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COMBINATORIAL MATRICES WITH SMALL DETERMINANTS

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1. Introduction. In this paper we will be concerned with the determinants of matrices whose elements are 0, 1 or -1, 1. Accordingly, let $S_{n,k}$ be the set of $n \times n$ 0, 1 matrices with exactly k ones in each row and column; and let H_n be the set of $n \times n$ -1, 1 matrices. Let $J = J_n$ denote (as usual) the $n \times n$ matrix all of whose elements are one. Then J is the only element of $S_{n,n}$, J also belongs to H_n, and the elements of $S_{n,k}$ may be characterized as those $n \times n$ 0, 1 matrices A such that $AJ = JA = kJ$.

Let

(1) $m_{n,k} = \min|\det(A)|, \quad A \in S_{n,k}, \det(A) \neq 0,$

(2) $M_{n,k} = \max|\det(A)|, \quad A \in S_{n,k}.$

We will show below that $m_{n,k}$ is well-defined except when $n = k > 1$, and $n = 4$, $k = 2$. These are the only values such that every element of $S_{n,k}$ is singular, and will be tacitly excluded from all discussions involving $m_{n,k}$.

The Hadamard bound for determinants shows that $M_{n,k} \leqq k^{n/2}$. Furthermore it is clear that $\det(A) \equiv 0 \bmod k$ for every $A \in S_{n,k}$, so that $m_{n,k} \geqq k$. Thus we have the crude bounds

$$k \leqq m_{n,k} \leqq M_{n,k} \leqq k^{n/2}.$$

The Hadamard bound is exact for an orthogonal matrix; and if A is "close" to an orthogonal matrix, it can be expected to have a large determinant. Thus if $k = q + 1$, $n = q^2 + q + 1$, where q is a prime power, then there is a finite projective plane of order q; and if A is the incidence matrix of this plane, A is "close" to an orthogonal matrix, and

$$|\det(A)| = q^{(q^2+q)/2}(q + 1) = (k - 1)^{(n-1)/2}k.$$

Thus

$$(k - 1)^{(n-1)/2}k \leqq M_{n,k} \leqq k^{n/2},$$

when $k = q + 1$, $n = q^2 + q + 1$, and q is a prime power. This question is treated in detail and essentially solved completely by H. J. Ryser in his paper [**1**]. Accordingly we will confine ourselves to the study of $m_{n,k}$, and also to the easier question of the behavior of m_n, defined below:

(3) $m_n = \min|\det(A)|, \quad A \in H_n, \det(A) \neq 0.$

It is not difficult to show that $m_n = 2^{n-1}$.

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2. The behavior of $m_{n,k}$. We first prove a preliminary lemma.

LEMMA 1. *Let A be an $n \times n$ matrix such that $AJ = sJ$, $s \neq 0$. Then*

(4) $\det(J + A) = ((n + s)/s)\det(A)$.

In particular, if $A \in S_{n,k}$, then

(5) $\det(J - A) = (-1)^{n-1}((n - k)/k)\det(A)$.

Proof. The following argument uses the multilinearity of the determinant. Write A as the matrix of its column vectors:

$$A = [A_1, A_2, \ldots, A_n].$$

Let δ be the $n \times 1$ vector all of whose entries are one. Then

$$A_1 + A_2 + \ldots + A_n = s\delta,$$

and

$$J + A = [A_1 + \delta, A_2 + \delta, \ldots, A_n + \delta].$$

If we subtract the first column of $J + A$ from all the other columns, we find that

$$\det(J + A) = \det[A_1 + \delta, A_2 - A_1, \ldots, A_n - A_1].$$

The multilinearity now implies that

$$\det(J + A) = \det(A) + \det[\delta, A_2 - A_1, \ldots, A_n - A_1].$$

Furthermore,

$$\begin{aligned}\det(A) &= \det[A_1, A_2, \ldots, A_n] = \det[A_1, A_2 - A_1, \ldots, A_n - A_1]\\ &= \det[s\delta - (n-1)A_1, A_2 - A_1, \ldots, A_n - A_1],\end{aligned}$$

as may be seen by adding columns $2, 3, \ldots, n$ to column 1. This readily implies that

$$\det[\delta, A_2 - A_1, \ldots, A_n - A_1] = (n/s)\det(A)$$

and it follows that

$$\det(J + A) = \det(A) + (n/s)\det(A) = ((n + s)/s)\det(A).$$

Hence (4) is proved, and (5) is an immediate corollary. This completes the proof.

Formula (4) also holds when $s = 0$, in the form

$$\det(J + A) = n\Lambda,$$

where Λ is the product of the eigenvalues of A other than 0.

We now prove the fact mentioned previously:

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THEOREM 1. *Let n, k be integers such that $n \geqq k \geqq 1$. Then $S_{n,k}$ always contains a nonsingular matrix, except when $n = k > 1$, and $n = 4$, $k = 2$.*

Proof. It is clear that if $n = k$, then $S_{n,k}$ consists of the J matrix alone, which is singular for $n > 1$. Furthermore every element of $S_{4,2}$ is permutation equivalent to

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{or to} \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix};$$

and each of these is singular. The cases noted are thus genuine exceptions.

Let $P = P_n$ denote the $n \times n$ full cycle

$$P = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then $I + P + P^2 + \dots + P^{n-1} = J$, so that the powers of P are disjoint. Put

$$C_k = I + P + P^3 + P^5 + \dots + P^{2k-3}, \quad k \leqq (n/2) + 1.$$

Then $C_k \in S_{n,k}$. Furthermore, C_k is singular if and only if there is a ζ such that $\zeta^n = 1$, and

$$(6) \quad 1 + \zeta + \zeta^3 + \zeta^5 + \dots + \zeta^{2k-3} = 0.$$

If $\zeta = 1$, then (6) becomes $k = 0$, which is impossible. If $\zeta = -1$, then (6) becomes $k - 2 = 0$. Let us assume that $k > 2$. Then (6) does not hold for $\zeta = \pm 1$, and is equivalent to

$$(7) \quad 1 + \zeta \frac{1 - \zeta^{2k-2}}{1 - \zeta^2} = 0.$$

Take complex conjugates in (7) and equate the results. We find that $\zeta^{2k-2} = 1$. But this is impossible, as is evident from (7). It follows that C_k is nonsingular for $3 \leqq k \leqq (n/2) + 1$.

Now set $D_{n-k} = J - C_k$. Then $D_{n-k} \in S_{n,n-k}$, and $(n/2) - 1 \leqq n - k \leqq n - 3$. Furthermore, Lemma 1 implies that

$$\det (D_{n-k}) = (-1)^{n-1}((n - k)/k) \det (C_k),$$

so that D_{n-k} is also nonsingular. We are left with the cases $k = 1, 2, n - 2,$

$n - 1$. Now $S_{n,1}$ consists of the permutation matrices, and all of these are nonsingular; and every element of $S_{n,n-1}$ is permutation equivalent to $J - I$, which is nonsingular for $n > 1$. We are thus left with the cases $k = 2, n - 2$.

Suppose first that $k = 2$. Note that if n is odd and >1, then $I + P_n \in S_{n,2}$, is nonsingular, and has determinant 2. If n is even and >4, then $n = m + 3$, where m is odd and >1. Furthermore, $Q = I + (P_3 \dotplus P_m)$ belongs to $S_{n,2}$, is nonsingular, and has determinant 4. This disposes of $k = 2$.

Now suppose that $k = n - 2$. Then if n is odd and >1, $J - I - P_n \in S_{n,n-2}$ and Lemma 1 implies that $\det(J - I - P_n) = n - 2$. If n is even and >4, then $J - Q \in S_{n,n-2}$ and Lemma 1 implies that $\det(J - Q) = 4 - 2n$. Thus all cases are covered and the proof is complete.

The next result supplies a lower bound for $m_{n,k}$ and determines it exactly when $(n, k) = 1$.

THEOREM 2. *Let $d = (n, k)$ denote the greatest common divisor of n and k. Let A be any matrix of $S_{n,k}$. Then* $\det(A) \equiv 0 \bmod kd$, *so that $m_{n,k} \geqq kd$. Furthermore, $m_{n,k} = k$ if and only if $d = 1$.*

Proof. Let A be any matrix of $S_{n,k}$. We perform the following elementary operations on A (which do not change the determinant):

i) add rows $2, 3, \ldots, n$ to row 1;
ii) add columns $2, 3, \ldots, n$ to column 1.

The first row of the resulting matrix is $\alpha = [nk, k, k, \ldots, k]$ and the first column is α^T. It follows at once that $\det(A)$ is divisible by kd, so that $m_{n,k} \geqq kd$. Thus the first part of the theorem is proved. To prove the second part, we note first that if $m_{n,k} = k$, then $d = 1$. Suppose then that $d = 1$, and consider the matrix

$$E = E_{n,k} = I + P + P^2 + \ldots + P^{k-1}. \tag{8}$$

Then $E \in S_{n,k}$, and

$$\det(E) = \prod_{\zeta^n=1} (1 + \zeta + \zeta^2 + \ldots + \zeta^{k+1}) = k \prod_{\zeta^n=1,\zeta\neq 1} \frac{1-\zeta^k}{1-\zeta}.$$

Since $(k, n) = 1$, ζ^k runs over all nth roots of unity other than 1 once and once only as ζ does. Hence

$$\prod_{\zeta^n=1,\zeta\neq 1} (1 - \zeta^k) = \prod_{\zeta^n=1,\zeta\neq 1} (1 - \zeta) \neq 0,$$

and it follows that $\det(E) = k$. This completes the proof.

There is another case when $m_{n,k}$ can be determined completely, which represents the other extreme. We have

THEOREM 3. *Suppose that k divides n, and $n > 2k$. Then $m_{n,k} = k^2$.*

Proof. We have $n = kt$, $t > 2$. Put

$$n = n_1 + n_2, \quad n_1 = (t-1)k - 1, \quad n_2 = k + 1.$$

Then $(n_1, k) = 1$, $n_1 > k$, $(n_2, k) = 1$, $n_2 > k$. Put $A = E_{n_1,k} \dotplus E_{n_2,k}$ (see formula (8)). Then $A \in S_{n,k}$ and $\det(A) = k^2$. Thus $m_{n,k} \leqq k^2$. But also $m_{n,k} \geqq k^2$, by Theorem 2. It follows that $m_{n,k} = k^2$, and the proof is complete.

We note that if k is odd, then $m_{2k,k}$ may be shown to be k^2, by consideration of the matrix

$$P + P^2 + \ldots + P^{k-1} + P^{k+1}.$$

These results prompt the reasonable conjecture that $m_{n,k} = kd = k(n, k)$. We are going to get further information on the size of $m_{n,k}$. The following lemma, which is of independent interest, will be useful for this purpose.

LEMMA 2. *Let m be a positive integer, c any integer. Then if m is odd, or if m is even and c is even, the congruence*

$$c \equiv x + y \mod m \tag{9}$$

has a solution such that

$$(x, m) = (y, m) = 1.$$

Proof. Suppose first that m_1, m_2 are relatively prime positive integers, and that x_1, y_1, x_2, y_2 may be found such that

$$c \equiv x_1 + y_1 \mod m_1, \quad c \equiv x_2 + y_2 \mod m_2,$$
$$(x_1, m_1) = (y_1, m_1) = 1, \quad (x_2, m_2) = (y_2, m_2) = 1.$$

Determine x, y by the Chinese Remainder Theorem so that

$$x \equiv x_1 \mod m_1, \quad x \equiv x_2 \mod m_2, \quad y \equiv y_1 \mod m_1, \quad y \equiv y_2 \mod m_2.$$

Then it is readily verified that $c \equiv x + y \mod m_1m_2$, and that $(x, m_1m_2) = (y, m_1m_2) = 1$. It follows that it is only necessary to prove the lemma when $m = p^e$, p prime. Here we proceed as follows: if $(p, c-1) = 1$, take $x = c - 1$, $y = 1$. If p divides $c - 1$, take $x = c - 2$, $y = 2$ (notice that $p \neq 2$ in this case). This completes the proof.

The case when c is odd and m is even is a genuine exception, since x and y cannot then both be odd in (9).

We now prove the following result.

THEOREM 4. *Suppose that k is odd, or that n and k are even, and that $n > 3k - 2$. Then $m_{n,k} \leqq k^2$.*

Proof. By Lemma 2 we may determine k_1, k_2 so that $0 \leqq k_1, k_2 \leqq k - 1$, $(k_1, k) = (k_2, k) = 1$, and $n \equiv k_1 + k_2 \mod k$. Put $n = k_1 + k_2 + tk$. Then

$$3k - 2 < n = k_1 + k_2 + tk \leqq 2k - 2 + tk,$$

so that $t > 1$. Thus $t \geqq 2$, and we may write $n = n_1 + n_2$, where $n_1 = (t-1)k + k_1$, $n_2 = k + k_2$, and $(n_1, k) = 1$, $n_1 > k$, $(n_2, k) = 1$, $n_2 > k$. Put $A = E_{n_1,k} \dotplus E_{n_2,k}$. Then $A \in S_{n,k}$ and $\det(A) = k^2$, so that $m_{n,k} \leqq k^2$. This completes the proof.

Similar bounds may be derived when $k > (2n-2)/3$, and n and k are of opposite parity, by consideration of the matrix $J - A$.

Finally, we prove the following in this section:

THEOREM 5. *The number $m_{n,k}$ is bounded above by a number which depends only on k.*

Proof. We may assume that $k > 2$, since the examples given in Theorem 1 together with Theorem 2 show that

$$m_{n,2} = \begin{cases} 2 & n \text{ odd}, n > 1, \\ 4 & n \text{ even}, n > 4. \end{cases}$$

Write

$$n = qk + r, \quad 0 \leqq r \leqq k - 1.$$

Suppose first that $q > 3$. Put

$$n_1 = (q-3)k + 1, \quad n_2 = k + 1, \quad n_3 = 2k + r - 2,$$

so that $n = n_1 + n_2 + n_3$. Then

$$(n_1, k) = 1, n_1 > k, \quad (n_2, k) = 1, \quad n_2 > k, \quad k < n_3 \leqq 3k - 3.$$

It follows by the method of Theorem 4 and by the Hadamard bound for determinants that

$$m_{n,k} \leqq k \cdot k \cdot k^{n_3/2} \leqq k^{(3k+1)/2}.$$

Next suppose that $q \leqq 3$. Then $n \leqq 4k - 1$, and so

$$m_{n,k} \leqq k^{n/2} \leqq k^{(4k-1)/2}.$$

These inequalities clearly imply the result.

3. The behavior of m_n. We now consider the easier question of the determination of m_n, defined by (3). We shall prove

THEOREM 6. *The number m_n defined by* (3) *is equal to* 2^{n-1}.

Proof. Let A be any matrix of H_n. Add the first row of A to all the other rows. In the resulting matrix, all elements other than those in the first row are even. It follows that $\det(A) \equiv 0 \bmod 2^{n-1}$. Now let $A = (a_{ij})$ be the matrix of H_n such that

$$a_{ij} = \begin{cases} 1 & i \leqq j \\ -1 & i > j. \end{cases}$$

Add the first row of A to all the other rows. The resulting matrix is upper triangular with diagonal elements $1, 2, 2, \ldots, 2$ and so has determinant 2^{n-1}. It follows that $m_n = 2^{n-1}$, and the proof is complete.

Reference

1. H. J. Ryser, *Maximal determinants in combinatorial investigations,* Can. J. Math. *8* (1956), 245–249.

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