{B}_n = \mathcal{S}_{m^n} \setminus \mathcal{W}_n$. For $n \geqslant 1$, we define $L_n = \bigcup_{W \in \mathcal{W}_n} W$. Therefore, $\{L_n\}_{n=1}^{\infty}$ is a monotonically decreasing sequence of compact sets. We write $L_\infty = \bigcap_{n=1}^{\infty} L_n$, i.e. the *limit set of* $\mathcal{W}_1$.

1.1.2. *Definition of labyrinth fractals*

We note that the graph concepts we use here were defined by Cristea and Steinsky [**5**].

For $n \geqslant 1$, we define $\mathcal{G}(\mathcal{W}_n) \equiv (\mathcal{V}(\mathcal{G}(\mathcal{W}_n)), \mathcal{E}(\mathcal{G}(\mathcal{W}_n)))$ to be the graph of $\mathcal{W}_n$, i.e. the graph whose vertices $\mathcal{V}(\mathcal{G}(\mathcal{W}_n))$ are the white squares in $\mathcal{W}_n$, and whose edges $\mathcal{E}(\mathcal{G}(\mathcal{W}_n))$ are the unordered pairs of white squares that have a common side. The *top row* of order n is the set of all white squares in $\{S_{i,m^n-1,m^n} \mid 0 \leqslant i \leqslant m^n - 1\}$. The bottom row, left column and right column are defined analogously. A *top exit* in $\mathcal{W}_n$ is a white square in the top row of order n, such that there is a white square in the same column in the bottom row of order n. A *bottom exit* is defined analogously. A *left exit* in $\mathcal{W}_n$ is a white square in the left column of order n, such that there is a white square in the same row in the right column of order n. A *right exit* is defined analogously. $\mathcal{W}_1$ has the following three properties, which we will use to define labyrinth sets. We call $\mathcal{W}_n$ an $m \times m$-*labyrinth set* if $\mathcal{W}_n$ satisfies Properties 1.1, 1.2 and 1.3.

Property 1.1. $\mathcal{G}(\mathcal{W}_n)$ is a tree.

Property 1.2. Exactly one top exit in $\mathcal{G}(\mathcal{W}_n)$ lies in the top row (of order n), exactly one bottom exit lies in the bottom row, exactly one left exit lies in the left column and exactly one right exit lies in the right column.

Property 1.3. If there is a white square in $\mathcal{W}_n$ at a corner of $\mathcal{W}_n$, then there is no white square in $\mathcal{W}_n$ at the diagonal opposite corner of $\mathcal{W}_n$.

We call the limit set L_∞ a *labyrinth fractal* and notice that for $\mathcal{W}_1 = \{W_1, \dots, W_n\}$ the limit set L_∞ is the attractor of the iterated function system (IFS) $\{P_{W_1}, \dots, P_{W_n}\}$, which satisfies the open set condition. Moreover, labyrinth fractals can also be obtained by Moran construction [**10**, **12**, **13**].

1.2. Overview

In § 2 we recapitulate proven results [**5**] that we need for this paper. One of the facts mentioned therein [**5**] is that labyrinth fractals are dendrites, which yields that between any two points in a labyrinth fractal there is a unique arc in the labyrinth fractal.

Cristea [**4**] introduced certain Moran fractals in the unit square, called limit net sets, and showed several connectedness properties as connectedness, local connectedness or arcwise connectedness. The construction of net sets is similar to that of labyrinth fractals, but net sets need not be self-similar and the black squares satisfy certain conditions, which in a sense say that they are 'well distributed'. In § 3, we define horizontally and vertically blocked labyrinth fractals, which are labyrinth fractals such that the set where the construction begins contains no row and no column of white squares: a property they have in common with net sets. An example of such a set is shown on the left of Figure 1. This section also contains the main results of this paper, which are summed up in Theorem 3.18. We show that between any two points in the limit set of a horizontally and vertically blocked $m \times m$-labyrinth set there is a unique arc a in the fractal, the length of a is infinite and the set of all points at which no tangent to a exists is dense in a. To gain this result, we first show that the path matrix (which is defined in § 2) of any horizontally and vertically blocked $m \times m$-labyrinth set is primitive. Then we use the Perron–Frobenius Theorem for primitive matrices to obtain the asymptotic behaviour of the path lengths in the nth step of the construction as n tends to ∞. Furthermore, we prove that the spectral radius r of the path matrix is greater than m, and that the box-counting dimension of a is $\log(r)/\log(m)$.

We note that the literature provides many examples of continuous curves with infinite length, like the von Koch curve [**16**, **17**], the Peano curve [**11**], the Hilbert curve [**8**] or the dragon curve [**2**, Example 5.1.6]. Following Akiyama *et al.* [**1**, Remark 3.2] or Hata [**7**, Remark 2, p. 391], we may find more examples of curves with infinite length. But we note that not all of these curves have the property that the distance between any two points of the curve (seen as a set of points) is infinite, as is the case for labyrinth fractals. Furthermore, we note that this property does not follow from the fact that labyrinth fractals are attractors of self-similar IFSs with open set condition and have Hausdorff dimension between 1 and 2, since this is also satisfied for the well-known Sierpiński carpet, where the distance between any two points is finite.

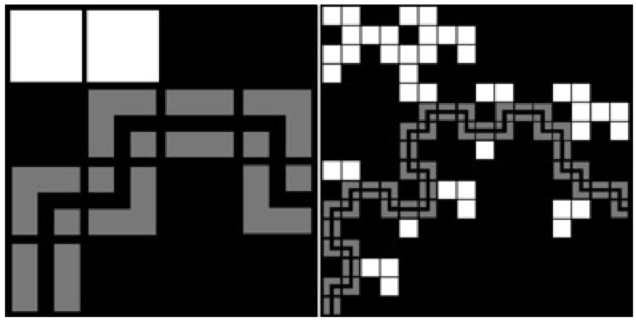

Figure 2. Paths from the right to the bottom exit of $\mathcal{W}_1$ and $\mathcal{W}_2$.

2. Previous results for labyrinth fractals

All the results in this section have been proven by Cristea and Steinsky [**5**]. Nevertheless, we present them here, as they will be used later.

A topological space X is an *arc* if there is a homeomorphism h from $[0,1]$ to X. We say that X is an arc between $h(0)$ and $h(1)$. A *curve* c is the image of a continuous function that is defined on a real interval. We say a topological space is a *simple closed curve* if it is homeomorphic to the unit cycle. A topological space X is *locally connected* if, for each point $x \in X$, each neighbourhood of x contains a connected open neighbourhood of x. A *continuum* is a compact connected $\mathcal{T}_2$-space, and a *dendrite* is a locally connected continuum that contains no simple closed curve. We refer to Bandt [**3**, Proposition 10] and note that between any pair of points $x \neq y$ in L_∞ there is a unique arc [**9**, Corollary 2, p. 301].

Theorem 2.1. *L_∞ is a dendrite.*

As a further remark, the Hausdorff dimension of L_∞ is

$$\dim_H = \frac{\log |\mathcal{W}_1|}{\log m},$$

which follows from known results for self-similar sets [**6**, Theorem 9.3]. Since in the case of labyrinth fractals we have $|\mathcal{W}_1| > m$, it follows that $\dim_H > 1$. We call a path in $\mathcal{G}(\mathcal{W}_n)$ a ◫-*path* if it leads from the top exit to the bottom exit of W_n. The ⊟-, ◰-, ◲-, ◱- and ◳-*paths* lead from left to right, top to right, right to bottom, bottom to left and left to top exits, respectively. We write ◫$_n$, ⊟$_n$, ◰$_n$, ◲$_n$, ◱$_n$ and ◳$_n$ for the length of the respective path.

We will now give a construction for the path between all possible pairs of exits. To demonstrate the idea, let us start with a path between the right exit and the bottom exit, as shown in Figure 2, but we note that the construction we describe here works for all labyrinth fractals.

First, we find the path between the right and the bottom exit of $\mathcal{W}_1$. Then we denote each white square in the path according to its neighbours within the path. If it has a top and a bottom neighbour within the path it is called ◫-*square* (with respect to the

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path). It is called ━-, ┗-, ┏-, ┓- and ┛-*square* if its neighbours are on the left and on the right, on the top and on the right, on the right and on the bottom, on the bottom and on the left and on the left and on the top, respectively. If the white square is an exit, it is supposed to have a neighbour outside the side of the exit. If it is a bottom exit, for example, then it has no bottom neighbour within the path but is supposed to have a neighbour below, outside the bottom, in addition to its inside neighbour.

We do this for all possible paths between two exits in $\mathcal{G}(\mathcal{W}_1)$ in the same way.

Lemma 2.2. *There is a non-negative* 6×6*-matrix* M *such that, for* $n \geqslant 1$,

$$\begin{pmatrix} \text{┃}_n \\ \text{━}_n \\ \text{┗}_n \\ \text{┏}_n \\ \text{┓}_n \\ \text{┛}_n \end{pmatrix} = M \cdot \begin{pmatrix} \text{┃}_{n-1} \\ \text{━}_{n-1} \\ \text{┗}_{n-1} \\ \text{┏}_{n-1} \\ \text{┓}_{n-1} \\ \text{┛}_{n-1} \end{pmatrix} = M^n \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{2.1}$$

We note that in the example shown in Figure 1, we have

$$M = \begin{pmatrix} 2 & 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 & 2 & 0 \\ 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

We call the matrix M in Lemma 2.2 *the path matrix of the labyrinth set* $\mathcal{W}_1$. For $n \geqslant 1$ and $W_1, W_2 \in \mathcal{G}(\mathcal{W}_n)$, let $p_n(W_1, W_2)$ be the path in $\mathcal{G}(\mathcal{W}_n)$ from W_1 to W_2.

Lemma 2.3 (arc construction). *Let* $a, b \in L_\infty$, *where* $a \neq b$. *For all* $n \geqslant 1$, *there are* $W_n(a), W_n(b) \in V(\mathcal{G}(\mathcal{W}_n))$ *such that*

(a) $W_1(a) \supseteq W_2(a) \supseteq \cdots$,

(b) $W_1(b) \supseteq W_2(b) \supseteq \cdots$,

(c) $a = \bigcap_{n=1}^{\infty} W_n(a)$,

(d) $b = \bigcap_{n=1}^{\infty} W_n(b)$.

(e) *the set* $\bigcap_{n=1}^{\infty}(\bigcup_{W \in p_n(W_n(a), W_n(b))} W)$ *is an arc between* a *and* b.

Let $T_n \in \mathcal{W}_n$ be the top exit of $\mathcal{W}_n$, for $n \geqslant 1$. The *top exit of* L_∞ is $\bigcap_{n=1}^{\infty} T_n$. The other exits of L_∞ are defined analogously. Let c be a curve. A surjective and continuous function $p : [a, b] \to c$ is called a *parametrization* of c. The length of p is defined as

$$L(p) = \sup \left\{ \sum_{i=1}^{n} (|p(t_{i+1}) - p(t_i)|) \;\middle|\; a = t_0 < \cdots < t_n = b \right\}.$$

We say an arc has *finite length* if it has a parametrization p with $L(p) < \infty$, and otherwise the arc has *infinite length*.

Lemma 2.4. *Let $n, k \geqslant 1$, $\{W_1, \dots, W_k\}$ be a (shortest) path between the exits W_1 and W_k in $\mathcal{G}(\mathcal{W}_n)$, $K_0 = W_1 \cap \partial([0,1] \times [0,1])$, $K_k = W_k \cap \partial([0,1] \times [0,1])$, and c be a curve in L_n from a point of K_0 to a point of K_k. The length of any parametrization of c is at least $(k-1)/(2m^n)$.*

Lemma 2.5.

(a) *For all $n \geqslant 1$, each exit e in L_∞ lies in exactly one square $W_n(e) \in \mathcal{W}_n$.*

(b) *If $n \geqslant 1$ and e is an exit of L_∞, then e is a fixed point of $P_{W_n(e)}$.*

(c) *Let $z \in [0,1] \times [0,1]$. Then $P_{W_n(e)}(z) - e$ and $z - e$ are linearly dependent for $n \geqslant 1$.*

Lemma 2.6. *If a is an arc between two exits e_1 and e_2 in L_∞, p is a path in $\mathcal{G}(\mathcal{W}_n)$ from $W_n(e_1)$ to $W_n(e_2)$, and $W_n \in \mathcal{W}_n$ is a $\mathbb{I}$-square with respect to p, then $W_n \cap a$ is an arc between the top exit and the bottom exit of W_n. If W_n is another type of square, the analogous statement holds.*

Lemma 2.7. *If a is an arc between the top and bottom exits in L_∞ and $W_n \in \mathcal{W}_n$, then $P_{W_n}(a)$ is an arc between the top and bottom exits in W_n. For the other pairs of exits, the analogous statement holds.*

Lemma 2.8. *If a is an arc between the top and bottom exits in L_∞, then*

$$\liminf_{k \to \infty} \frac{\log(\mathbb{I}_k)}{k \log(m)} \leqslant \underline{\dim}_B(a) \leqslant \overline{\dim}_B(a) \leqslant \limsup_{k \to \infty} \frac{\log(\mathbb{I}_k)}{k \log(m)}.$$

For the other pairs of exits, the analogous statement holds.

3. Horizontally and vertically blocked labyrinth fractals

In this section we present the main results of this paper. An $m \times m$-labyrinth set $\mathcal{W}_1$ is called *horizontally blocked* if the row (of squares) from the left exit to the right exit contains at least one black square. It is called *vertically blocked* if the column (of squares) from the top exit to the bottom exit contains at least one black square. We note that there is no horizontally or vertically blocked $m \times m$-labyrinth set for $m \leqslant 3$. The main statement is contained in Theorem 3.18. We show that between any two points in the limit set of a horizontally and vertically blocked $m \times m$-labyrinth set there is a unique arc a in the fractal, the length of a is infinite and the set of all points at which no tangent to a exists is dense in a. We prove that the path matrix of any horizontally and vertically blocked $m \times m$-labyrinth set is primitive and use the Perron–Frobenius Theorem for primitive matrices to obtain the asymptotic behaviour of the path lengths in the nth step of the construction, as $n \to \infty$.

From now on, let $m \geqslant 4$, let $\mathcal{W}_1$ be a horizontally and vertically blocked $m \times m$-labyrinth set and let $\{L_k\}_{k=1}^{\infty}$ be defined as in § 1.1.1.

Lemma 3.1. *Let $P = \{W_1, \dots, W_n\}$ be a path between the top exit and the bottom exit of $\mathcal{W}_1$. The* ┗*-squares and the* ┓*-squares occur alternately in P, beginning with a* ┗*-square and ending with a* ┓*-square, i.e.*

(a) *if W_j is a* ┓*-square in P, then there is an $i < j$ such that W_i is a* ┗*-square in P,*

(b) *if W_j is a* ┗*-square in P, then there is an $i > j$ such that W_i is a* ┓*-square in P,*

(c) *if W_i and W_j are* ┓*-squares in P, for $i < j$, then there is a k such that $i < k < j$ and W_k is a* ┗*-square in P,*

(d) *if W_i and W_j are* ┗*-squares in P, for $i < j$, then there is a k such that $i < k < j$ and W_k is a* ┓*-square in P.*

Proof. (a) If we do not use a ┗-square and start at W_j, which is a ┓-square, it is only possible to go downwards or to the left, such that we cannot reach the top exit W_1.

(b) This follows from the same arguments as (a).

(c) If the two ┓-squares lie in different rows, then it is not possible to reach the upper ┓-square from the lower ┓-square without a ┗-square, by the same argument as in (a). If the two ┓-squares lie in different columns, then it is not possible to reach the right ┓-square from the left ┓-square without a ┗-square, by the same argument as in (a).

(d) The proof is analogous to that of (c). □

We note that Lemma 3.1 also holds if ┗-square is replaced by ┛-square and ┓-square is replaced by ┏-square. Also, for both versions of Lemma 3.1, an analogous lemma for paths from the left exit to the right exit holds.

Lemma 3.2. *Let $k < n$, $\{W_1, \dots, W_k\}$ be a path between the top exit and the right exit of $\mathcal{W}_1$, and let $\{W_{k+1}, \dots, W_n\}$ be a path between the left exit and the bottom exit of $\mathcal{W}_1$. Let $\mathcal{G}$ be the graph that results if we horizontally combine two copies of $\mathcal{G}(\mathcal{W}_1)$ such that the right exit of the left copy is neighboured to the left exit of the right copy. Then $P = \{W_1, \dots, W_k, W_{k+1}, \dots, W_n\}$ is a path in $\mathcal{G}$ between the top exit of the left copy and the bottom exit of the right copy in $\mathcal{G}$. The* ┗*-squares and the* ┓*-squares occur alternately in P, beginning with a* ┗*-square and ending with a* ┓*-square, i.e. the items (a)–(d) are the same as in Lemma 3.1.*

Proof. The proof is similar to that of Lemma 3.1. □

We note that Lemma 3.2, like Lemma 3.1, also exists in four versions.

For $u, v \in$ {┃, ━, ┗, ┏, ┓, ┛}, let $n(u, v)$ be the number of u-squares in a v-path of $\mathcal{G}(\mathcal{W}_1)$.

Lemma 3.3.

(a) n(┗, ┃) $=$ n(┓, ┃) *and* n(┏, ┃) $=$ n(┛, ┃).

(b) n(┗, ━) $=$ n(┓, ━) *and* n(┏, ━) $=$ n(┛, ━).

(c) $n(\text{┗},\text{┗}) + n(\text{┗},\text{┓}) = n(\text{┓},\text{┗}) + n(\text{┓},\text{┓})$ *and* $n(\text{┏},\text{┗}) + n(\text{┏},\text{┓}) = n(\text{┛},\text{┗}) + n(\text{┛},\text{┓})$.

(d) $n(\text{┗},\text{┏}) + n(\text{┗},\text{┛}) = n(\text{┓},\text{┏}) + n(\text{┓},\text{┛})$ *and* $n(\text{┏},\text{┏}) + n(\text{┏},\text{┛}) = n(\text{┛},\text{┏}) + n(\text{┛},\text{┛})$.

Proof. (a) $n(\text{┗},\text{┃}) = n(\text{┓},\text{┃})$ follows from Lemma 3.1 and $n(\text{┏},\text{┃}) = n(\text{┛},\text{┃})$ is deduced from the notes after Lemma 3.1.

(b) The arguments are the same as in (a), except that we use the remarks after Lemma 3.1.

(c) To show $n(\text{┗},\text{┗}) + n(\text{┗},\text{┓}) = n(\text{┓},\text{┗}) + n(\text{┓},\text{┓})$ and $n(\text{┏},\text{┗}) + n(\text{┏},\text{┓}) = n(\text{┛},\text{┗}) + n(\text{┛},\text{┓})$ we use Lemma 3.2.

(d) We prove this in the same way as (c). □

For a matrix $Q = (q_{i,j})_{i,j=1}^{r} \geqslant 0$, we define $Q' = (q'_{i,j})_{i,j=1}^{r}$ by

$$q'_{i,j} = \begin{cases} 1 & \text{if } q_{i,j} > 0, \\ 0 & \text{if } q_{i,j} = 0. \end{cases}$$

A *digraph* G is a pair (V, E), where $V = V(G)$ is a finite set and $E = E(G)$ is a subset of $(V \times V) \setminus \{(a, a) \mid a \in V\}$. Now, we define the digraph $\mathcal{G} \equiv (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{\text{┃},\text{━},\text{┗},\text{┏},\text{┓},\text{┛}\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is defined by the adjacency matrix M', where M is the path matrix of $\mathcal{W}_1$. We write $u \to v$ if $(u, v) \in \mathcal{E}$ and note that $u \to v$ if and only if a u-square is contained in a v-path in L_1.

Lemma 3.4.

$$\text{┃} \to \text{┃}, \quad \text{┗} \to \text{┃}, \quad \text{┏} \to \text{┃}, \quad \text{┓} \to \text{┃} \quad \textit{and} \quad \text{┛} \to \text{┃},$$
$$\text{━} \to \text{━}, \quad \text{┗} \to \text{━}, \quad \text{┏} \to \text{━}, \quad \text{┓} \to \text{━} \quad \textit{and} \quad \text{┛} \to \text{━}.$$

Proof. By symmetry, it suffices to show the first line. Let P be a ┃-path. We indirectly assume that P contains no ┃-square. Without loss of generality, we assume that the top exit is a ┗-square. If P contains a ┓-square W, then the square below W in P must be a ┗-square, to avoid a cycle in $\mathcal{G}(\mathcal{W}_1)$. Thus, P cannot reach the bottom exit, which is a contradiction. Therefore, $\text{┃} \to \text{┃}$.

Since $\mathcal{W}_1$ is vertically blocked, there is a first square S in P, starting from the top exit, that is not a ┃-square. Thus, S must be a ┗- or a ┛-square. Without loss of generality, we assume that S is a ┗-square. With Lemma 3.3 (a) we obtain that P also contains a ┓-square. If P neither contains a ┏-square nor a ┛-square, then the path could only go downwards or to the right and therefore, it would not reach the bottom exit, which is a contradiction. Therefore, Lemma 3.3 (a) yields that there is a ┏-square and a ┛-square in P. □

Lemma 3.5.

$$\text{┃} \to \text{┏} \quad \textit{or} \quad \text{┃} \to \text{┛},$$
$$\text{┃} \to \text{┗} \quad \textit{or} \quad \text{┃} \to \text{┓},$$
$$\text{━} \to \text{┏} \quad \textit{or} \quad \text{━} \to \text{┛},$$
$$\text{━} \to \text{┗} \quad \textit{or} \quad \text{━} \to \text{┓}.$$

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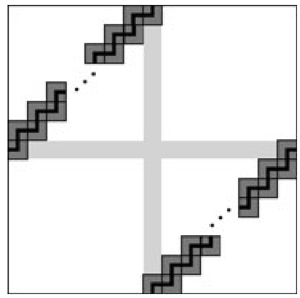

Figure 3. For the proof of Lemma 3.5.

Proof. By symmetry, it suffices to show the first line. We indirectly assume that neither ◫ $\to$ ◲ nor ◫ $\to$ ◰. The top exit and the left exit of $\mathcal{W}_1$ are denoted by T and L, respectively. We number the columns and rows of $\mathcal{W}_1$ from 1 to m, beginning at the left and at the top. Now, we write $\mathrm{col}(T)$ for the column where T lies, and we write $\mathrm{row}(L)$ for the row in which L appears. If $\mathrm{row}(L) > \mathrm{col}(T)$, then we take a look at the ◰-path P. The top exit T must be a ◰-square in P, because otherwise, as P leads to L and we have assumed that ◫ $\to$ ◰ does not hold, a cycle must occur in $\mathcal{G}(\mathcal{W}_1)$. Since $\mathrm{row}(L) > \mathrm{col}(T)$, P must contain a ◫-square, since otherwise P will not reach the left exit L (if we say P starts in the top exit T). Thus, $\mathrm{row}(L) > \mathrm{col}(T)$ is not possible. In the same way we show that the ◲-path contains a ◫-square if $\mathrm{row}(L) < \mathrm{col}(T)$, which makes $\mathrm{row}(L) < \mathrm{col}(T)$ impossible. Therefore, $\mathrm{row}(T) = \mathrm{col}(L)$, which implies that there is no ⊟-square in the ◰-path and no ⊟-square in the ◲-path. Property 1.1 yields that there is a path P_0 between the bottom exit and L. P_0 has to use a square that is adjacent to P. As Figure 3 shows, this produces a contradiction either to Property 1.1 or to Property 1.2. □

A non-negative square matrix $Q = (Q_{i,j})_{i,j=1}^{k}$ is called *irreducible* if for every pair $1 \leqslant i, j \leqslant k$, there is a positive integer p such that $Q_{i,j}^{p} > 0$. A *path* of length $n \geqslant 1$ from a to b in a digraph $G \equiv (V, E)$ is a sequence $(a_0, a_1, \dots, a_n)$ such that $a_0, a_1, \dots, a_n \in V$, $a_0 = a$, $a_n = b$, $a_i \neq a_j$ for $0 \leqslant i < j \leqslant n$, and $(a_{i-1}, a_i) \in E$ for $i = 1, \dots, n$. We say a digraph is *strongly connected* if there is a path from every vertex to every other vertex. We note that the non-negative square matrix Q is irreducible if and only if the digraph with vertex set $\{1, \dots, k\}$ and adjacency matrix Q' is strongly connected.

Theorem 3.6. *The path matrix M of an $m \times m$-labyrinth set is irreducible if and only if the labyrinth set is horizontally and vertically blocked.*

Proof.

(1) If, without loss of generality, the labyrinth set is not horizontally blocked, then the ⊟-path contains only ⊟-squares. Therefore, it is not possible to reach ⊟ in $\mathcal{G}$ from any vertex but ⊟ by a path. Thus, M is not irreducible.

(2) Let the labyrinth set be horizontally and vertically blocked. We indirectly assume that M is not irreducible. Thus, $\mathcal{G}$ is not strongly connected, which implies that there are vertices $u, v \in \mathcal{V}$ such that there is no path from u and v in $\mathcal{G}$.

As the first case we treat $u \in$ {◰, ◲, ◱, ◳}. Lemma 3.4 gives $u \to$ ◫ and $u \to$ ⊟. Thus, if $v \in$ {◫, ⊟}, then $u \to v$ yields a contradiction. If $v \in$ {◰, ◲, ◱, ◳}, we assume without loss of generality that $v =$ ◰. If ◫ $\to$ ◰, then $u \to$ ◫ $\to$ ◰ $= v$ is a contradiction. If ⊟ $\to$ ◰, then $u \to$ ⊟ $\to$ ◰ $= v$ is a contradiction. Thus, neither ◫ $\to$ ◰ nor ⊟ $\to$ ◰ holds. Now, we indirectly assume that ◱ $\to$ ◰ does not hold. The top exit is a ◰ square in the ◰-path, because otherwise, as P leads to the right exit, a cycle must occur in $\mathcal{G}(\mathcal{W}_1)$. Thus, the ◰-path consists only of one ◰-square, which means that there is an exit, which is a right exit and a top exit. This is impossible by Property 1.3, such that we may assume ◱ $\to$ ◰. Since ◫ $\to$ ◰ does not hold, ◫ $\to$ ◱ must be satisfied by Lemma 3.5. Therefore, $u \to$ ◫ $\to$ ◱ $\to$ ◰ $= v$ is a contradiction.

The second case is $u \in$ {◫, ⊟}. We assume without loss of generality that $u =$ ◫. If $v \in$ {◫, ⊟}, a contradiction follows, since then there must be a path from u to v, as Lemma 3.5 yields ◫ $\to$ ◲ or ◫ $\to$ ◳ and Lemma 3.4 gives ◲ $\to$ ◫, ◲ $\to$ ⊟, ◳ $\to$ ◫ and ◳ $\to$ ⊟. If $v \in$ {◰, ◲, ◱, ◳}, we assume without loss of generality that $v =$ ◰. If ◫ $\to$ ◰, then $u =$ ◫ $\to$ ◰ $= v$ is a contradiction. So we state that ◫ $\to$ ◰ does not hold and therefore Lemma 3.5 yields ◫ $\to$ ◱. Furthermore, Lemma 3.4 gives ◱ $\to$ ⊟. If ⊟ $\to$ ◰, then $u =$ ◫ $\to$ ◱ $\to$ ⊟ $\to$ ◰ $= v$ is a contradiction. So we state that ⊟ $\to$ ◰ does not hold. As in the first case, it follows that ◱ $\to$ ◰ and finally $u =$ ◫ $\to$ ◱ $\to$ ◰ $= v$ yields a contradiction. □

A non-negative square matrix Q is called *primitive* if there is a positive integer p such that $Q^p > 0$, i.e. all entries of Q^p are greater than 0. We note that a primitive matrix has to be irreducible and that an irreducible matrix with at least one non-negative diagonal element is primitive.

Theorem 3.7. *The path matrix M of an $m \times m$-labyrinth set is primitive if and only if the labyrinth set is horizontally and vertically blocked.*

Proof.

(1) If the labyrinth set is not horizontally and vertically blocked, then M is not irreducible, by Theorem 3.6. Thus, M is not primitive.

(2) If the labyrinth set is horizontally and vertically blocked, then M is irreducible, by Theorem 3.6. Since Lemma 3.4 provides ◫ $\to$ ◫, M is primitive.

□

Let M be the path matrix of a horizontally and vertically blocked $m \times m$-labyrinth set $\mathcal{W}_1$. The *reduced path matrix* $\bar{M}$ of $\mathcal{W}_1$ arises from M in the following way. We add the sixth row to the fourth row and the fifth row to the third row in M. From the result we delete the fifth and sixth rows and columns. Let ◰◱$_n$ = ◰$_n$ + ◱$_n$ and ◲◳$_n$ = ◲$_n$ + ◳$_n$. We need the reduced path matrix $\bar{M}$ for technical reasons which we will explain later.

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Lemma 3.8. *For $n \geqslant 1$,*

$$\begin{pmatrix} \text{◫}_n \\ \text{⊟}_n \\ \text{◰◳}_n \\ \text{◲◱}_n \end{pmatrix} = \bar{M} \begin{pmatrix} \text{◫}_{n-1} \\ \text{⊟}_{n-1} \\ \text{◰◳}_{n-1} \\ \text{◲◱}_{n-1} \end{pmatrix}, \quad \textit{where} \quad \begin{pmatrix} \text{◫}_0 \\ \text{⊟}_0 \\ \text{◰◳}_0 \\ \text{◲◱}_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}.$$

Proof. Let $M = (m_{ij})_{i,j=1}^{6}$ and let M^* be

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\ m_{31}+m_{51} & m_{32}+m_{52} & m_{33}+m_{53} & m_{34}+m_{54} & m_{35}+m_{55} & m_{36}+m_{56} \\ m_{41}+m_{61} & m_{42}+m_{62} & m_{43}+m_{63} & m_{44}+m_{64} & m_{45}+m_{65} & m_{46}+m_{66} \end{pmatrix}.$$

We use Lemma 2.2 to obtain

$$\begin{pmatrix} \text{◫}_n \\ \text{⊟}_n \\ \text{◰◳}_n \\ \text{◲◱}_n \end{pmatrix} = M^* \begin{pmatrix} \text{◫}_{n-1} \\ \text{⊟}_{n-1} \\ \text{◰}_{n-1} \\ \text{◲}_{n-1} \\ \text{◳}_{n-1} \\ \text{◱}_{n-1} \end{pmatrix},$$

which is equal to

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31}+m_{51} & m_{32}+m_{52} & m_{33}+m_{53} & m_{34}+m_{54} \\ m_{41}+m_{61} & m_{42}+m_{62} & m_{43}+m_{63} & m_{44}+m_{64} \end{pmatrix} \begin{pmatrix} \text{◫}_{n-1} \\ \text{⊟}_{n-1} \\ \text{◰}_{n-1}+\text{◳}_{n-1} \\ \text{◲}_{n-1}+\text{◱}_{n-1} \end{pmatrix},$$

since the third column of M^* is equal to its fifth column and the fourth column of M^* is equal to its sixth column, by Lemma 3.3. □

Lemmas 3.1 and 3.2 allow us to combine the ◰-squares and ◳-squares of a path (between exits) into ◰◳-*squares* and to combine the ◲-squares and ◱-squares of a path (between exits) into ◲◱-*squares*. If the columns and rows of $\bar{M}$ correspond to ◫, ⊟, ◰◳ and ◲◱, then the element in row x and column y is the number of y-squares in the x-path. The ◰◳-path and the ◲◱-path arise according to Lemma 3.2.

Lemma 3.9. *The reduced path matrix $\bar{M}$ of a horizontally and vertically blocked $m \times m$-labyrinth set is primitive.*

Proof. We define the digraph $\mathcal{G}^* \equiv (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{\text{◫}, \text{⊟}, \text{◰◳}, \text{◲◱}\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is defined by the adjacency matrix $\bar{M}'$.

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(1) If, without loss of generality, the labyrinth set is not horizontally blocked, then the ⊟-path contains only ⊟-squares. Therefore, it is not possible to reach ⊟ in $\mathcal{G}^*$ from any vertex but ⊟ by a path. Thus, $\bar{M}$ is not irreducible.

(2) If the $m \times m$-labyrinth set is horizontally and vertically blocked, then Lemmas 3.4 and 3.5 allow us to show that the graph $\mathcal{G}^*$ is strongly connected.

□

From the Perron–Frobenius Theorem for primitive matrices, e.g. in the book of Seneta [**14**, Theorem 1.1, p. 3], we have the following result.

Theorem 3.10. *Let M be the path matrix of a horizontally and vertically blocked $m \times m$-labyrinth set. Then there is an eigenvalue r of M such that*

(a) *r is real and greater than 0,*

(b) *strictly positive left and right eigenvectors can be associated with r,*

(c) *$r > |\lambda|$ for any eigenvalue $\lambda \neq r$ of M.*

Let λ be an eigenvalue of M with $|\lambda| < r$, and the property that for any other eigenvalue λ' of M with $|\lambda'| < r$ we have either $|\lambda'| < |\lambda|$ or $|\lambda| = |\lambda'|$ and the algebraic multiplicity m_λ of λ is not lower than the algebraic multiplicity of λ'. Let w, u be positive right and left eigenvectors to r, in accordance with Theorem 3.10, normed so that $uw = 1$. The following theorem [**14**, Theorem 1.2, p. 9] yields the asymptotic behaviour of M^k.

Theorem 3.11. *For a primitive matrix M:*

(a) *if $\lambda \neq 0$, then as $n \to \infty$*

$$M^n = r^n wu + \mathcal{O}(n^{(m_\lambda - 1)}|\lambda|^n);$$

(b) *if $\lambda = 0$, then for $n \geqslant 1$*

$$M^n = r^n wu.$$

We note that we actually need the reduced path matrix $\bar{M}$ to give a lower bound of the spectral radius of M. The argument which we use for $\bar{M}$ in Theorem 3.12 does not work for M, since, in general, neither the minimal row sum nor the minimal column sum of M has to be greater than or equal to m.

Theorem 3.12. *Let M be the path matrix and $\bar{M}$ be the reduced path matrix of a horizontally and vertically blocked $m \times m$-labyrinth set. The spectral radius of M is equal to the spectral radius of $\bar{M}$ and is greater than m.*

Proof. We first show that the spectral radius of $\bar{M}$ is greater than m. The row sum of the first row of $\bar{M}$ is not lower than m, since a ◫-path needs at least one ◫-square or one ▙▟-square or one ▛▜-square to go from one row to the row beneath. By the same argument, the row sum of the other rows of $\bar{M}$ is not lower than m. Now, we will show that the row sum of the third row of $\bar{M}$ is greater than m, by indirectly assuming that

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the row sum is equal to m. Let P be a -path between the top exit of the -square and the bottom exit of the neighbouring -square. If P consists of m squares, where the combined -squares count as one square and the combined -squares also count as one square, P has to go at least one step downwards and at least one step to the right in each of its squares. Therefore, P consists of -squares only, which is impossible by Lemma 3.5. As this is a contradiction, we conclude that the sum of the third row is greater than m. It is known [**14**, Corollary 1, p.8] that the spectral radius is not lower than the minimal row sum, and is greater than the minimal row sum if not all row sums are equal. This implies that the spectral radius of $\bar{M}$ is greater than m.

Now, we prove that the spectral radius of M is equal to the spectral radius of $\bar{M}$. We indirectly assume that the spectral radius r of M is not equal to the spectral radius $\bar{r}$ of $\bar{M}$. For $n \geqslant 1$, let $M^n = (m_{ij,n})_{i,j=0}^{6}$ and $\bar{M}^n = (\bar{m}_{ij,n})_{ij=0}^{4}$. Theorem 3.11 yields

$$\lim_{n\to\infty} \frac{m_{11,n} + m_{12,n} + m_{13,n} + m_{14,n} + m_{15,n} + m_{16,n}}{r^n} = c > 0$$

and

$$\lim_{n\to\infty} \frac{\bar{m}_{11,n} + \bar{m}_{12,n} + 2\bar{m}_{13,n} + 2\bar{m}_{14,n}}{\bar{r}^n} = \bar{c} > 0.$$

This is impossible since

 $_n = m_{11,n} + m_{12,n} + m_{13,n} + m_{14,n} + m_{15,n} + m_{16,n} = \bar{m}_{11,n} + \bar{m}_{12,n} + 2\bar{m}_{13,n} + 2\bar{m}_{14,n},$

by Lemma 3.8. □

Let $T_n \in \mathcal{W}_n$ be the top exit of $\mathcal{W}_n$ for $n \geqslant 1$. The *top exit of* L_∞ is $\bigcap_{n=1}^{\infty} T_n$. The other exits of L_∞ are defined analogously. Let c be a curve. A surjective and continuous function $p\colon [a,b] \to c$ is called a *parametrization* of c. The length of p is defined as

$$L(p) = \sup\left\{ \sum_{i=1}^{n} (|p(t_{i+1}) - p(t_i)|) \;\middle|\; a = t_0 < \cdots < t_n = b \right\}.$$

We say an arc has *finite length* if it has a parametrization p with $L(p) < \infty$ and otherwise the arc has *infinite length*. Let t be the top exit of L_∞, $n \geqslant 1$, and $W \in \mathcal{W}_n$. Then we call $P_W(t)$ the *top exit* of W. The other exits of W are defined analogously. We note that we have now defined exits for three different types of object, i.e. for $\mathcal{W}_n$, for L_∞ and for squares in $\mathcal{W}_n$.

Lemma 3.13. *Between any two different exits of L_∞ there is no arc of finite length in L_∞. Furthermore, between two different exits of a square $W \in \mathcal{W}_n$, for $n \geqslant 1$, there is no arc of finite length in L_∞.*

Proof. Let r be the spectral radius of M. M is primitive, by Theorem 3.7. Since Theorem 3.12 guarantees $r > m$, it follows from Theorem 3.11 that each entry of M^n/m^n tends to ∞. Now, Lemmas 2.2 and 2.4 imply that there is no arc of finite length between two different exits of L_∞. To prove the second statement, we assume that there is an

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arc a between two different exits of W whose length is finite. The function P_W is a homeomorphism between $[0,1]\times[0,1]$ and W, and $P_W^{-1}(W\cap L_\infty)=L_\infty$. Thus, $P_W^{-1}(a)$ is an arc in L_∞ between two different exits of L_∞, whose length is infinite, as we have just showed. As this is a contradiction, the proof is completed. □

Theorem 3.14. *The length of the arc a in L_∞ between any two distinct points x and y in L_∞ is infinite.*

Proof. Let p_n be the path between $W_n(x)$ and $W_n(y)$, constructed as in Lemma 2.3. Since, $x\neq y$, there must be a $k\geqslant 1$, such that p_k consists of at least three squares. Let $W\in p_n$ be a square that is different from $W_n(x)$ and $W_n(y)$. By the construction of the arc a between x and y in Lemma 2.3, it follows that $a\cap W$ is an arc between two exits of W. From Lemma 3.13 it follows that the length of $a\cap W$ cannot be finite. □

Following [**15**, § 7.2, p. 73], we say that *there exists a tangent T at a point x_0* to an arc a if for every positive angle ϕ, there exists an ϵ, such that, for all x in a that satisfy $|x_0-x|<\epsilon$, the angle between the line T and the line through x and x_0 is not greater than ϕ.

Lemma 3.15. *If a is an arc between two exits e_1 and e_2 in L_∞, then there exists no tangent at e_1 or e_2 to a.*

Proof. We will show this for e_1. Without loss of generality, we assume that e_1 is the left exit. By Lemma 2.5 (a), for all $n\geqslant 1$, e_1 lies in exactly one square $W_n(e_1)\in\mathcal{W}_n$ and e_2 lies in exactly in one square $W_n(e_2)\in\mathcal{W}_n$. Let p_n be the path in $\mathcal{G}(\mathcal{W}_n)$ from $W_n(e_1)$ to $W_n(e_2)$. The square $W_n(e_1)$ can only be of type ⊟, ◰ or ◱, with respect to p_n, for each $n\geqslant 1$. Thus, at least one of the types ⊟, ◰ or ◱ occurs infinitely often in the sequence $\{W_n(e_1)\}_{n=1}^\infty$, with respect to p_n. Without loss of generality we assume that there is a sequence $\{n_j\}_{j=1}^\infty$ such that $W_{n_j}(e_1)$ is a ⊟-square, for all $j\geqslant 1$. Thus, $W_{n_1}(e_1)\cap a$ is an arc between the left exit and the right exit of $W_{n_1}(e_1)$, by Lemma 2.6. The path matrix of a horizontally and vertically blocked labyrinth set is irreducible, by Theorem 3.6. This implies that there is an $n\geqslant 1$ such that projections of all four exits of L_∞ into squares of $\mathcal{W}_n$ must be contained in any arc in L_∞ between two exits of L_∞. Therefore, an arc between two exits of a square cannot be a straight line segment. Thus, we may choose two points p_1 and p_2 in $W_{n_1}(e_1)\cap a$ which are both different from e_1, such that p_1-e_1 and p_2-e_1 are not collinear. On the one hand, $a_j=P_{W_{n_j}(e_1)}(p_1)$ and $b_j=P_{W_{n_j}(e_1)}(p_2)$ define sequences for $j\geqslant 2$, which converge to e_1 and there is no $j\geqslant 1$ such that $a_j=e_1$ or $b_j=e_1$. Lemmas 2.6 and 2.7 yield that $a_j,b_j\in a$ for $j\geqslant 1$. On the other hand, a_j lies on a straight line l_a for all $j\geqslant 1$ and b_j lies on a straight line l_b for all $j\geqslant 1$, by Lemma 2.5 (c). As p_1-e_1 and p_2-e_1 are not collinear, we have $l_a\neq l_b$. It follows that there exists no tangent to a at e_1. □

Corollary 3.16. *If $a\subset L_\infty$ is an arc, then the set of all points at which no tangent to a exists is dense in a.*

Proof. For $n \geqslant 1$, let $p_n(W_n(a), W_n(b)) = \{W_1, \dots, W_k\}$ be constructed according to Lemma 2.3. Let e_1, e_2, e_3 and e_4 be the exits of L_∞ and let $1 \leqslant j \leqslant k$. From Lemmas 2.6 and 2.7, we obtain that $W_j \cap a$ is an arc from $P_{W_j}(e_{i_1})$ to $P_{W_j}(e_{i_2})$ for some $1 \leqslant i_1 < i_2 \leqslant 4$. The set

$$a \cap \bigcup_{n=1}^{\infty} \bigcup_{W_n \in \mathcal{W}_n} P_{W_n}(\{e_1, e_2, e_3, e_4\})$$

is dense in a. Lemma 3.15 implies that

$$a \cap \bigcup_{n=1}^{\infty} \bigcup_{W_n \in \mathcal{W}_n} P_{W_n}(\{e_1, e_2, e_3, e_4\})$$

is a subset of the set of all points at which no tangent to a exists. □

Lemma 3.17. *Let L_∞ be the limit set of a horizontally and vertically blocked $m \times m$-labyrinth set with path matrix M and let r be the spectral radius of M. If a is an arc between two exits in L_∞, then*

$$\dim_B(a) = \frac{\log(r)}{\log(m)}.$$

Proof. We use Lemma 2.8 and Theorem 3.11. □

As a summary, we now state our main result.

Theorem 3.18. *Let L_∞ be the limit set of a horizontally and vertically blocked $m \times m$-labyrinth set with path matrix M and let r be the spectral radius of M. Between any two points in L_∞ there is a unique arc a; the length of a is infinite,* $\dim_B(a) = \log(r)/\log(m)$ *and the set of all points at which no tangent to a exists is dense in a.*

Proof. Let x and y be two points in L_∞. Since L_∞ is a dendrite, by Theorem 2.1, there is a unique arc a from x to y [**9**, Corollary 2, p. 301]. The length of a is infinite, by Theorem 3.14, and Corollary 3.16 says that the set of points at which no tangent to a exists is dense in a. Lemma 3.17 yields the box-counting dimension of a. □

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