ON SOME STEADY STATE THERMOELASTIC STRESS DISTRIBUTIONS IN A SLAB

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1. Introduction

The calculation of the steady state thermal stresses in an isotropic elastic half space or slab with traction free faces has been the subject of several investigations. Sternberg and McDowell (1), using an extension of the Bousinesq-Papkowitch method of isothermal elasticity, first derived the now well-known result that in such a body which contains no heat sources there exists a plane state of stress parallel to the boundary planes. Sneddon and Lockett (2) approached this class of problems by direct solution of the equations of thermoelasticity using a double Fourier integral transform method, the results being transformed to Hankel type integrals in the case of axial symmetry. A further approach due to Nowinski (3) exploits the fact that in steady state thermoelasticity each component of the displacement vector is a biharmonic function which can be expressed as a combination of harmonics. However, possibly the most economical method of solution of this type of problem is that of Williams (4) who expressed the displacement vector in terms of two scalar potential functions, one of which is directly related to the temperature field. The same principle has also been used by Fox (5) in treating thermoelastic distributions in a slab containing a spherical cavity.

Recently Martin and Payton (6) have considered a mathematically more complicated " mixed boundary condition " problem which arises in the study of thermal stresses in a missile heat shield. Here one face of an infinite slab of finite thickness is rigidly held while the other face is stress free. The clamped face is at zero temperature and on the stress free face is imposed a temperature distribution which varies only in one direction on the face. The problem then reduces to one of plane strain. The equations of plane strain are solved directly by means of Fourier transforms, a process which is both algebraically unwieldy and analytically complicated.

In this paper we shall give a simple treatment of this and other non-axially symmetric mixed boundary condition problems using a suitable solution of the equations of thermoelasticity in terms of harmonic functions. In § 2 we give the basic formulae required and express the displacement vector in terms of three harmonic functions, the derivative with respect to z of one of these being proportional to the temperature field. In § 3 the mixed boundary condition slab problem is formulated using a cylindrical polar coordinate system, the harmonics being expressed as standard type Hankel integrals. The case of a non-axially symmetric temperature distribution of the form $f_n(\rho)\cos n(\theta+\theta_0)$ on the stress free face is treated in detail. In general it is not possible to evaluate in closed form the integrals occurring in the displacement and stress components and numerical methods must be used. § **3** concludes with a short treatment of the axisymmetric analogue of the problem of Martin and Payton; specifically the temperature on $z = 1$ takes a constant value over a circular region of radius a, being zero elsewhere.

In § **4** the problem of Martin and Payton is considered and it is shown that, replacing the Hankel integrals in the harmonics by Fourier integrals, the problem is mathematically equivalent to that of § **3**. In § **5** some further mixed condition problems are briefly considered in which the rigid fixing of the zero temperature face is replaced by a frictionless constraint.

2. Basic equations

We consider an isotropic infinite flat elastic slab of finite thickness, one of whose faces is rigidly held. We choose a set of Cartesian coordinates (x, y, z) in which the clamped face is the plane $z = 0$, the origin of coordinates being at some point O in this plane. We also introduce at O a set of cylindrical polar coordinates (ρ, θ, z) and choose our unit of length so that the stress free face of the slab is $z = 1$. There is established in the solid a steady temperature field $T(\rho, \theta, z)$ where T is the deviation of the absolute temperature from the temperature of the solid in a state of zero stress and strain which we take to be zero. In the absence of body forces or heat sources within the medium the steady state equations of thermoelasticity are

$$(1-2\eta)\nabla^2\boldsymbol{u}+\text{grad}\,[\text{div}\,\boldsymbol{u}-2\alpha(1+\eta)T]=0 \tag{1}$$

and

$$\nabla^2 T = 0, \tag{2}$$

where $\boldsymbol{u}$ is the displacement vector, η is Poisson's ratio and α is the coefficient of linear expansion of the solid. Further, the stress vector $\boldsymbol{Z}$ across an element of surface area whose normal is parallel to Oz is given by

$$\boldsymbol{Z}=\mu\left\{\frac{[2\eta\,\text{div}\,\boldsymbol{u}-2\alpha(1+\eta)T]}{(1-2\eta)}\boldsymbol{k}+\frac{\partial\boldsymbol{u}}{\partial z}+\text{grad}\,(\boldsymbol{u}\,.\,\boldsymbol{k})\right\}, \tag{3}$$

where μ is the modulus of rigidity and $\boldsymbol{k}$ a unit vector along $0z$.

We next construct the solution of (1) and (2) most convenient for our purposes. It is readily verified by direct substitution that a solution of these equations is given by

$$\boldsymbol{u}_1=\phi\boldsymbol{k} \tag{4}$$

and

$$T=\frac{1}{2\alpha(1+\eta)}\frac{\partial\phi}{\partial z}, \tag{5}$$

where ϕ is a harmonic function. The corresponding stress vector $\boldsymbol{Z}_1$ is

$$\boldsymbol{Z}_1 = \mu \text{ grad } \phi. \tag{6}$$

A more general solution can now be obtained by adding to (4) an appropriate isothermal ($T \equiv 0$) displacement which (Green and Zerna (7), p. 167) can be written in terms of harmonic functions ψ and χ as

$$\boldsymbol{u}_2 = (3-4\eta)\psi\boldsymbol{k} - z \text{ grad } \psi + \text{grad } \chi, \tag{7}$$

the stress vector $\boldsymbol{Z}_2$ being

$$\boldsymbol{Z}_2 = 2\mu\left\{(1-2\eta) \text{ grad } \psi + \frac{\partial\psi}{\partial z}\boldsymbol{k} - z \text{ grad } \frac{\partial\psi}{\partial z} + \text{grad } \frac{\partial\chi}{\partial z}\right\}. \tag{8}$$

Thus we take for our solution of equations (1) and (2) the expressions

$$\boldsymbol{u} = (\phi + \beta\psi)\boldsymbol{k} - z \text{ grad } \psi + \text{grad } \chi \tag{9}$$

and

$$T = \frac{1}{2\alpha(1+\eta)}\frac{\partial\phi}{\partial z}, \tag{10}$$

where $\beta = 3-4\eta$, the associated stress vector being

$$\boldsymbol{Z} = \mu\left\{\text{grad } \phi + (\beta-1) \text{ grad } \psi + 2\frac{\partial\psi}{\partial z}\boldsymbol{k} - 2z \text{ grad } \frac{\partial\psi}{\partial z} + 2 \text{ grad } \frac{\partial\chi}{\partial z}\right\}. \tag{11}$$

3. The thermoelastic problem referred to cylindrical polar coordinates

Suppose that the thermoelastic stress system in the slab is set up by a temperature field which is zero on the clamped face $z = 0$ and takes a prescribed value $f_n(\rho) \cos n(\theta+\theta_0)$ on the traction free face $z = 1$. Here n is an arbitrary positive integer on zero and more general temperature distributions on $z = 1$ can be obtained by Fourier superposition. It is easily seen that the temperature field satisfying these boundary conditions is

$$T(\rho, \theta, z) = \cos n(\theta+\theta_0)\int_0^\infty \lambda A(\lambda) \sinh \lambda z J_n(\lambda\rho)d\lambda \tag{12}$$

where $A(\lambda)$ is determined by the requirement that

$$f_n(\rho) = \int_0^\infty \lambda A(\lambda) \sinh \lambda J_n(\lambda\rho)d\lambda. \tag{13}$$

Thus, using the Hankel inversion theorem,

$$A(\lambda) \sinh \lambda = \int_0^\infty \rho f_n(\rho)J_n(\lambda\rho)d\rho; \tag{14}$$

further, from (10) the function ϕ is found as

$$\phi = 2\alpha(1+\eta) \cos n(\theta+\theta_0)\int_0^\infty A(\lambda) \cosh \lambda z J_n(\lambda\rho)d\lambda. \tag{15}$$

It should be noted that if $A(\lambda)$ is $O(\lambda^{-n-1})$ or greater at the origin $\lambda = 0$ the integrand in (15) becomes singular there. We shall assume here and elsewhere that such singularities are excluded by the reinterpretation of the integral as a contour integral with a suitable indentation in the contour at the origin. In any event all real integrals derived from (15) which represent quantities of physical interest have integrands which are analytic at $\lambda = 0$.

In order to solve the elastic problem we now represent the harmonics ψ and χ in (9) as

$$\psi = \cos n(\theta+\theta_0) \int_0^\infty \{P_1(\lambda)\cosh\lambda z + P_2(\lambda)\sinh\lambda z\} J_n(\lambda\rho)d\lambda, \tag{16}$$

and

$$\chi = \cos n(\theta+\theta_0) \int_0^\infty Q(\lambda)\sinh\lambda z J_n(\lambda\rho)d\lambda. \tag{17}$$

With this choice of ψ and χ the conditions $u_\rho = u_\theta = 0$ on the clamped face are satisfied identically and the condition $u_z = 0$ on $z = 0$ gives

$$2\alpha(1+\eta)A(\lambda)+\beta P_1(\lambda)+2\lambda Q(\lambda) = 0. \tag{18}$$

On the face $z = 1$ the mechanical boundary conditions are

$$\tau_{\rho z} = \tau_{\theta z} = \sigma_z = 0 \tag{19}$$

where, in the usual notation, $\tau_{\rho z}$, $\tau_{\theta z}$ and σ_z are the components of $\boldsymbol{Z}$ in the directions of ρ, θ and z increasing, respectively. Referring to (11), the conditions $\tau_{\rho z} = \tau_{\theta z} = 0$ on $z = 1$ lead to the same relation, namely

$$\begin{aligned}2\alpha(1+\eta)A(\lambda)\cosh\lambda + P_1(\lambda)\{(\beta-1)\cosh\lambda - 2\lambda\sinh\lambda\} \\ +P_2(\lambda)\{(\beta-1)\sinh\lambda-2\lambda\cosh\lambda\} \\ +2\lambda Q(\lambda)\cosh\lambda = 0,\end{aligned} \tag{20}$$

and the condition $\sigma_z = 0$ on $z = 1$ gives

$$\begin{aligned}2\alpha(1+\eta)A(\lambda)\sinh\lambda + P_1(\lambda)\{(\beta+1)\sinh\lambda-2\lambda\cosh\lambda\} \\ +P_2(\lambda)\{(\beta+1)\cosh\lambda-2\lambda\sinh\lambda\}+2\lambda Q(\lambda)\sinh\lambda = 0.\end{aligned} \tag{21}$$

On solving (18), (20) and (21) we find

$$\left.\begin{aligned}P_1(\lambda) &= -2\alpha(1+\eta)A(\lambda)(\beta+\cosh 2\lambda)/\Delta(2\lambda), \\ P_2(\lambda) &= -2\alpha(1+\eta)A(\lambda)(2\lambda-\sinh 2\lambda)/\Delta(2\lambda), \\ \text{and}\quad Q(\lambda) &= -2\alpha(1+\eta)A(\lambda)(1+4\lambda^2+\beta\cosh 2\lambda)/\lambda\Delta(2\lambda)\end{aligned}\right\} \tag{22}$$

where $\Delta(x) = 1+\beta^2+x^2+2\beta\cosh x$, thus determining the harmonics ψ and χ in terms of the known function $A(\lambda)$. In general, for a given $f_n(\rho)$, the evaluation of the integrals occurring in the displacement and stress components is complicated by the existence of the factor $\{\Delta(2\lambda)\}^{-1}$ in the integrands and it is not possible to find closed forms.

Let us consider briefly the axisymmetric analogue of the problem treated in (6). Suppose that the temperature distribution on $z = 1$ is given by

$$T(\rho, 1) = \begin{cases} T_0, & 0 \leqq \rho \leqq a \\ 0, & \rho > a. \end{cases} \tag{23}$$

Then, from (14),

$$A(\lambda) = \frac{T_0 a J_1(\lambda a)}{\lambda \sinh \lambda}, \tag{24}$$

and the temperature field in the slab is

$$T(\rho, z) = T_0 a \int_0^\infty \frac{\sinh \lambda z}{\sinh \lambda} J_1(\lambda a) J_0(\lambda \rho) d\lambda. \tag{25}$$

For given ρ, a and z the infinite integral in (25) must be evaluated numerically; an alternative series expansion, rapidly convergent for large ρ, can be found by considering the contour integral

$$\int_C \frac{\sinh \zeta z}{\sinh \zeta} F(\zeta) d\zeta$$

where

$$F(\zeta) = \begin{cases} H_1^{(1)}(\zeta a) J_0(\zeta \rho), & \text{if } \rho < a, \\ J_1(\zeta a) H_0^{(1)}(\zeta \rho), & \text{if } \rho > a. \end{cases}$$

Here $\zeta = \lambda + i\sigma$ and the contour C consists of a large semi-circle in the upper ζ-half-plane, and the real axis indented in the upper half-plane at the origin. We find that

$$T = \tfrac{1}{2} T_0 a \int_C \frac{\sinh \zeta z}{\sinh \zeta} F(\zeta) d\zeta \tag{26}$$

$$= \begin{cases} T_0 \left(1 - 2a \sum_{n=1}^{\infty} (-1)^{n-1} \sin n\pi z K_1(n\pi a) I_0(n\pi\rho) \right), & \rho < a, \\ 2T_0 a \sum_{n=1}^{\infty} (-1)^{n-1} \sin n\pi z I_1(n\pi a) K_0(n\pi\rho), & \rho > a. \end{cases} \tag{27}$$

Integration of (25) with respect to z to compute ϕ produces a singular integrand of the type mentioned earlier in this section and hence we must use instead the contour integral in (26). We have

$$\phi = \alpha(1+\eta) T_0 a \int_C \frac{\cosh \zeta z}{\zeta \sinh \zeta} \mathrm{F}(\zeta) d\zeta$$

$$= \begin{cases} 2\alpha(1+\eta) T_0 \left(z - 2a \sum_{n=1}^{\infty} (-1)^n \dfrac{\cos n\pi z}{n\pi} K_1(n\pi a) I_0(n\pi\rho) \right), & \rho < a, \\ 4\alpha(1+\eta) T_0 a \sum_{n=1}^{\infty} (-1)^n \dfrac{\cos n\pi z}{n\pi} I_1(n\pi a) K_0(n\pi\rho), & \rho > a. \end{cases} \tag{28}$$

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Further, from (16), (17), (22) and (24) the harmonics ψ and χ are given by

$$\psi = -\tfrac{1}{2}\alpha(1+\eta)T_0 a \int_C \frac{\{(\beta+\cosh 2\zeta)\cosh \zeta z+(2\zeta-\sinh 2\zeta)\sinh \zeta z\}}{\zeta\Delta(2\zeta)\sinh \zeta} F(\zeta)d\zeta, \tag{29}$$

and

$$\chi = -\tfrac{1}{2}\alpha(1+\eta)T_0 a \int_C \frac{(1+4\zeta^2+\beta\cosh 2\zeta)\sinh \zeta z}{\zeta^2\Delta(2\zeta)\sinh \zeta} F(\zeta)d\zeta, \tag{30}$$

from which infinite series expansions can be derived, leading to formulae for the stress and displacement components analogous to those given in (6). However, these formulae contain summations over those roots of $\Delta(2\zeta) = 0$ (see (6)) with positive imaginary parts and from the practical point of view it is probably better to evaluate numerically the real integral forms of these quantities. For example, the stress components on the plane $z = 0$, quantities of particular physical interest ((6), page 2), are given by

$$\tau_{\rho z} = 2\alpha(1+\eta)T_0 a\mu \int_0^\infty \frac{\{(1-\beta)(1-\cosh 2\lambda)+4\lambda^2\}}{\Delta(2\lambda)\sinh \lambda} J_1(\lambda a)J_1(\lambda\rho)d\lambda, \tag{31}$$

and

$$\sigma_z = -8\alpha(1-\eta^2)T_0 a\mu \int_0^\infty \frac{(2\lambda-\sinh 2\lambda)}{\Delta(2\lambda)\sinh \lambda} J_1(\lambda a)J_0(\lambda\rho)d\lambda. \tag{32}$$

4. The problem of Martin and Payton

We now apply the solutions (9) and (10) of the thermoelastic equations to the problem considered by Martin and Payton (6). Again, our analysis is formal and any apparent singularities at the origin introduced by the manipulation of Fourier integrals are assumed to be removed, without comment, by suitable modification of the contour of integration.

Relative to the Cartesian axes introduced in § 2 we now suppose that on the stress free face the prescribed temperature distribution is a function $f(x)$ of the coordinate x only. The Fourier representation

$$T(x, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(\lambda)e^{i\lambda x}\sinh \lambda z d\lambda \tag{33}$$

gives zero temperature on the clamped face $z = 0$ and $a(\lambda)$ is determined by the equation

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(\lambda)e^{i\lambda x}\sinh \lambda d\lambda. \tag{34}$$

Fourier inverting,

$$a(\lambda)\sinh \lambda = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx \tag{35}$$

and the function ϕ is found from (10) as

$$\phi(x, z) = \frac{\alpha(1+\eta)}{\pi}\int_{-\infty}^{\infty} \frac{a(\lambda)}{\lambda} e^{i\lambda x}\cosh \lambda z d\lambda. \tag{36}$$

(If $a(\lambda)$ is $O(1)$ at $\lambda = 0$, (36) is an example of a Fourier integral with a singular integrand referred to above.)

The mechanical boundary conditions to be satisfied are

$$\left.\begin{aligned} u_x = u_z &= 0 \text{ on } z = 0 \\ \text{and} \qquad \tau_{xy} = \sigma_z &= 0 \text{ on } z = 1 \end{aligned}\right\} \tag{37}$$

Thus we represent the harmonics ψ and χ as

$$\psi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \{p_1(\lambda)\cosh\lambda z + p_2(\lambda)\sinh\lambda z\} e^{i\lambda x} d\lambda \tag{38}$$

and

$$\chi = \frac{1}{2\pi}\int_{-\infty}^{\infty} q(\lambda)\sinh\lambda z e^{i\lambda x} d\lambda, \tag{39}$$

the condition $u_x = 0$ on $x = 0$ thereby being satisfied. Application of the three remaining boundary conditions in (37) leads to equations of the form (18), (20) and (21) with $A(\lambda)$ replaced by $\frac{a(\lambda)}{\lambda}$. Thus we find

$$\left.\begin{aligned} p_1(\lambda) &= -2\alpha(1+\eta)a(\lambda)(\beta+\cosh 2\lambda)/\lambda\Delta(2\lambda), \\ p_2(\lambda) &= -2\alpha(1+\eta)a(\lambda)(2\lambda-\sinh 2\lambda)/\lambda\Delta(2\lambda), \\ \text{and} \qquad q(\lambda) &= -2\alpha(1+\eta)a(\lambda)(1+4\lambda^2+\beta\cosh 2\lambda)/\lambda^2\Delta(2\lambda), \end{aligned}\right\} \tag{40}$$

and the Fourier transforms of the displacement and stress components are readily shown to agree with those calculated in (6) by a much more complicated procedure.

5. A further class of mixed boundary condition problems

A further class of mixed boundary condition problems arises if, instead of being rigidly held, the face $z = 0$ of the slab rests against a rigid frictionless foundation (i.e. the shear stress and normal component of displacement are zero on $z = 0$). It was observed by Sneddon (8) that, if the foundation is thermally insulated, a prescribed temperature on the stress free face produces a plane state of stress parallel to the boundary faces. In terms of our solution the condition $\frac{\partial T}{\partial z} = 0$ on $z = 0$ implies the vanishing of ϕ on $z = 0$, and all boundary conditions can be satisfied by setting $\psi \equiv 0$ and

$$-2\frac{\partial \chi}{\partial z} = \phi.$$

Thus, from (11) we have $Z \equiv 0$ and

$$\boldsymbol{u} = -2\frac{\partial \chi}{\partial z}\boldsymbol{k} + \operatorname{grad}\chi.$$

Suppose now that the foundation is maintained at zero temperature. With a prescribed surface temperature of the form $f_n(\rho)\cos n(\theta+\theta_0)$, T is again given by equations (12), (13) and (14), and now a three-dimensional state of stress results. The boundary conditions are

$$\left.\begin{aligned} &\tau_{\rho z} = \tau_{\theta z} = u_z = 0 \text{ on } z = 0, \\ \text{and} \quad &\tau_{\rho z} = \tau_{\theta z} = \sigma_z = 0 \text{ on } z = 1. \end{aligned}\right\} \tag{41}$$

Representing ψ as in (16) and χ by

$$\chi = \cos n(\theta+\theta_0) \int_0^\infty \{Q_1(\lambda)\cosh \lambda z + Q_2(\lambda)\sinh \lambda z\} J_n(\lambda\rho)\,d\lambda, \tag{42}$$

the conditions (41) lead to a set of four simultaneous equations for P_1, P_2, Q_1 and Q_2 with solution

$$\left.\begin{aligned} P_1(\lambda) &= -2\alpha(1+\eta)A(\lambda)/(1+\beta), \\ P_2(\lambda) &= -2\alpha(1+\eta)A(\lambda)(\cosh 2\lambda-1)/(1+\beta)\Sigma, \\ Q_1(\lambda) &= -2\alpha(1+\eta)A(\lambda)\{(\beta-1)\sinh^2\lambda+2\lambda^2\}/(1+\beta)\Sigma, \\ \text{and} \quad Q_2(\lambda) &= -2\alpha(1+\eta)A(\lambda)/\lambda(1+\beta), \end{aligned}\right\} \tag{43}$$

where $\Sigma = 2\lambda+\sinh 2\lambda$. The function Σ^{-1} in the integrands of the stress and displacement components is of common occurrence in thick plate theory; approximate methods of handling such integrals have been given by Sneddon ((9), p. 476).

Finally, we remark that if the temperature distribution in the slab is that of § **4**, the solution of the thermoelastic problem may be deduced from the previous paragraph as in § **4**, Fourier integrals replacing the Hankel transforms.

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