A SET THEORY FOUNDED ON UNIQUE GENERATING PRINCIPLE

KATUZI ONO

The most important thing for a set theory seems to be that it can generate new mathematical objects, and I think that there must be an underlying principle, simple and unique, which unifies the acts of generating. The naive set theory has a unique generating principle, which defines, by any proposition on a variable x, the set of all x's satisfying the proposition. Certainly, we must restrict this generating principle so as to exclude all contradictions it contains, without losing its essential rôle as logic of mathematics, and at the same time we would like to keep its uniqueness and simplicity. Two ways of approach, the one from the axiomatic set theory and the other from logical side, were found by mathematicians. A system of axiomatic set theory was first given by Zermelo [1], and the works of Fraenkel [2], von Neumann [3], Bernays [4], Bourbaki [5], Goedel [6], and Ackermann [7] followed along this line. The axiom systems consist of some generating principles together with the equality axioms, the axiom of extensionality, the axiom of infinity, and the axiom of choice. Another way of approach was given by the type theory of Whitehead and Russell [8], followed by the stratification theory of Quine [9], [10]. These theories gave suitable restriction on the principle of abstraction itself, and so they have a unique generating principle. My present work follows along the line of axiomatic set theory, but it has only one generating principle G together with two equality axioms E1 and E2, and the axiom of infinity described in two formulas P1 and P2. This axiom system is equivalent to Zermelo-Fraenkel's system excluding the axiom of choice. Precise description of the system of axioms is given in §2, and a proof of equivalence is given in §3.

1. Logic

Our logic is the ordinary predicate logic of the first order. As for the logical symbols, $(Ux)\mathfrak{A}(x)$ (Uniqueness of x satisfying $\mathfrak{A}(x)$) and $(UEx)\mathfrak{A}(x)$

Received July 15, 1957.

https://doi.org/10.1017/S0027763000022017 Published online by Cambridge University Press

(Unique existence of x satisfying $\mathfrak{A}(x)$) are used together with the ordinary notations. Exact definitions of these quantifiers will be given later.

In the course of our reasoning, we *may* or *may not* use Ackermann's ε-symbol. ($\varepsilon_x \mathfrak{A}(x)$ denotes one of the x's which satisfies $\mathfrak{A}(x)$). Moreover, we *may* or *may not* assume for ε-symbol

$$(x)[\mathfrak{A}(x) \Longleftrightarrow \mathfrak{B}(x)] \Longrightarrow \varepsilon_x \mathfrak{A}(x) = \varepsilon_x \mathfrak{B}(x).$$

The only thing we have to do is to fix the system we are considering. If we allow to use ε-symbol with suitable axioms or inference schemes, we have a set theory with the axiom of choice.

2. Set theory

As fundamental notions of our set theory, we take up three: "$\in$", "$\{ \}$", and "P".

$x \in y$ denotes that x is an element of the set y.

$\{x\}$ denotes the set which has x as its only element.

P denotes a fixed set which is an example of infinite sets.

Our system of axioms consists of five axioms E1, E2, P1, P2, and G. E1 and E2 are essentially equivalent to the equality axioms, P1 and P2 are essentially two out of five Peano's axioms, and the axiom G is the only generating principle of our set theory. All the theories of mathematics and their axioms can be formalized in our system, if we only adjoin the axiom of choice in a suitable way.

Equality axioms:

E1. $x \in \{x\}$,

E2. $x \in \{y\} \Longrightarrow [\mathfrak{A}(x) \Longrightarrow \mathfrak{A}(y)]$.

If we replace in these axioms "$x \in \{y\}$" by "$x = y$", we get the ordinary equality axioms:

E′1. $x = x$,

E′2. $x = y \Longrightarrow [\mathfrak{A}(x) \Longrightarrow \mathfrak{A}(y)]$.

In connection with defining "$x = y$" by "$x \in \{y\}$", let us here mention a few words on the definitions used in our system. Namely, there are two types of definitions. The one consists of definitions which introduce abbreviations for formulas or terms. In this case, it is sufficient that for every defined formula or term we can distinctly know which formula or term it denotes. By the definition of this type we have the following:

Definition 1. $x = y$ denotes $x \in \{y\}$.

Definition 2. $(Ux)\mathfrak{A}(x)$ denotes $(x, y)[(\mathfrak{A}(x)$ and $\mathfrak{A}(y)) \Rightarrow x = y]$.

Definition 3. $(UEx)\mathfrak{A}(x)$ denotes $[(Ux)\mathfrak{A}(x)$ and $(Ex)\mathfrak{A}(x)]$.

Definition 4. $m \subseteqq n$ denotes $(x)(x \in m \Rightarrow x \in n)$.

The other consists of definitions of terms $\mathfrak{t}(x, \ldots, w)$ by $\mathfrak{A}(t, x, \ldots, w)$ after showing $(x, \ldots, w)(UEt)\mathfrak{A}(t, x, \ldots, w)$. $\mathfrak{t}(x, \ldots, w)$ denotes the t determined by the condition $\mathfrak{A}(t, x, \ldots, w)$. We need only definitions of these two types.

Axioms for existence of infinite sets:

P1. $$x \in P \Rightarrow \{x\} \in P,$$

P2. $$(x)(x \notin v) \Rightarrow v \in P.$$

These axioms correspond to the Peano's axioms:

P′1. If x is a natural number, then x' is so too.

P′2. 0 is a natural number.

When we want to construct the theory of natural numbers in our system, the simplest way for us may be to express 0, 1, 2, . . . by $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, . . . , where $\emptyset$ expresses the null set. The existence of the null set can be proved in our system.

Generating principle:

G. $$(x)(Uy)\mathfrak{A}(x, y) \Rightarrow (UEq)(y)[y \in q \Rightarrow (Ex, u)^{1)}(x \subseteqq u \in p \text{ and } \mathfrak{A}(x, y))]$$

(Bound variables q and u are to be supposed to occur only in the indicated places).

This axiom is formally similar to the axiom of replacement:

$$(x)(Uy)\mathfrak{A}(x, y) \Rightarrow (Eq)(y)[y \in q \Leftrightarrow (Ex)(x \in p \text{ and } \mathfrak{A}(x, y))].$$

However, the axiom G is much stronger than the axiom of replacement.

1) (Ex, u) stands for $(Ex)(Eu)$.

3. Equivalence

To make sure that our system of axioms

E1. $x \in \{x\}$,

E2. $x \in \{y\} \Rightarrow [\mathfrak{A}(x) \Rightarrow \mathfrak{A}(y)]$,

P1. $x \in P \Rightarrow \{x\} \in P$,

P2. $(x)(x \notin v) \Rightarrow v \in P$,

G. $(x)(Uy)\mathfrak{A}(x, y) \Rightarrow (UEq)(y)[y \in q \Longleftrightarrow (Ex, u)(x \subseteqq u \in p \text{ and } \mathfrak{A}(x, y))]$

is sufficient for our purpose, I will show that it is equivalent to an axiom system for Zermelo-Fraenkel's set theory without the axiom of choice.

Let us now write down an example of axiom systems for Zermelo-Fraenkel's set theory. In this theory the fundamental notions are "$\in$", "$=$", and "P", and the axioms are as follows (the bound variable q in S9 and S10 is supposed to occur only in the indicated places):

S1. $x = x$

(The first equality axiom)

S2. $x = y \Rightarrow [\mathfrak{A}(x) \Rightarrow \mathfrak{A}(y)]$.

(The second equality axiom)

S3. $(t)[t \in u \Longleftrightarrow t = x] \Rightarrow [x \in P \Rightarrow u \in P]$.

(The successor of any element of P is also contained in P)

S4. $(x)(x \notin v) \Rightarrow v \in P$.

(P contains the null set)

S5. $(t)[t \in x \Longleftrightarrow t \in y] \Longleftrightarrow x = y$.

(Any set is perfectly determined by its elements)

S6. $(Ep)(u)[(u = x \text{ or } u = y) \Rightarrow u \in p]$.

(There is a set which contains x and y)

S7. $(Eq)(u)[(Ex)(u \in x \in p) \Rightarrow u \in q]$.

(There is a set which contains all the elements of elements of p)

S8. $(Eq)(u)[u \subseteqq p \Rightarrow u \in q]$,

where $u \subseteqq p$ stands for $(x)[x \in u \Rightarrow x \in p]$.

(There is a set which contains all the subsets of p)

S9. $(Eq)(u)[u \in q \Longleftrightarrow (u \in p \text{ and } \mathfrak{A}(u))]$.

(The aussonderungsaxiom: There is a set q which consists of all the elements of a given set p satisfying a given condition $\mathfrak{A}(u)$)

S10. $(x)(Uy)\mathfrak{A}(x, y) \Longrightarrow (Eq)(y)[(Ex)(x \in p \text{ and } \mathfrak{A}(x, y)) \Longrightarrow y \in q]$

(The axiom of replacement: There is a set q which contains all the images of a given set p by a given function).

Hereafter we denote the axiom system E1, E2, P1, P2, and G shortly by EPG and the axiom system S1-S10 by S. In the following lines we show the equivalence of EPG and S by suitable interpretation.

3.1. *EPG is provable in S*

By S5, S6 and S9, we can easily prove

S(1) $(UEp)(u)[u \in p \Longleftrightarrow u = x]$,

so we can define $\{x\}$ by

S(2) $\{x\}$ denotes the p determined by $(u)[u \in p \Longleftrightarrow u = x]$.

By the definition S(2) we can conclude directly

S(3) $u \in \{x\} \Longleftrightarrow u = x$.

From S(3) and the axioms S1 and S2, we have

S(4) $x \in \{x\}$,

S(5) $x \in \{y\} \Longrightarrow [\mathfrak{A}(x) \Longrightarrow \mathfrak{A}(y)]$.

These are the axioms E1 and E2 of EPG.

From S(3) and the axiom S3, we have

S(6) $x \in P \Longrightarrow \{x\} \in P$.

This is the axiom P1 of EPG. The axiom S4 itself is the axiom P2 of EPG.

We are now only to prove the axiom G in S. Clearly

S(7) $(t)[(Es)(t \in s \in p) \Longrightarrow t \in q] \Longrightarrow (x)[(Eu)(x \subseteqq u \in p) \Longrightarrow x \subseteqq q]$

is provable; so we can prove from this and the axioms S7, S8 and S10

S(8) $(x)(Uy)\mathfrak{A}(x, y) \Longrightarrow (Eq)(y)[(Ex, u)(x \subseteqq u \in p \text{ and } \mathfrak{A}(x, y)) \Longrightarrow y \in q]$

From S(8) and the axioms S5 and S9, we have finally

S(9) $(x)(Uy)\mathfrak{A}(x,y) \Rightarrow (UEq)(y)[y \in q \Longleftrightarrow (Ex, u)(x \subseteqq u \in p \text{ and } \mathfrak{A}(x,y))]$,

which is the axiom G of EPG itself.

3.2. *S is provable in EPG*

As has been already mentioned, the interpretation for "=" is:

EPG(1) $x = y$ denotes $x \in \{y\}$.

From this definition and the axioms E1 and E2 we have immediately

EPG(2) $x = x$,

EPG(3) $x = y \Rightarrow [\mathfrak{A}(x) \Rightarrow \mathfrak{A}(y)]$,

which are the axiom S1 and S2 of S. By these theorems and E1, we can easily prove

EPG(4) $(x)(Uy)[x = y \text{ and } \mathfrak{B}(y)]$,

$(Ex, u)[[x \subseteqq u \in \{p\} \text{ and } x = y \text{ and } \mathfrak{B}(y)] \Longleftrightarrow [y \subseteqq p \text{ and } \mathfrak{B}(y)]]$,

by which we have, from the axiom G by substituting $[x = y \text{ and } \mathfrak{B}(y)]$ for $\mathfrak{A}(x, y)$ and $\{p\}$ for p,

EPG(5) $(UEq)(y)[y \in q \Longleftrightarrow (y \subseteqq p \text{ and } \mathfrak{B}(y))]$.

From the definition EPG(1) and the axioms E1 and E2, we have also one of the Peano's axioms,

EPG(6) $(Ux)[\{x\} = y]$.

By making use of this formula and the axiom G, we can prove

EPG(7) $(x)(Uy)\mathfrak{C}(x, y) \Rightarrow (UEq)(y)[y \in q \Longleftrightarrow (Eu)(u \in p \text{ and } \mathfrak{C}(u, y))]$,

because

$(v)(Uy)\mathfrak{C}(v, y) \Rightarrow (x)(Uy)(Ev)[\{v\} = x \text{ and } \mathfrak{C}(v, y))]$,

$(Ex, u)[x \subseteqq u \in \{p\} \text{ and } (Ev)[\{v\} = x \text{ and } \mathfrak{C}(v, y)]]$

$\Longleftrightarrow (Ev)[v \in p \text{ and } \mathfrak{C}(v, y)]$

are easily provable. The axiom S10 of S is a weaker form of the formula EPG(7).

If we substitute $[x = y \text{ and } \mathfrak{D}(y)]$ for $\mathfrak{C}(x, y)$ in EPG (7), we have

EPG(8) $(UEq)(y)[y \in q \Longleftrightarrow (y \in p \text{ and } \mathfrak{D}(y))]$,

because

$(x)(Uy)[x = y \text{ and } \mathfrak{D}(y)]$,

$(Eu)[u \in p \text{ and } u = y \text{ and } \mathfrak{D}(y)] \Longleftrightarrow [y \in p \text{ and } \mathfrak{D}(y)]$

are easily provable in EPG. The axiom S9 of S is a weaker form of EPG(8).

If we substitute P for p and $y \neq y$ for $\mathfrak{D}(y)$ in EPG(8), we have

EPG(9) $(UEv)(x)\, x \notin v$,

hence we can define the null set $\emptyset$ by

EPG(10) $\emptyset$ denotes the v determined by $(x)(x \notin v)$.

Therefore we obtain

EPG(11) $x \notin \emptyset$.

For the null set $\emptyset$, we can prove easily one of the Peano's axioms

EPG(12) $\emptyset \neq \{x\}$ especially $\emptyset \neq \{\emptyset\}$.

By EPG(11) and the axiom P2, we can prove

EPG(13) $\emptyset \in P$,

and by making use of P1, we can prove also

EPG(14) $\{\emptyset\} \in P$.

Substituting P for p and $[y = \emptyset \text{ or } y = \{\emptyset\}]$ for $\mathfrak{D}(y)$ in EPG(8), we have

EPG(15) $(UEp)(x)[x \in p \Longleftrightarrow (x = \emptyset \text{ or } x = \{\emptyset\})]$,

and by this we define $\{\emptyset, \{\emptyset\}\}$ as follows:

EPG(16) $\{\emptyset, \{\emptyset\}\}$ denotes the p determined by

$$(x)[x \in p \Longleftrightarrow (x = \emptyset \text{ or } x = \{\emptyset\})].$$

Because, by using EPG(12), we can prove the following propositions

$(x)(Uy)[(x = \emptyset \text{ and } y = s) \text{ or } (x = \{\emptyset\} \text{ and } y = t)]$,

$(Ex)[x \in \{\emptyset, \{\emptyset\}\} \text{ and } ((x = \emptyset \text{ and } y = s) \text{ or } (x = \{\emptyset\} \text{ and } y = t))]$

$\Longleftrightarrow [y = s \text{ or } y = t]$,

we get, by substituting $\{\emptyset, \{\emptyset\}\}$ for p and $[(x = \emptyset \text{ and } y = s) \text{ or } (x = \{\emptyset\} \text{ and } y = t)]$ for $\mathfrak{S}(x, y)$ in EPG(7),

EPG(17) $(UEq)(y)[y \in q \Longleftrightarrow (y = s \text{ or } y = t)]$.

The axiom S6 of S is a weaker form of EPG(17).

https://doi.org/10.1017/S0027763000022017 Published online by Cambridge University Press

Further, if we substitute $y=y$ for $\mathfrak{D}(y)$ in EPG(8), we get

EPG(18) $(UEq)(y)[y \in q \Longleftrightarrow y \in p]$.

This is logically equivalent to the axiom S5 of S.

By EPG(18) and the definition EPG(1), we have from P1 and EPG(3)

EPG(19) $(t)[t \in u \Longleftrightarrow t = x] \Longrightarrow [x \in P \Longrightarrow u \in P]$,

thus the axiom S3 is proved.

Substituting $x = \{y\}$ for $\mathfrak{A}(x, y)$ in the axiom G and using EPG(6) and the easily provable proposition

$$(Ex, u)[x \subseteqq u \in p \text{ and } x = \{y\}] \Longleftrightarrow (Eu)(y \in u \in p),$$

we have

EPG(20) $(Eq)(y)[y \in q \Longleftrightarrow (Eu)(y \in u \in p)]$.

The axiom S7 of S is a weaker form of EPG(20).

Then again, if we substitute $y=y$ for $\mathfrak{B}(y)$ in EPG(5), we get

EPG(21) $(UEq)(y)[y \in q \Longleftrightarrow y \subseteqq p]$.

The axiom S8 of S is a weaker form of this proposition.

Thus we have proved that all the axioms S1-S10 of S are provable in EPG.

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Mathematical Institute
Nagoya University

https://doi.org/10.1017/S0027763000022017 Published online by Cambridge University Press