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Holomorphic Functions of Slow Growth on Nested Covering Spaces of Compact Manifolds

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Abstract. Let Y be an infinite covering space of a projective manifold M in $\mathbb{P}^N$ of dimension $n \geq 2$. Let C be the intersection with M of at most $n-1$ generic hypersurfaces of degree d in $\mathbb{P}^N$. The preimage X of C in Y is a connected submanifold. Let ϕ be the smoothed distance from a fixed point in Y in a metric pulled up from M. Let $\mathcal{O}_\phi(X)$ be the Hilbert space of holomorphic functions f on X such that $f^2 e^{-\phi}$ is integrable on X, and define $\mathcal{O}_\phi(Y)$ similarly. Our main result is that (under more general hypotheses than described here) the restriction $\mathcal{O}_\phi(Y) \to \mathcal{O}_\phi(X)$ is an isomorphism for d large enough.

This yields new examples of Riemann surfaces and domains of holomorphy in $\mathbb{C}^n$ with corona. We consider the important special case when Y is the unit ball $\mathbb{B}$ in $\mathbb{C}^n$, and show that for d large enough, every bounded holomorphic function on X extends to a unique function in the intersection of all the nontrivial weighted Bergman spaces on $\mathbb{B}$. Finally, assuming that the covering group is arithmetic, we establish three dichotomies concerning the extension of bounded holomorphic and harmonic functions from X to $\mathbb{B}$.

Introduction

Let $Y \to M$ be an infinite covering space of an n-dimensional projective manifold, $n \geq 2$. The function theory of such spaces is still not well understood. The central problem in this area is the conjecture of Shafarevich that the universal covering space of any projective manifold is holomorphically convex. This is a higher-dimensional variation on the venerable theme of uniformization. There are no known counterexamples to the conjecture, and it has been verified only in a number of fairly special cases.

Suppose M is embedded into a projective space by sections of a very ample line bundle L. The generic linear subspace of codimension $n-1$ intersects M in a 1-dimensional connected submanifold C called an L-curve. The preimage X of C is a connected Riemann surface embedded in Y. A natural approach to constructing holomorphic functions on Y is to extend them from X. This has the advantage of reducing certain questions to the 1-dimensional case, but the price one pays is having to work with functions of slow growth. Here, slow growth means slow exponential growth with respect to the distance from a fixed base point or a similar well-behaved exhaustion, in an L^2 or L^∞ sense. Functions in the Hardy class $H^p(X)$ grow slowly in this sense for p large enough.

In Section 1, we improve upon the main result of our earlier paper [Lár1] and show that if L is sufficiently ample, then the restriction map $\cdot|X$ is an isomorphism of the Hilbert

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spaces of holomorphic functions of slow growth. As before, the proof is based on the L^2 method of solving the $\bar{\partial}$-equation. This may be viewed as a sampling and interpolation theorem, related to those of Seip; Berndtsson and Ortega Cerdà; and others. See [BO], [Sei], and the references therein.

In Section 2, we use the isomorphism theorem to construct new examples of Riemann surfaces with corona. These easily defined surfaces have many symmetries, we have a simple description of characters in the corona, and the corona is large in the sense that it contains a domain in euclidean space of arbitrarily high dimension.

In Section 3, we adapt results of Hörmander on generating algebras of holomorphic functions of exponential growth to the case of covering spaces. Consequently, as shown in Section 4, the restriction map $\cdot|X$ may fail to be an isomorphism if L is not sufficiently ample compared to the exhaustion.

Under the mild assumption that the covering group is Gromov hyperbolic, we found in [Lár2] that the only obstruction to every positive harmonic function on X being the real part of a holomorphic function (in which case X has many holomorphic functions of slow growth that extend to Y) is a geometric condition involving the Martin boundary, characteristic of the higher-dimensional case. There are examples of infinitely connected X for which the obstruction is not present, but these have 1-dimensional boundary, whereas in general the curves X of interest to us do not: they have the same boundary as the ambient space Y. No examples with higher dimensional boundary are known.

In hopes of shedding some light on the dichotomy in [Lár2], we restrict ourselves from Section 4 onwards to what seems to be the most auspicious setting possible and let Y be the unit ball $\mathbb{B}$ in $\mathbb{C}^n$, $n \geq 2$. We present the results of the previous sections in a more explicit form. We obtain a sampling and interpolation theorem for the weighted Bergman spaces on $\mathbb{B}$. For each weight, the restriction to X induces an isomorphism from the weighted Bergman space on $\mathbb{B}$ to the one on X if L is sufficiently ample. This is in contrast to Seip's result that no sequence in the disc is both sampling and interpolating for any weighted Bergman space [Sei]. Also, every bounded holomorphic function on X extends to a unique function of infraexponential growth on $\mathbb{B}$, *i.e.*, a function in the intersection of all the nontrivial weighted Bergman spaces on $\mathbb{B}$, when L is sufficiently ample, for instance when L is the m-th tensor power of the canonical bundle K with $m \geq 2$. Whether the extension is itself bounded is an important open question.

In Section 5, assuming that the covering group is an arithmetic subgroup of the automorphism group $PU(1, n)$ of $\mathbb{B}$, we establish two dichotomies related to that in [Lár2] but using very different means. One of them says that either every holomorphic function f continuous up to the boundary on the preimage of a $K^{\otimes m}$-curve in a finite covering of M extends to a continuous function on $\bar{\mathbb{B}}$ which is holomorphic on $\mathbb{B}$, or the boundary functions of such functions f generate a dense subspace of the space of continuous functions $\partial\mathbb{B} \to \mathbb{C}$. This is probably another manifestation of the elusive phenomenon discovered in [Lár2]. Section 6 contains an analogous dichotomy for harmonic functions. In contrast to the case of holomorphic functions, it is easy to see that a harmonic function on X, continuous up to the boundary, generally does not extend to a plurisubharmonic function bounded above on $\mathbb{B}$.

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1 An Extension Theorem

We will be working with the following objects.

1. A covering space $\pi\colon Y \to M$ of a compact n-dimensional Kähler manifold M with a Kähler form ω.
2. A smooth function $\phi\colon Y \to \mathbb{R}$ such that $d\phi$ is bounded.
3. A line bundle L on M with canonical connection ∇ and curvature Θ in a hermitian metric h.
4. A section s of L over M with $\nabla s \neq 0$ at each point of its nonempty zero locus C. Then C is a smooth (possibly disconnected) hypersurface in M. Let $X = \pi^{-1}(C)$.

We denote the pullbacks to Y of ω, L, and s by the same letters.

Let $\mathcal{O}_\phi(X)$ be the vector space of holomorphic functions f on X such that $f^2 e^{-\phi}$ is integrable on X with respect to the volume form of the induced Kähler metric on X. This is a Hilbert space with respect to the inner product

$$(f, g) \mapsto \int_X f\bar{g}e^{-\phi}\omega^{n-1}.$$

We define $\mathcal{O}_\phi(Y)$ similarly.

Let $U_1, \dots, U_m$ be the pullbacks of shrunk coordinate polydiscs in which $C = \{z_n = 0\}$, covering a neighbourhood of C. If $f \in \mathcal{O}_\phi(Y)$ and $x = (a_1, \dots, a_{n-1}, 0) \in X \cap U_k$, then

$$|f(x)|^2 \leq c \int_D |f|^2,$$

where D is the disc with $z_j = a_j$ for $j = 1, \dots, n-1$ in U_k, and $c > 0$ is a constant independent of f. Hence,

$$\int_{X\cap U_k} |f|^2 e^{-\phi} \leq c \int_{U_k} |f|^2 e^{-\phi},$$

so

$$\int_X |f|^2 e^{-\phi} \leq c \int_Y |f|^2 e^{-\phi}.$$

This shows that we have a continuous linear restriction map

$$\rho\colon \mathcal{O}_\phi(Y) \to \mathcal{O}_\phi(X), \quad f \mapsto f|X.$$

In a previous paper we showed that under suitable curvature assumptions, ρ is surjective when $n \geq 2$.

Theorem 1.1 **[Lár1, Theorem 3.1]** *If $n \geq 2$ and $\Theta \geq i\partial\bar{\partial}\phi + \varepsilon\omega$ for some $\varepsilon > 0$, then ρ is surjective.*

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By [Lár1, Corollary 2.4], since the weighted metric $e^{\phi}h$ in L has curvature $-i\partial\bar\partial\phi + \Theta \geq \varepsilon\omega$, the k-th L^2 cohomology group $H^k_{(2)}(Y, L^\vee)$ of Y with coefficients in the dual bundle $L^\vee$ with the dual metric $e^{-\phi}h^\vee$ vanishes for $k < n$. The proof of Theorem 1.1 is based on vanishing for $k = 1$, for which we need $n \geq 2$. We will now use vanishing for $k = 0$ to show that ρ is injective.

Let $f \in \mathcal{O}_\phi(Y)$ such that $f|X = 0$. Then $\alpha = fs^\vee$ is a holomorphic section of $L^\vee$ on Y. We will show that α is square-integrable with respect to $e^{-\phi}h^\vee$. Then vanishing of $H^0_{(2)}(Y, L^\vee)$ implies that $\alpha = 0$, so $f = 0$.

Since $\nabla s \neq 0$ on C, there is a constant $c > 0$ such that $\operatorname{dist}(\cdot, C) \leq c|s|$ on M. For $y \in Y \setminus X$ let $x \in X$ have $\operatorname{dist}(y, x) = \operatorname{dist}(y, X)$. Then

$$\begin{aligned} |\alpha(y)| &= |f(y)||s(y)|^{-1} \leq c|f(x) - f(y)|/\operatorname{dist}(x, y) \\ &\leq c \sup |df| \leq c \int_B |f| \leq c\Big(\int_B |f|^2\Big)^{1/2}, \end{aligned}$$

where the supremum is taken over a ball centred at y covering all of M, and B is a ball of larger radius. Hence,

$$|\alpha(y)|^2 e^{-\phi(y)} \leq c \int_B |f|^2 e^{-\phi},$$

so

$$\int_Y |\alpha|^2 e^{-\phi} \leq c \int_Y |f|^2 e^{-\phi} < \infty.$$

We have proved the following theorem.

Theorem 1.2 *Suppose*

$$\Theta \geq i\partial\bar\partial\phi + \varepsilon\omega$$

for some $\varepsilon > 0$. Then ρ is injective. If $\dim X \geq 1$, then ρ is an isomorphism.

By induction, the theorem generalizes to the case when C is the common zero locus of sections $s_1, \ldots, s_k$, $k \leq n$, of L over M which, in a trivialization, can be completed to a set of local coordinates at each point of C. When $k = n - 1$, such C will be referred to as L-curves. If L is very ample, and therefore the pullback of the hyperplane bundle by an embedding of M into some projective space, then this condition means that the linear subspace $\{s_1, \ldots, s_k = 0\}$ intersects M transversely in a smooth subvariety C of codimension k. By Bertini's theorem, this holds for the generic linear subspace of codimension k. If $k \leq n - 1$, then C is connected and the map $\pi_1(C) \to \pi_1(M)$ is surjective by the Lefschetz hyperplane theorem, which implies that X is connected.

An important example of a function ϕ as above is obtained by smoothing the distance δ from a fixed point in Y. By a result of Napier [Nap], there is a smooth function τ on Y such that

(1) $c_1\delta \leq \tau \leq c_2\delta + c_3$ for some $c_1, c_2, c_3 > 0$,
(2) $d\tau$ is bounded, and
(3) $i\partial\bar\partial\tau$ is bounded.

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Furthermore, by (1) and since the curvature of Y is bounded below, there is $c > 0$ such that $e^{-c\tau}$ is integrable on Y. Then $e^{-c\tau}$ is also integrable on X.

If L is positive, then $k\Theta \geq i\partial\bar\partial\tau$ for $k \in \mathbb{N}$ sufficiently large by (3), so the curvature inequality in Theorem 1.2 holds if L is replaced by a sufficiently high tensor power of itself.

Example 1.3 There is an open hyperbolic Riemann surface X such that for some $\varepsilon > 0$, any $f \in \mathcal{O}(X)$ with $|f| \leq ce^{\varepsilon\delta}$ is constant, where δ is the distance from a fixed point in the Poincaré metric.

Namely, there is an example due to Cousin of a projective 2-dimensional torus (abelian surface) M with a $\mathbb{Z}$-covering space $Y \to M$ such that Y has no nonconstant holomorphic functions [NR, 3.9]. Let τ be a smooth function on Y satisfying (1), (2), and (3), such that $e^{-\tau/2}$ is integrable. Let L be a very ample line bundle on M such that $\Theta \geq i\partial\bar\partial\tau + \varepsilon\omega$. Let X be the pullback in Y of an L-curve in M. Then $\mathcal{O}_\tau(X) = \mathbb{C}$ by Theorem 1.2. If $f \in \mathcal{O}(X)$ and $|f| \leq ce^{\varepsilon\delta}$ with $\varepsilon > 0$ sufficiently small, then $|f| \leq ce^{\tau/4}$, so $f \in \mathcal{O}_\tau(X)$ and f is constant.

2 New Examples of Riemann Surfaces With Corona

Let X be a complex manifold. Let $H^\infty(X)$ be the space of bounded holomorphic functions on X, which is a Banach algebra in the supremum norm. Let $\mathcal{M}$ be the character space of $H^\infty(X)$, which is a compact Hausdorff space in the weak-star topology. There is a continuous map $\iota\colon X \to \mathcal{M}$ taking $x \in X$ to the evaluation character $f \mapsto f(x)$. The Corona Problem asks whether $\iota(X)$ is dense in $\mathcal{M}$. The complement of the closure of $\iota(X)$ in $\mathcal{M}$ is referred to as the corona, so if $\iota(X)$ is not dense in $\mathcal{M}$, then X is said to have corona. It is well known that the following are equivalent.

(1) $\iota(X)$ is dense in $\mathcal{M}$.

(2) If $f_1, \dots, f_m \in H^\infty(X)$ and $|f_1| + \cdots + |f_m| > \varepsilon > 0$, then there are $g_1, \dots, g_m \in H^\infty(X)$ such that $f_1g_1 + \cdots + f_mg_m = 1$.

By Carleson's famous Corona Theorem (1962), the disc has no corona. The Corona Theorem holds for Riemann surfaces of finite type and planar domains of various kinds. The Corona Problem for arbitrary planar domains is open. Around 1970, Cole constructed the first example of a Riemann surface with corona [Gam]. By modifying Cole's example, Nakai obtained a regular Parreau-Widom surface with corona [Nak], [Has, p. 229]. Recently, Barrett and Diller showed that the homology covering spaces of domains in the Riemann sphere, whose complement has positive logarithmic capacity and zero length, have corona [BD]. See also [EP, 7.3]. Sibony [Sib] found the first example of a domain of holomorphy in $\mathbb{C}^n$, $n \geq 2$, with corona. There are no known examples of such domains without corona.

We will now present a new class of Riemann surfaces with corona (see also Theorem 4.2). We remind the reader that if Y is a bounded domain in $\mathbb{C}^n$ covering a compact complex manifold M, then Y is a domain of holomorphy [Sie, p. 136] and M is projective. In fact, the canonical bundle of M is ample [Kol, 5.22].

Theorem 2.1 *Let $\pi\colon Y \to M$ be a covering map, where Y is a bounded domain in $\mathbb{C}^n$, $n \geq 2$, and M is compact. Let L be an ample line bundle on M. If C is an $L^{\otimes m}$-curve in M with m sufficiently large, then the Riemann surface $X = \pi^{-1}(C)$ has corona. In fact, the natural map* *from X into the character space $\mathcal{M}$ of $H^\infty(X)$ extends to an embedding of Y into $\mathcal{M}$ which maps $Y \setminus X$ into the corona of X.*

Proof Let $\tau \geq 0$ be a smoothed distance function on Y as described in Section 1, such that $e^{-\tau}$ is integrable on Y, and hence on X, so $H^\infty(X) \subset \mathcal{O}_\tau(X)$.

We claim that $\mathcal{O}_\tau(X) \cdot \mathcal{O}_\tau(X) \subset \mathcal{O}_{3\tau}(X)$. Namely, suppose $f \in \mathcal{O}_\tau(X)$. For $p \in X$, let B be the ball of radius 1 centred at p in a metric pulled up from C. Then

$$|f(p)| \leq c \int_B |f| \leq c\Big(\int_B |f|^2\Big)^{1/2},$$

where the constants are independent of p, so

$$|f(p)|e^{-\tau(p)/2} \leq c\Big(\int_B |f|^2 e^{-\tau}\Big)^{1/2} < \infty.$$

Hence, $|f| \leq ce^{\tau/2}$, so

$$\int_X |f|^4 e^{-3\tau} \leq c \int_X e^{2\tau} e^{-3\tau} < \infty,$$

and $f^2 \in \mathcal{O}_{3\tau}(X)$.

If m is sufficiently large, then the curvature of $L^{\otimes m}$ is at least $3i\partial\bar{\partial}\tau + \omega$ for some Kähler form ω on M, so the restriction map $\rho\colon \mathcal{O}_{3\tau}(Y) \to \mathcal{O}_{3\tau}(X)$ is an isomorphism by Theorem 1.2. Clearly, $\rho\mathcal{O}_\tau(Y) = \mathcal{O}_\tau(X)$. For $p \in Y$, let λ_p be the linear functional $f \mapsto \rho^{-1}(f)(p)$ on $H^\infty(X)$. If $f, g \in H^\infty(X)$, then $\rho^{-1}(f)\rho^{-1}(g) \in \mathcal{O}_\tau(X) \cdot \mathcal{O}_\tau(X) \subset \mathcal{O}_{3\tau}(X)$, and $\rho^{-1}(f)\rho^{-1}(g)|X = fg$, so $\rho^{-1}(f)\rho^{-1}(g) = \rho^{-1}(fg)$. This shows that λ_p is a character on $H^\infty(X)$.

We have obtained a map $\iota\colon Y \to \mathcal{M}$, $p \mapsto \lambda_p$, extending the natural map from X into $\mathcal{M}$. We claim that ι is a homeomorphism onto its image with the induced topology, and that $\overline{\iota(X)} \cap \iota(Y) = \iota(X)$. First of all, since $H^\infty(Y)$ separates points, ι is injective. The topology on $\mathcal{M}$ is the weakest topology that makes all the Gelfand transforms $\hat{f}\colon \chi \mapsto \chi(f)$, $f \in H^\infty(X)$, continuous, so ι is continuous if and only if $\hat{f} \circ \iota$ is continuous for all $f \in H^\infty(X)$, but $\hat{f} \circ \iota = \rho^{-1}(f)$, so this is clear. Let $p \in Y$ and suppose the polydisc $P = \{z \in \mathbb{C}^n \colon |f_i(z)| < \varepsilon, i = 1, \ldots, n\}$, where $f_i(z) = z_i - p_i$, is contained in Y. Let $V = \{\chi \in \mathcal{M} : |\chi(f_i)| < \varepsilon, i = 1, \ldots, n\}$. Then V is open in $\mathcal{M}$, and $\iota(P) = V \cap \iota(Y)$. This shows that $\iota\colon Y \to \iota(Y)$ is open, so ι is a homeomorphism onto its image. Finally, suppose $p \in Y \setminus X$. Say $\max |f_i| > \varepsilon > 0$ on X. Then V is an open neighbourhood of $\iota(p)$ which does not intersect $\iota(X)$. Hence, $\overline{\iota(X)} \cap \iota(Y) = \iota(X)$. This shows that the corona of X contains an embedded image of $Y \setminus X$. ■

Let us remark that a Riemann surface as in the theorem is not Parreau-Widom, since a Parreau-Widom surface X embeds into the character space of $H^\infty(X)$ as an open subset [Has, p. 222]. See also the proof of Theorem 5.2.

By the same argument we easily obtain the following more general result.

Theorem 2.2 *Let Y be a covering space of a projective manifold M with $\dim M \geq 2$, L be a line bundle on M, and C be an L-curve in M with preimage X in Y. If*

(1) L is sufficiently ample, and
(2) there is a bounded holomorphic map $g\colon Y \to \mathbb{C}^m$ with $\overline{g(X)} \neq g(Y)$,

then X is a Riemann surface with corona.

Finally, under the hypotheses of Theorem 2.1, if $U \subset Y$ is a neighbourhood of X with $Y \not\subset \bar{U}$, then we can show by the same argument as in the proof of Theorem 2.1 that U has corona. Since X is Stein, U may be chosen to be a domain of holomorphy by Siu's theorem [Siu]. Thus we obtain new examples of bounded domains of holomorphy in $\mathbb{C}^n$ with corona.

3 Generating Hörmander Algebras on Covering Spaces

In this section, we adapt results of Hörmander [Hör] on generating algebras of holomorphic functions of exponential growth to the case of covering spaces over compact manifolds. We let $X \to M$ be a covering space of an n-dimensional compact hermitian manifold M. Let $\phi\colon X \to [0, \infty)$ be a smooth function such that

(1) $d\phi$ is bounded, and
(2) $e^{-c\phi}$ is integrable on X for some $c > 0$.

Let
$$\mathcal{A}_\phi = \mathcal{A}_\phi(X) = \bigcup_{c>0} \mathcal{O}_{c\phi}(X)$$
be the vector space of holomorphic functions f on X such that $f^2 e^{-c\phi}$ is integrable on X for some $c > 0$. By (2), $\mathcal{A}_\phi$ contains all bounded holomorphic functions on X. The following is easy to see by an argument similar to that in the proof of Theorem 2.1.

Proposition 3.1 *A holomorphic function f on X is in $\mathcal{A}_\phi$ if and only if $|f| \leq ce^{a\phi}$ for some $a > 0$. Hence, $\mathcal{A}_\phi$ is a $\mathbb{C}$-algebra, called a Hörmander algebra.*

If functions $f_1, \dots, f_m$ in $\mathcal{A}_\phi$ generate $\mathcal{A}_\phi$, then there are $g_1, \dots, g_m \in \mathcal{A}_\phi$ such that $f_1 g_1 + \dots + f_m g_m = 1$, so
$$\max_{i=1,\dots,m} |f_i| \geq ce^{-a\phi} \quad \text{for some } a > 0.$$

We will establish an effective converse to this observation. Our proof is a straightforward adaptation of Hörmander's Koszul complex argument in [Hör]. See also [EP, 7.3].

Let $m \geq 1$ and $r, s \geq 0$ be integers, and $t \in [0, \infty)$. Choose a basis $\{e_1, \dots, e_m\}$ for $\mathbb{C}^m$. Let $L_r^s(t)$ be the space of smooth $\Lambda^s\mathbb{C}^m$-valued $(0, r)$-forms on X which are square-integrable with respect to $e^{-t\phi}$.

Lemma 3.2 *Suppose M is Kähler with Kähler form ω, and $\operatorname{Ric}(X) + ti\partial\bar\partial\phi \geq \varepsilon\omega$ for some $\varepsilon > 0$. If $\eta \in L_{r+1}^s(t)$ and $\bar\partial\eta = 0$, then there is $\xi \in L_r^s(t)$ with $\bar\partial\xi = \eta$.*

Proof This follows directly from standard L^2 theory. See for instance [Dem, Section 14]. ∎

Now let $f_1, \dots, f_m$ be holomorphic functions on X such that

$$ce^{-c_2\phi} \le \max_i |f_i|^2 \le ce^{c_1\phi}, \quad c_1, c_2 > 0.$$

Define a linear operator $\alpha\colon L_r^{s+1}(t) \to L_r^s(t + c_1)$ by the formula

$$\alpha(e_{i_1} \wedge \dots \wedge e_{i_{s+1}}) = \sum_{k=1}^{s+1} (-1)^{k+1} f_{i_k} e_{i_1} \wedge \dots \wedge \hat{e}_{i_k} \wedge \dots \wedge e_{i_{s+1}},$$

and set $\alpha = 0$ on $L_r^0(t)$. Then $\alpha^2 = 0$, and α commutes with $\bar{\partial}$. Also define a linear operator $\beta\colon L_r^s(t) \to L_r^{s+1}(t + c_1 + 2c_2)$ by the formula

$$\beta(\xi) = \frac{1}{|f_1|^2 + \dots + |f_m|^2} \Big(\sum_{i=1}^m \bar{f}_i e_i \Big) \wedge \xi.$$

Then $\alpha\beta + \beta\alpha$ is the inclusion $L_r^s(t) \hookrightarrow L_r^s\big(t + 2(c_1 + c_2)\big)$.

Lemma 3.3 *Suppose $e^{-a\phi}$ is integrable on X. If $\xi \in L_r^s(t)$ and $\alpha(\xi) = 0$, then there is $\eta \in L_r^{s+1}(t + c_1 + 2c_2)$ such that $\alpha(\eta) = \xi$ and in addition $\bar{\partial}\eta \in L_{r+1}^{s+1}(t + 2c_1 + 3c_2 + a)$ if $\bar{\partial}\xi = 0$.*

Proof Take $\eta = \beta(\xi)$. Say $\bar{\partial}\xi = 0$. Then

$$|\bar{\partial}\eta| \le c|\xi| \max_i |\bar{\partial} \frac{\bar{f}_i}{|f_1|^2 + \dots + |f_m|^2}| \le c|\xi| \max_i |\partial f_i| e^{(c_1+3c_2)\phi/2}.$$

If $f \in \mathcal{O}(X)$ and $|f|^2 \le ce^{c_1\phi}$, then $|\partial f|^2 \le ce^{(c_1+a)\phi}$. Hence, $|\bar{\partial}\eta|^2 \le c|\xi|^2 e^{(2c_1+3c_2+a)\phi}$, and the lemma follows. ■

Theorem 3.4 *Let $X \to M$ be a covering space of a compact Kähler manifold with Kähler form ω. Let $\phi\colon X \to [0, \infty)$ be a smooth plurisubharmonic function such that*

(1) $d\phi$ is bounded,
(2) $e^{-a\phi}$ is integrable on X, $a > 0$, and
(3) $\mathrm{Ric}(X) + bi\partial\bar{\partial}\phi \ge \varepsilon\omega$ for some $\varepsilon > 0$.

Let $f_1, \dots, f_m$ be holomorphic functions on X such that

$$ce^{-c_2\phi} \le \max_i |f_i|^2 \le ce^{c_1\phi}, \quad c_1, c_2 > 0.$$

Let $r, s \ge 0$ be integers. If $\xi \in L_r^s(t)$, $\bar{\partial}\xi = 0$, $\alpha(\xi) = 0$, and $t \ge b - 2c_2$, then there is $\eta \in L_r^{s+1}(u)$, where

$$u = t + c_1 + 2c_2 + (m - s - 1)\big(a + 3(c_1 + c_2)\big),$$

such that $\bar{\partial}\eta = 0$ and $\alpha(\eta) = \xi$.

Taking $r = s = 0$ and $\xi = 1$, we obtain the following corollary.

Corollary 3.5 *Let the hypotheses be as in Theorem 3.4. There are $g_1, \dots, g_m$ in $\mathcal{O}_{t\phi}(X)$ where*

$$t = \max\{a, b - 2c_2\} + c_1 + 2c_2 + (m-1)\big(a + 3(c_1 + c_2)\big),$$

such that $f_1 g_1 + \cdots + f_m g_m = 1$.

The following corollary is our analogue of [Hör, Theorem 1].

Corollary 3.6 *Let $X \to M$ be a covering space of a compact Kähler manifold with Kähler form ω. Let $\phi\colon X \to [0, \infty)$ be a smooth function such that*

(1) $d\phi$ is bounded,
(2) $e^{-c\phi}$ is integrable on X for some $c > 0$, and
(3) $i\partial\bar\partial\phi \geq \varepsilon\omega$ for some $\varepsilon > 0$.

Then functions $f_1, \dots, f_m$ in $\mathcal{A}_\phi$ generate $\mathcal{A}_\phi$ if and only if

$$\max_{i=1,\dots,m} |f_i| \geq ce^{-a\phi} \quad \text{for some } a > 0.$$

The hypotheses of the corollary are satisfied for example when X is the unit ball in $\mathbb{C}^n$ and $\phi = -\log(1 - |\cdot|^2)$, which is comparable to the Bergman distance from the origin (see Section 4).

Proof of Theorem 3.4 If $s \geq m$ or $r > n$, then $\xi = 0$ and we take $\eta = 0$. Assume that $s < m$ and $r \leq n$, and that the theorem has been proved with r, s replaced by $r + 1$, $s + 1$. By Lemma 3.3, there is $\eta_1 \in L_r^{s+1}(t + c_1 + 2c_2)$ such that $\alpha(\eta_1) = \xi$ and $\bar\partial\eta_1 \in L_{r+1}^{s+1}(t + 2c_1 + 3c_2 + a)$. Now $\bar\partial\bar\partial\eta_1 = 0$ and $\alpha(\bar\partial\eta_1) = \bar\partial\alpha(\eta_1) = \bar\partial\xi = 0$, so by the induction hypothesis, there is $\eta_2 \in L_{r+1}^{s+2}(u - c_1)$ such that $\bar\partial\eta_2 = 0$ and $\alpha(\eta_2) = \bar\partial\eta_1$. By Lemma 3.2, there is $\eta_3 \in L_r^{s+2}(u - c_1)$ such that $\bar\partial\eta_3 = \eta_2$. Now let $\eta = \eta_1 - \alpha(\eta_3) \in L_r^{s+1}(u)$. Then $\bar\partial\eta = \bar\partial\eta_1 - \alpha(\bar\partial\eta_3) = \bar\partial\eta_1 - \alpha(\eta_2) = 0$ and $\alpha(\eta) = \alpha(\eta_1) = \xi$. ■

4 The Case of the Ball

In this section, we will consider the results of the previous sections in the explicit setting of the unit ball $\mathbb{B}$ in $\mathbb{C}^n$, $n \geq 2$. For an instructive discussion of compact ball quotients, see [Kol, Chapter 8].

Let M be a projective manifold covered by $\mathbb{B}$ with a positive line bundle L with curvature Θ. Let X be the preimage in $\mathbb{B}$ of an L-curve C in M. We are particularly interested in the extension problem for bounded holomorphic functions on X.

The restriction map $H^\infty(\mathbb{B}) \to H^\infty(X)$ is injective, so we can consider $H^\infty(\mathbb{B})$ as a subspace of $H^\infty(X)$, which is closed in the sup-norm. In the locally uniform topology, however, $H^\infty(\mathbb{B})$ is dense in $H^\infty(X)$. Namely, say $f \in H^\infty(X)$, and let $F \in \mathcal{O}(\mathbb{B})$ be an extension of f. For $r < 1$, the functions $z \mapsto F(rz)$ are bounded in $\mathbb{B}$, and they converge locally uniformly to f on X as $r \to 1$.

It is well known that if Y is a complex submanifold of a neighbourhood of $\bar{\mathbb{B}}$, then every bounded holomorphic function on $Y \cap \mathbb{B}$ extends to a bounded holomorphic function on

$\mathbb{B}$ [HL, 4.11.1]. Our X is of course far from extending to a submanifold of a larger ball, and the extension problem for bounded holomorphic functions on X is very much open. We will show, however, that when L is sufficiently positive, bounded holomorphic functions on X extend to holomorphic functions on $\mathbb{B}$ whose growth is, in a precise sense, at most just barely exponential.

First we collect a few formulas concerning the geometry of $\mathbb{B}$. The Bergman metric of $\mathbb{B}$ is

$$ds^2 = \sum_{j,k=1}^{n} g_{jk} dz_j \otimes d\bar{z}_k,$$

where

$$g_{jk} = \left\langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k} \right\rangle = \frac{n+1}{(1-|z|^2)^2}\big((1-|z|^2)\delta_{jk} + \bar{z}_j z_k\big).$$

This is a common convention. It is used for instance in [Kra] and [Sto]; other authors may use a constant scalar multiple of the above. The Kähler form of the Bergman metric is

$$\omega = -\frac{1}{2}\operatorname{Im} ds^2 = \frac{i}{2}\sum_{j,k=1}^{n} g_{jk} dz_j \wedge d\bar{z}_k = -\frac{i(n+1)}{2}\partial\bar{\partial}\log(1-|z|^2).$$

The distance from the origin in the Bergman metric is

$$\delta(z) = \frac{\sqrt{n+1}}{2}\log\frac{1+|z|}{1-|z|}, \quad z \in \mathbb{B}.$$

The Ricci curvature of the Bergman metric, *i.e.*, the curvature form of the induced metric in the cocanonical bundle (the top exterior power of the tangent bundle) is

$$\operatorname{Ric}(\omega) = -\frac{i}{2}\partial\bar{\partial}\log\det(g_{jk}) = -\omega.$$

Let

$$\tau(z) = -\frac{n+1}{2}\log(1-|z|^2).$$

Then $i\partial\bar{\partial}\tau = \omega$, and τ is a nonnegative strictly plurisubharmonic exhaustion of $\mathbb{B}$. Also, τ is comparable to δ. More precisely,

$$\sqrt{n+1}\,\delta \le \tau \le \sqrt{n+1}\,\delta + (n+1)\log 2.$$

It may be shown that $|d\tau|$ is bounded. Let $\Omega = \omega^n/n!$ be the volume form of the Bergman metric. Then

$$\Omega = c(1-|z|^2)^{-(n+1)}\Omega_0,$$

where $c > 0$ is constant, and Ω_0 is the euclidean volume form on $\mathbb{C}^n$. We have

$$\int_{\mathbb{B}} e^{-c\tau}\Omega < \infty \quad \text{if and only if } c > \frac{2n}{n+1}.$$

The weighted Bergman space $\mathcal{B}_p^a$ is the space of holomorphic functions g on $\mathbb{B}$ such that

$$\int_{\mathbb{B}} |g|^p (1-|z|)^a \omega^n < \infty.$$

See [Sto, Chapter 10]. We have $\mathcal{B}_2^n = 0$. The intersection $\mathcal{B}_2^{n+0} = \bigcap_{\varepsilon>0} \mathcal{B}_2^{n+\varepsilon}$ contains the Hardy space $H^2(\mathbb{B})$ of holomorphic functions f on $\mathbb{B}$ such that $|f|^2$ has a harmonic majorant.

The boundary behaviour of the functions in $\mathcal{B}_2^{n+0}$ may be rather wild. A theorem of Bagemihl, Erdös, and Seidel [BES], [Mac], states that if $\mu\colon [0,1) \to [0,\infty)$ goes to infinity at 1, then there exists a holomorphic function f on the unit disc with $|f| \le \mu(|\cdot|)$, such that for some sequence $r_n \nearrow 1$, we have $\min_{|z|=r_n} |f(z)| \to \infty$. In particular, f does not have a finite limit along any curve that intersects every neighbourhood of the boundary. Taking $\mu(r) = -\log(1-r)$, we get f in $\mathcal{B}_2^{1+0}$.

We have

$$\mathcal{B}_2^a = \mathcal{O}_{\frac{2a}{n+1}\tau}(\mathbb{B}).$$

Let

$$\mathcal{E}(\mathbb{B}) = \bigcap_{c>\frac{2n}{n+1}} \mathcal{O}_{c\tau}(\mathbb{B}),$$

and define $\mathcal{E}(X)$ similarly. These are Fréchet-Hilbert spaces.

From now on we let $n \ge 2$. Theorem 1.2 yields the following result.

Theorem 4.1 *If $\Theta > c\omega$, then the restriction map $\rho\colon \mathcal{O}_{c\tau}(\mathbb{B}) \to \mathcal{O}_{c\tau}(X)$ is an isomorphism.*

Suppose

$$\Theta > \frac{2n}{n+1}\omega.$$

This holds for instance if $L = K^{\otimes k}$ with $k \ge 2$. Then we have a well-defined restriction map $\rho\colon \mathcal{E}(\mathbb{B}) \to \mathcal{E}(X)$, which is an isomorphism. Hence, every bounded holomorphic function on X extends to a unique function in $\mathcal{E}(\mathbb{B})$.

The 1-dimensional case is very different. Seip has shown that no submanifold of the disc is both sampling and interpolating for any weighted Bergman space [Sei].

Now we can easily obtain fairly explicit examples of Riemann surfaces with corona.

Theorem 4.2 *If $\Theta > \frac{2n}{n+1}\omega$, then X has corona.*

Proof Let $p \in \mathbb{B} \setminus X$ and $f_i(z) = z_i - p_i$, $i = 1, \dots, n$. Then $\sum |f_i| > \varepsilon > 0$ on X. Suppose X has no corona. Then there are $g_1, \dots, g_n \in H^\infty(X)$ with $\sum f_i g_i = 1$. By Theorem 4.1, g_i extends to a function $G_i \in \mathcal{E}(\mathbb{B})$. Then $h = \sum f_i G_i \in \mathcal{E}(\mathbb{B})$, and $h|X = 1$. Again by Theorem 4.1, $h = 1$, which is absurd since $h(p) = 0$. ■

The proof of Theorem 2.1 shows that if $\Theta > \frac{6n}{n+1}\omega$, then $\mathbb{B}$ embeds into the character space $\mathcal{M}$ of $H^\infty(X)$, taking $\mathbb{B} \setminus X$ into the corona.

Now τ satisfies the hypotheses of Corollary 3.6, both on $\mathbb{B}$ and on X. Hence, functions $f_1 \dots, f_m \in \mathcal{A}_\tau(\mathbb{B})$ generate $\mathcal{A}_\tau(\mathbb{B})$ if and only if

$$\max_{i=1,\dots,m} |f_i| \ge ce^{-a\tau} \quad \text{for some } a > 0,$$

and similarly for X.

In contrast to the first part of Theorem 4.1, we we obtain the following result from Corollary 3.5.

Theorem 4.3 *If $s \geq 1$ and $\Theta \leq s\omega$, then the restriction map $\rho\colon \mathcal{O}_{c\tau}(\mathbb{B}) \to \mathcal{O}_{c\tau}(X)$ is not an isomorphism for*

$$c > s + \frac{2n^2 - n + 1}{n+1}.$$

Proof By adjunction, the Ricci curvature of X in the metric induced by the Bergman metric on $\mathbb{B}$ is $\operatorname{Ric}(X) = -\Theta - \omega$. We apply Corollary 3.5 with $X \to C$, $\phi = \tau$, $a = 2n/(n+1) + \varepsilon$, $b = s + 1 + \varepsilon$, $\varepsilon > 0$, $f_i = z_i - p_i$, $i = 1, \dots, n$, where $p \in \mathbb{B} \setminus X$, and $c_1, c_2 = 0$. For every $c > s + (2n^2 - n + 1)/(n+1)$ we obtain $g_1, \dots, g_n$ in $\mathcal{O}_{c\tau}(X)$ such that $f_1 g_1 + \cdots + f_n g_n = 1$. If $\rho\colon \mathcal{O}_{c\tau}(\mathbb{B}) \to \mathcal{O}_{c\tau}(X)$ was an isomorphism, there would be $G_1, \dots, G_n$ in $\mathcal{O}_{c\tau}(\mathbb{B})$ with $\sum (z_i - p_i) G_i = 1$ on $\mathbb{B}$, which is absurd. ■

Consider the special case when $L = K^{\otimes m}$, $m \geq 1$. We have proved that $\rho\colon \mathcal{O}_{c\tau}(\mathbb{B}) \to \mathcal{O}_{c\tau}(X)$ is an isomorphism if $c < m$, but not if $c > m + \frac{2n^2 - n + 1}{n+1}$. Furthermore, we can easily show that ρ is not injective if $c > 2m + \frac{2n}{n+1}$.

Namely, $e = dz_1 \wedge \cdots \wedge dz_n$ is a zero-free section of K with norm $ce^{-\tau}$. Let $s \neq 0$ be a holomorphic section of L on M, vanishing on C. Then $f = s/e^{\otimes m}$ is a holomorphic function in the kernel of ρ with $|f| \leq ce^{m\tau}$, so $f \in \mathcal{O}_{c\tau}(\mathbb{B})$ for all $c > 2m + \frac{2n}{n+1}$.

5 A Dichotomy

As before, we consider a projective manifold M covered by the unit ball $\mathbb{B}$ in $\mathbb{C}^n$, $n \geq 2$, with a positive line bundle L, and the preimage X in $\mathbb{B}$ of an L-curve C in M. We will denote the covering group by Γ. The bounded extension problem for holomorphic functions is related to the question of which bounded harmonic functions on X are real parts of holomorphic functions. This question was studied in [Lár2], where the following dichotomy was established in the more general setting of a nonelementary Gromov hyperbolic covering space of a compact Kähler manifold.

Theorem 5.1 **[Lár2, Theorem 4.2]** *One of the following holds.*

(1) Every positive harmonic function on X is the real part of a holomorphic function.
(2) If $u \geq 0$ is the real part of an H^1 function on X, then the boundary decay of u at a zero on the Martin boundary of X is no faster than its radial decay.

By results of Ancona [Anc], the Martin compactification of X is naturally homeomorphic to $X \cup \mathbb{S}$, where $\mathbb{S} = \partial\mathbb{B}$ is the unit sphere.

Clearly, if (1) holds, then there are holomorphic functions on X with a bounded real part that do not extend to a holomorphic function on $\mathbb{B}$ with a bounded real part.

If (1) holds, then each Martin function k_p, $p \in \mathbb{S}$, is the real part of a holomorphic function f_p on X. Then the holomorphic map $\exp(-f_p)\colon X \to \mathbb{D}$ is proper at every boundary point except p. Here, $\mathbb{D}$ denotes the unit disc. Also, if $p, q \in \mathbb{S}$, $p \neq q$, then the holomorphic map $\big(\exp(-f_p), \exp(-f_q)\big)\colon X \to \mathbb{D} \times \mathbb{D}$ is proper. However, we have the following result.

Theorem 5.2 *There is no proper holomorphic map $X \to \mathbb{D}$.*

Proof Bounded holomorphic functions separate points on X, so if there is a proper holomorphic map $X \to \mathbb{D}$, then X is Parreau-Widom by a theorem of Hasumi [Has, p. 209]. By [Lár2, Theorem 5.1], if X is Parreau-Widom, then X is either isomorphic to $\mathbb{D}$ or homeomorphic to the 2-sphere with a Cantor set removed. Both possibilities are excluded by the Martin boundary of X being $\mathbb{S}$. ■

When L is sufficiently ample, we can prove a stronger result.

Theorem 5.3 *If L is sufficiently ample and f is a holomorphic function on X, then $f^{-1}(U)$ is not relatively compact in X for any nonempty open subset U of the image $f(X)$. In other words, every value of f is taken at infinity.*

Proof Suppose there is a holomorphic function f on X such that $f^{-1}(U)$ is relatively compact in X for some nonempty open subset U of $f(X)$. We may assume that $0 \in U$. Then $1/f$ is a meromorphic function on X which has a pole p and is bounded outside the compact closure of $f^{-1}(U)$. Since bounded holomorphic functions separate points on X, a theorem of Hayashi [Hay] now implies that the natural map from X into the character space of $H^\infty(X)$ is open when restricted to some neighbourhood of p. By Theorem 2.1, this is absurd when L is sufficiently ample. ■

We will now present another dichotomy in a similar vein. Let $\mathcal{C}_K(\mathbb{S})$ denote the space of continuous functions $\mathbb{S} \to K$, $K = \mathbb{R}$ or $K = \mathbb{C}$, with the supremum norm. Let P be the subspace of $\mathcal{C}_{\mathbb{R}}(\mathbb{S})$ of boundary values of pluriharmonic functions on $\mathbb{B}$ which are continuous on $\bar{\mathbb{B}}$, and $\mathcal{O}$ be the subspace of $\mathcal{C}_{\mathbb{C}}(\mathbb{S})$ of boundary values of holomorphic functions on $\mathbb{B}$ which are continuous on $\bar{\mathbb{B}}$. It is known that if V is a proper closed subspace of $\mathcal{C}_{\mathbb{R}}(\mathbb{S})$ and V is invariant under the action of the automorphism group $G = PU(1,n)$ of $\mathbb{B}$, then $V = \mathbb{R}$ or $V = P$. Also, if V is a proper closed G-invariant subspace of $\mathcal{C}_{\mathbb{C}}(\mathbb{S})$, then V is one of the following: $\mathbb{C}$, $\mathcal{O}$, $\bar{\mathcal{O}}$, $P + iP$ [Rud, 13.1.4].

If C is an L-curve in a finite covering of M with preimage X in $\mathbb{B}$, then we denote by $E(C)$ the space of functions $\alpha \in \mathcal{C}_{\mathbb{R}}(\mathbb{S})$ such that the harmonic extension $H[\alpha] = H_X[\alpha]$ of α to X is the real part of a holomorphic function on X. Clearly, $P \subset E(C)$. Now $\alpha \in E(C)$ if and only if all the periods of $H[\alpha]$ vanish, so $E(C)$ is closed in $\mathcal{C}_{\mathbb{R}}(\mathbb{S})$.

Theorem 5.4 *Suppose that the covering group Γ is arithmetic, and that L is a tensor power of the canonical bundle. Then one (and only one) of the following holds.*

(1) $E(C) = P$ for every L-curve in a finite covering of M.
(2) The subspace of $\mathcal{C}_{\mathbb{R}}(\mathbb{S})$ generated by $E(C)$ for all L-curves C in finite coverings of M is dense in $\mathcal{C}_{\mathbb{R}}(\mathbb{S})$.

Note that (2) holds if (1) in Theorem 5.1 holds for the preimage X of some L-curve in a finite covering of M.

Proof Let C be an L-curve in a finite covering M_1 of M, with preimage X in $\mathbb{B}$. Then $M_1 = \mathbb{B}/\Gamma_1$, where Γ_1 is a subgroup of finite index in Γ. Let g be an element of the commensurability subgroup $\operatorname{Comm}(\Gamma)$ in G. This means that Γ and $g\Gamma g^{-1}$ are commensurable, *i.e.*, their intersection is of finite index in both of them. Then $\Gamma_2 = \Gamma_1 \cap g\Gamma_1 g^{-1}$ is a subgroup of finite index in Γ_1. If $\alpha \in E(C)$, so $H_X[\alpha] = \operatorname{Re} f$ with f holomorphic on X, then $f \circ g$ is holomorphic on $g^{-1}X$ and $H_{g^{-1}X}[\alpha \circ g] = \operatorname{Re} f \circ g$. If $\gamma \in \Gamma_2$, then $\gamma = g^{-1}\gamma_1 g$ for some $\gamma_1 \in \Gamma_1$, so $\gamma g^{-1}X = g^{-1}\gamma_1 g g^{-1}X = g^{-1}\gamma_1 X = g^{-1}X$.

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Hence, $g^{-1}X$ is Γ_2-invariant, so $g^{-1}X$ is the preimage of an L-curve C' in the finite covering $\mathbb{B}/\Gamma_2$ of M (here is where we use the assumption that L is a tensor power of the canonical bundle), and $\alpha \circ g \in E(C')$.

This shows that the subspace E of $\mathcal{C}_{\mathbb{R}}(\mathbb{S})$ described in (2) is invariant under $\operatorname{Comm}(\Gamma)$. Since Γ is arithmetic, $\operatorname{Comm}(\Gamma)$ is Hausdorff-dense in G [Zim, 6.2.4] (and in fact conversely), so the closure $\bar{E}$ of E is a G-invariant subspace of $\mathcal{C}_{\mathbb{R}}(\mathbb{S})$. Hence, $\bar{E}$ is either P or $\mathcal{C}_{\mathbb{R}}(\mathbb{S})$, and the theorem follows. ■

If the spaces $E(C)$ are rigid in the sense that they do not change when C is varied in its linear equivalency class, then the theorem yields a strong dichotomy.

Corollary 5.5 *Suppose that Γ is arithmetic, and that L is a tensor power of the canonical bundle. Suppose also that if C_1 and C_2 are L-curves in the same finite covering of M, then $E(C_1) = E(C_2)$. Then $E(C)$ is either P or $\mathcal{C}_{\mathbb{R}}(\mathbb{S})$ for every L-curve C in a finite covering of M.*

We obtain analogous results for holomorphic functions. If C is an L-curve in a finite covering of M with preimage X in $\mathbb{B}$, let us denote by $F(C)$ the closed subspace of functions $\alpha \in \mathcal{C}_{\mathbb{C}}(\mathbb{S})$ that extend to a holomorphic function on X. Clearly, $\mathcal{O} \subset F(C)$, but $F(C)$ is considerably smaller than $\mathcal{C}_{\mathbb{C}}(\mathbb{S})$.

Lemma 5.6 $F(C) \cap \big(P + i\mathcal{C}_{\mathbb{R}}(\mathbb{S})\big) = \mathcal{O}$.

Proof Let $\alpha \in F(C) \cap \big(P + i\mathcal{C}_{\mathbb{R}}(\mathbb{S})\big)$, so $H[\alpha] = f \in \mathcal{O}(X)$ and there is $u \in \mathcal{C}_{\mathbb{R}}(\bar{\mathbb{B}})$ such that $u|\mathbb{B}$ is pluriharmonic and $u|\mathbb{S} = \operatorname{Re}\alpha$, so $u|X = \operatorname{Re} f$. There is $F \in \mathcal{O}(\mathbb{B})$ such that $u = \operatorname{Re} F$ and $F|X = f$. We need to show that F extends continuously to $\bar{\mathbb{B}}$.

Now F maps $\mathbb{B}$ into a vertical strip. Let σ be an isomorphism from a neighbourhood of the closure of this strip in the Riemann sphere onto $\mathbb{D}$. Then $\sigma \circ F$ is a bounded holomorphic function on $\mathbb{B}$ and $\sigma \circ F|X = \sigma \circ f$, so $\sigma \circ F$ has the same nontangential boundary function $\sigma \circ \alpha$ as $\sigma \circ f$. Since $\sigma \circ \alpha$ is continuous, $\sigma \circ F$ extends continuously to $\bar{\mathbb{B}}$, and so does $F = \sigma^{-1} \circ \sigma \circ F$. ■

Theorem 5.7 *Suppose that Γ is arithmetic, and that L is a tensor power of the canonical bundle. Then one of the following holds.*

(1) $F(C) = \mathcal{O}$ for every L-curve in a finite covering of M.
(2) The subspace of $\mathcal{C}_{\mathbb{C}}(\mathbb{S})$ generated by $F(C)$ for all L-curves C in finite coverings of M is dense in $\mathcal{C}_{\mathbb{C}}(\mathbb{S})$.

Suppose furthermore that if C_1 and C_2 are L-curves in the same finite covering of M, then $F(C_1) = F(C_2)$. Then $F(C) = \mathcal{O}$ for every L-curve C in a finite covering of M.

Proof By the same argument as in the proof of Theorem 5.4, the closure of the subspace F of $\mathcal{C}_{\mathbb{C}}(\mathbb{S})$ described in (2) is G-invariant, so it is $\mathcal{O}$, $P + iP$, or $\mathcal{C}_{\mathbb{C}}(\mathbb{S})$ itself. Lemma 5.6 shows that if $F \subset P + iP$, then $F = \mathcal{O}$. ■

Loosely speaking, either the spaces $F(C)$ are all the same, and equal to $\mathcal{O}$, for all L-curves in finite coverings of M, or they are diverse enough that every complex continuous function on $\mathbb{S}$ can be uniformly approximated by functions of the form $\alpha_1 + \cdots + \alpha_k$, where $\alpha_i \in F(C_i)$ for L-curves $C_1, \ldots, C_k$ in some finite covering of M.

Clearly, 5.7(2) implies 5.4(2), and 5.4(1) implies 5.7(1). It is not clear if the reverse implications hold, *i.e.*, if Theorems 5.4 and 5.7 actually express the same dichotomy.

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6 Harmonic Functions

Since $\mathbb{B}$ is Stein, every subharmonic function on X extends to a plurisubharmonic function on $\mathbb{B}$. Whereas the bounded extension problem for holomorphic functions is hard to fathom, it is fairly easy to see, using a little potential theory, that a bounded-above subharmonic function on X need not extend to a bounded-above plurisubharmonic function on $\mathbb{B}$.

Recall that if $\alpha \in \mathcal{C}_{\mathbb{R}}(\mathbb{S})$, then there exists a unique $u \in \mathcal{C}_{\mathbb{R}}(\bar{\mathbb{B}})$ such that

(1) u is plurisubharmonic on $\mathbb{B}$,
(2) $(\partial\bar{\partial}u)^n = 0$, *i.e.*, u is maximal, and
(3) $u|\mathbb{S} = \alpha$.

Let us write $u = M[\alpha] = M_{\mathbb{B}}[\alpha]$. This is the solution of the Dirichlet problem for the Monge-Ampère operator, due to Bedford and Taylor [BT]. See also earlier work of Bremermann [Bre] and Walsh [Wal]. In fact, u is given by the Perron-Bremermann formula $u = \sup \mathcal{F}_\alpha$, where $\mathcal{F}_\alpha$ is the set of all plurisubharmonic functions v on $\mathbb{B}$ with

$$\limsup_{z \to x} v(z) \leq \alpha(x), \quad x \in \mathbb{S}.$$

The operator $M\colon \mathcal{C}_{\mathbb{R}}(\mathbb{S}) \to \mathcal{C}_{\mathbb{R}}(\bar{\mathbb{B}})$ is continuous in the sense that if $\alpha_i \to \alpha$ uniformly on $\mathbb{S}$, then $M[\alpha_i] \to M[\alpha]$ uniformly on $\bar{\mathbb{B}}$. Namely, if $\varepsilon > 0$, then $\alpha - \varepsilon \leq \alpha_i \leq \alpha + \varepsilon$ for i sufficiently large, and then $\mathcal{F}_{\alpha-\varepsilon} \subset \mathcal{F}_{\alpha_i} \subset \mathcal{F}_{\alpha+\varepsilon}$, so $M[\alpha] - \varepsilon \leq M[\alpha_i] \leq M[\alpha] + \varepsilon$.

Theorem 6.1 *Let $\alpha \in \mathcal{C}_{\mathbb{R}}(\mathbb{S})$. The following are equivalent.*

(1) The harmonic extension $H[\alpha]$ of α to X extends to a bounded-above plurisubharmonic function on $\mathbb{B}$.
(2) $H[\alpha]$ extends to a function in $\mathcal{C}_{\mathbb{R}}(\bar{\mathbb{B}})$ which is maximal plurisubharmonic on $\mathbb{B}$.
(3) $M[\alpha]$ is harmonic on X.
(4) $M[\alpha]|X = H[\alpha]$.

Proof Clearly, (4) $\Leftrightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). We need to show that (1) $\Rightarrow$ (4). Suppose v is plurisubharmonic and bounded above on $\mathbb{B}$ such that $v|X = H[\alpha]$. Now v has a non-tangential boundary function $\hat{v}$, and since $v|X = H[\alpha]$, we have $\hat{v} = \widehat{H[\alpha]} = \alpha$ almost everywhere. Since v is bounded above and subharmonic, we have $v = h + p$, where h is harmonic and $p \leq 0$ is a subharmonic potential. In fact, $h = H_{\mathbb{B}}[\alpha]$. Hence,

$$\limsup_{z \to x} v(z) \leq \limsup_{z \to x} h(z) = \lim_{z \to x} h(z) = \alpha(x)$$

for all $x \in \mathbb{S}$, so $v \leq M[\alpha]$. On the other hand, the Perron formulas for $H[\alpha]$ and $M[\alpha]$ show that $M[\alpha]|X \leq H[\alpha]$. Hence, $H[\alpha] = M[\alpha]|X$. ■

This shows that if $H[\alpha]$, $\alpha \in \mathcal{C}_{\mathbb{R}}(\mathbb{S})$, extends to a bounded-above plurisubharmonic function on $\mathbb{B}$, then $M[\alpha]$ extends it, so $M[\alpha]|X$ is harmonic. Clearly, this fails for *most* α. It would be interesting to know if such α can be the boundary function of the real part of a holomorphic function f on X. Then f would not extend to a bounded holomorphic function on $\mathbb{B}$.

Corollary 6.2 *The set of functions $\alpha \in \mathcal{C}_{\mathbb{R}}(\mathbb{S})$ such that $M[\alpha]|X$ is harmonic is a closed Γ-invariant subspace of $\mathcal{C}_{\mathbb{R}}(\mathbb{S})$.*

Proof Let $\alpha, \beta \in \mathcal{C}_{\mathbb{R}}(\mathbb{S})$, and suppose $M[\alpha]$, $M[\beta]$ are harmonic on X. By the theorem, $H[\alpha]$, $H[\beta]$ extend to bounded-above plurisubharmonic functions on $\mathbb{B}$, but then so does $H[\alpha+\beta] = H[\alpha] + H[\beta]$, so $M[\alpha+\beta]$ is harmonic on X.

If $\alpha_i \to \alpha$ uniformly on $\mathbb{S}$, then $M[\alpha_i] \to M[\alpha]$ uniformly on $\bar{\mathbb{B}}$, so if $M[\alpha_i]|X$ are harmonic, then so is $M[\alpha]|X$. ∎

We obtain a dichotomy analogous to those in Section 5. If C is an L-curve in a finite covering of M with preimage X in $\mathbb{B}$, let us denote by $D(C)$ the space of functions $\alpha \in \mathcal{C}_{\mathbb{R}}(\mathbb{S})$ such that $M[\alpha]|X$ is harmonic. Clearly, $P \subset D(C)$ but, as noted above, $D(C)$ is considerably smaller than $\mathcal{C}_{\mathbb{R}}(\mathbb{S})$.

Theorem 6.3 *Suppose that Γ is arithmetic, and that L is a tensor power of the canonical bundle. Then one of the following holds.*

(1) $D(C) = P$ for every L-curve in a finite covering of M.
(2) The subspace of $\mathcal{C}_{\mathbb{R}}(\mathbb{S})$ generated by $D(C)$ for all L-curves C in finite coverings of M is dense in $\mathcal{C}_{\mathbb{R}}(\mathbb{S})$.

Suppose furthermore that if C_1 and C_2 are L-curves in the same finite covering of M, then $D(C_1) = D(C_2)$. Then $D(C) = P$ for every L-curve C in a finite covering of M.

There are no examples for which it is known which alternative holds in any of the four dichotomies 5.1, 5.4, 5.7, and 6.3, nor is it known if these dichotomies are actually different.

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