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SOME RINGS OF INVARIANTS THAT ARE COHEN-MACAULAY

LARRY SMITH

ABSTRACT. Let $\rho: G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of the finite group G over the field $\mathbb{F}$. If the order $|G|$ of G is relatively prime to the characteristic of $\mathbb{F}$ or $n = 1$ or 2, then it is known that the ring of invariants $\mathbb{F}[V]^G$ is Cohen-Macaulay. There are examples to show that $\mathbb{F}[V]^G$ need not be Cohen-Macaulay when $|G|$ is divisible by the characteristic of $\mathbb{F}$. In all such examples $\dim_{\mathbb{F}}(V)$ is at least 4. In this note we fill the gap between these results and show that rings of invariants in three variables are always Cohen-Macaulay.

0. **Introduction.** Let $\rho: G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of the finite group G over the field $\mathbb{F}$. Denote by $\mathbb{F}[V]$ the graded algebra of homogenous polynomial functions on the vector space $V = \mathbb{F}^n$ and let $\mathbb{F}[V]^G$ be the subalgebra of polynomials invariant under the action of G. For a general reference on invariant theory we refer to [7].

The ring $\mathbb{F}[V]^G$ is known to be Cohen-Macaulay if the order of G is relatively prime to the characteristic of $\mathbb{F}$ ([5] or [7], Theorem 6.7.8) or if $\dim_{\mathbb{F}}(V) = 2$ ([7], Corollary 8.2.3). If the characteristic of $\mathbb{F}$ divides the order of G this may fail: see for example [1], [4], or [7], Section 6.7, Example 2. In all of these examples the dimension of V is at least 4. In this note we prove that rings of invariants in three variables are always Cohen-Macaulay. This shows that the non-Cohen-Macaulay 20 year old example of Bertin is of minimal dimension.

1. **Rings of Invariants in Three Variables.** Let $\rho: G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of the finite group over the field $\mathbb{F}$. If $\mathbb{F}$ is a field of characteristic zero then $\mathbb{F}[V]^G$ is always Cohen-Macaulay, so we assume that $\mathbb{F}$ is a field of characteristic $p \neq 0$. The proof of the main result makes use of a reduction to the case of p–groups. (See [2] or [7], Theorem 8.3.1.)

PROPOSITION 1.1. *Let $\mathbb{F}$ be a field of characteristic $p \neq 0$ and $\rho: G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ a representation of a finite group G. If $\mathbb{F}[V]^{\mathrm{Syl}_p(G)}$ is Cohen-Macaulay then so is $\mathbb{F}[V]^G$.*

PROOF. The index $|G : \mathrm{Syl}_p(G)|$ is relatively prime to p. Therefore we have the projection operator (see [7], Section 2.4)

$$\pi^G_{\mathrm{Syl}_p(G)}: \mathbb{F}[V]^{\mathrm{Syl}_p(G)} \longrightarrow \mathbb{F}[V]^G$$

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which is an $\mathbb{F}[V]^G$-module homomorphism such that the composite

$$(*) \qquad \mathbb{F}[V]^G \hookrightarrow \mathbb{F}[V]^{\mathrm{Syl}_p(G)} \xrightarrow{\pi^G_{\mathrm{Syl}_p(G)}} \mathbb{F}[V]^G$$

is the identity. Let $f_1, \ldots, f_n \in \mathbb{F}[V]^G$ be a system of parameters. $\mathbb{F}[V]^{\mathrm{Syl}_p(G)}$ is finitely generated as $\mathbb{F}[V]^G$-module. Hence $f_1, \ldots, f_n \in \mathbb{F}[V]^{\mathrm{Syl}_p(G)}$ is also a system of parameters for $\mathbb{F}[V]^{\mathrm{Syl}_p(G)}$. By hypothesis $\mathbb{F}[V]^{\mathrm{Syl}_p(G)}$ is Cohen-Macaulay, so by Macaulay's Theorem ([7], Corollary 6.7.7) $f_1, \ldots, f_n \in \mathbb{F}[V]^{\mathrm{Syl}_p(G)}$ is a regular sequence. Therefore by [7], Corollary 6.7.11, $\mathbb{F}[V]^{\mathrm{Syl}_p(G)}$ is a free $\mathbb{F}[f_1, \ldots, f_n]$-module. It follows from $(*)$ that $\mathbb{F}[V]^G$ is a direct summand of $\mathbb{F}[V]^{\mathrm{Syl}_p(G)}$ as $\mathbb{F}[V]^G$-module, and hence *a fortiori* as $\mathbb{F}[f_1, \ldots, f_n]$-module. Therefore $\mathbb{F}[V]^G$ is a projective $\mathbb{F}[f_1, \ldots, f_n]$-module, and hence by [7], Proposition 6.1.1, a free $\mathbb{F}[f_1, \ldots, f_n]$-module, which makes it Cohen-Macaulay. ∎

THEOREM 1.2. *Let* $\rho\colon G \hookrightarrow \mathrm{GL}(3, \mathbb{F})$ *be a representation of the finite group* G *over the field* $\mathbb{F}$. *Then* $\mathbb{F}[x, y, z]^G$ *is Cohen-Macaulay.*

PROOF. From the theorem of Hochster and Eagon, *loc. cit.*, and Proposition 1.1 we may suppose that $\mathbb{F}$ has characteristic $p \neq 0$ and $G = P$ is a finite p-group. Since P is finite there is no loss in generality in assuming that $\mathbb{F}$ is a Galois field with $q = p^m$ elements.

Let $V = \mathbb{F}^3$ and consider the fixed point set V^P of the action of P on V. There is the decomposition

$$V = V^P \sqcup B_1 \sqcup \cdots \sqcup B_m$$

where $B_1, \ldots, B_m \subset V$ are the nontrivial orbits of P on V (*i.e.*, the orbits with more than one point). Since $q^3 \equiv 0 \equiv |B_i| \bmod p$ for $i = 1, \ldots, m$ it follows that $|V^P| \equiv 0 \bmod p$. The zero vector is a fixed point so V^P is nonempty, and therefore contains at least q elements, so $\dim_F(V^P) > 0$. Since the action of P in V is faithful $\dim_{\mathbb{F}}(V^P) \neq 3$ and hence $\dim(V^P) = 1, 2$.

If $\dim_{\mathbb{F}}(V^P) = 2$ then every nonidentity element of P acts on V as a transvection and hence by [6] or [7], Theorem 8.2.13, $\mathbb{F}[x, y, z]^P$ is a polynomial ring, and therefore Cohen-Macaulay. If $\dim_{\mathbb{F}}(V^P) = 1$ then $\mathrm{codim}_{\mathbb{F}}(V^P \subset V) = 2$ so by the theorem of Ellingsrud and Skjelbred [3]

$$\mathrm{codim}(\mathbb{F}[x, y, z]^P) \geq 2 + \dim_{\mathbb{F}}(V^P) = 3$$

where codim$(-)$ denotes the homological codimension of $-$. Since the Krull dimension $\dim(\mathbb{F}[V]^P) = \dim_{\mathbb{F}}(V) = 3$, $\mathbb{F}[x, y, z]^P$ is Cohen-Macaulay. ∎

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Mathematisches Institut
der Universität Göttingen
Bunsenstrasse 3 - 5
D 37073 Göttingen
Germany
e-mail: larry@cfgauss.uni-math.gwdg.de