ON R_0-CLOSED CLASSES, AND FINITELY GENERATED GROUPS

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1. Introduction.

1.1. If a group satisfies the maximal condition for normal subgroups, then all its central factors are necessarily finitely generated. In [**2**], Hall asked whether there exist finitely generated soluble groups which do not satisfy the maximal condition for normal subgroups but all of whose central factors are finitely generated. We shall answer this question in the affirmative. We shall also construct a finitely generated group all of whose subnormal subgroups are perfect (and which therefore has no non-trivial central factors), but which does not satisfy the maximal condition for normal subgroups. Related to these examples is the question of which classes of finitely generated groups satisfy the maximal condition for normal subgroups. A characterization of such classes has been obtained by Hall, and we shall include his result as our first theorem. This, in turn, is related to our second theorem, which gives examples of classes of groups having unique maximal R_0-closed subclasses. Before we can state either of these results, we must define R_0.

1.2. By a *class* $\mathfrak{X}$ we mean a family of groups, which contains all groups of order 1, and which has the property that $G \in \mathfrak{X}$ and $G_1 \cong G$ together imply that $G_1 \in \mathfrak{X}$. Given any class $\mathfrak{X}$, we define another class $R_0\mathfrak{X}$ as follows: $G \in R_0\mathfrak{X}$ if and only if G has a finite number of normal subgroups $N_1, \ldots, N_m$ whose intersection is 1 and $G/N_i \in \mathfrak{X}$ $(i = 1, \ldots, m)$. If $\mathfrak{X} = R_0\mathfrak{X}$, we say that $\mathfrak{X}$ is R_0-closed. Then R_0 is a closure operation in the sense of Hall [**4**, §§ 1–3].

It follows from Zorn's Lemma that every class $\mathfrak{X}$ has one or more maximal R_0-closed subclasses. For let $\mathfrak{X}_\alpha$ be the class of all $\mathfrak{X}$-groups of cardinal at most $\aleph_\alpha$. Then the number of isomorphism classes of groups in $\mathfrak{X}_\alpha$ is a well-defined cardinal number so that Zorn's Lemma coupled with the fact that the union $\bigcup_{\lambda \in \Lambda} \mathfrak{Y}_\lambda$ of any nested system of R_0-closed classes $\mathfrak{Y}_\lambda$ is again R_0-closed, (Λ being a set), shows that every R_0-closed subclass of $\mathfrak{X}_\alpha$ is contained in a maximal R_0-closed subclass of $\mathfrak{X}_\alpha$. Thus $\mathfrak{Z} = \bigcup_\alpha \mathfrak{Z}_\alpha$, where $\mathfrak{Z}_{\alpha+1}$ is a maximal R_0-closed subclass of $\mathfrak{X}_{\alpha+1}$ containing $\mathfrak{Z}_\alpha$ and $\mathfrak{Z}_\mu = \bigcup_{\alpha<\mu} \mathfrak{Z}_\alpha$ for a limit ordinal μ, is a maximal R_0-closed subclass of $\mathfrak{X}$.

Certain classes, however, contain a unique maximal R_0-closed subclass, and Hall's result states that the class $\mathfrak{G}$ of all finitely generated groups gives

Received November 18, 1968.

an example of this phenomenon. We shall adopt the notation of [**2**], and use max-n to denote the maximal condition for normal subgroups.

THEOREM 1 (P. Hall). *The class of finitely generated groups which satisfy* max-n *is the unique maximal R_0-closed subclass of* $\mathfrak{G}$.

1.3. We shall find other examples of classes which have a unique maximal R_0-closed subclass. If $\mathfrak{X}$ is any class, we say that $\mathfrak{X}$ is *Q-closed* if every quotient of an $\mathfrak{X}$-group is in $\mathfrak{X}$, and we define the class $\mathfrak{X}^{-Q}$ of $\mathfrak{X}$ *perfect* groups by the condition that $G \in \mathfrak{X}^{-Q}$ if and only if G has no non-trivial quotients in $\mathfrak{X}$.

For example, if $\mathfrak{A}$ is the class of abelian groups, then $\mathfrak{A}^{-Q}$ consists of all perfect groups. We shall denote by $C_{\mathfrak{X}}$ the class of all $\mathfrak{X}^{-Q}$ groups with no non-trivial central factors in $\mathfrak{X}$, and we shall prove the following result.

THEOREM 2. *If $\mathfrak{X}$ is any Q-closed class, then $C_{\mathfrak{X}}$ is the unique maximal R_0-closed subclass of $\mathfrak{X}^{-Q}$.*

1.4. The next result gives the example quoted at the beginning of this introduction.

THEOREM 3. *There is a finitely generated soluble group which does not satisfy* max-n, *but all of whose central factors are finitely generated.*

In view of Theorem 1, this has the following immediate corollary.

COROLLARY. *The class of finitely generated soluble groups, with all their central factors finitely generated, is not R_0-closed.*

A subgroup K is said to be *subnormal* in a group G if there is a finite chain of the form $K = K_0 \lhd K_1 \lhd \ldots \lhd K_n = G$. If $\mathfrak{X}$ is any class, we define an $\mathfrak{X}$ *pluperfect* group G to be one all of whose subnormal subgroups are $\mathfrak{X}$ perfect; this implies that $G \in C_{\mathfrak{X}}$. A group will be called a *Camm* group if it is a finitely generated simple group containing an infinite cyclic subgroup; such groups have been constructed in [**1**]. Finally, a $\{Q, S_n\}$-closed class $\mathfrak{X}$ is one with the property that every quotient, and every subnormal subgroup of an $\mathfrak{X}$-group is in $\mathfrak{X}$. With these definitions, we can state our last result.

THEOREM 4. *Let $\mathfrak{X}$ be any $\{Q, S_n\}$-closed class, and suppose that $\mathfrak{X}$ does not contain every Camm group. Then there is a finitely generated $\mathfrak{X}$ pluperfect group which does not satisfy* max-n.

By taking $\mathfrak{X}$ to be the class of all abelian groups, we obtain the following result.

COROLLARY. *There exists a finitely generated group, all of whose subnormal subgroups are perfect, but which does not satisfy the maximal condition for normal subgroups.*

Another consequence of Theorem 4, in view of Theorem 1, is the following result.

COROLLARY. *The class of finitely generated $\mathfrak{X}$ pluperfect groups is not R_0-closed.*

1.5. The classes $\mathfrak{X}^{-Q}$ and the class $\mathfrak{G}$ are closed under the operations of taking extensions and quotients. In the notation of [**4**] this means that they are $\{E, Q\}$-closed. This prompts the question: Is there an $\{E, Q\}$-closed class which does not have a unique maximal R_0-closed subclass?

We are very grateful to Professor P. Hall for permission to include Theorem 1.

2. Proof of Theorem 1. In showing that a class $\mathfrak{X}$ is R_0-closed, it is clearly sufficient to suppose that H and K are normal subgroups of G such that $H \cap K = 1$, $G/H \in \mathfrak{X}$, and $G/K \in \mathfrak{X}$, and to prove that $G \in \mathfrak{X}$.

LEMMA 2.1. *The class of all groups satisfying* max-n *is R_0-closed.*

Proof. Let H and K be two normal subgroups of G such that G/H and G/K both satisfy max-n, and $H \cap K = 1$, and suppose that $M_1 \leqq M_2 \leqq M_3 \leqq \ldots$ is an ascending chain of normal subgroups of G. Since G/H satisfies max-n, there is an integer i such that $M_i H = M_{i+r} H$ for all $r > 0$. Similarly,

$$(M_j \cap H)K = (M_{j+r} \cap H)K$$

for some j and all $r > 0$; then the fact that $K \cap H = 1$ implies that $M_j \cap H = M_{j+r} \cap H$. Thus $M_k = M_{k+r}$ for all $r > 0$, where $k = \max(i, j)$.

LEMMA 2.2. *The class of finitely generated groups satisfying* max-n *is R_0-closed.*

Proof. Suppose that G/H and G/K are both finitely generated and satisfy max-n, and assume that $H \cap K = 1$. Lemma 2.1 shows that G satisfies max-n, and hence it is sufficient to prove that $G \in \mathfrak{G}$. Since G/H and G/K are both finitely generated, there exists a finitely generated subgroup L of G such that $G = LH = LK$. Also G satisfies max-n, and thus $H = \langle a_1^G, \ldots, a_n^G \rangle$, where n is a finite integer, and $a_1, \ldots, a_n$ are suitably chosen elements of H. But $[H, K] = 1$, and it follows that $H = \langle a_1^L, \ldots, a_n^L \rangle$. Thus

$$G = \langle L, a_1, \ldots, a_n \rangle \in \mathfrak{G},$$

as required.

We can now prove Theorem 1. Because of Lemma 2.2, we have only to show that if $\mathfrak{X} = R_0\mathfrak{X} \leqq \mathfrak{G}$, then every $\mathfrak{X}$-group satisfies max-n. Suppose, on the contrary, that there is an $\mathfrak{X}$-group G containing a normal subgroup K which cannot be generated by a finite number of conjugacy classes of G. Let $D = G_1 \times G_2$ be the direct product of two copies G_1 and G_2 of G, and let $G^* = \{(a, a): a \in G\}$ be the diagonal subgroup of D. If K_1 and K_2 are the normal subgroups of G_1 and G_2 which correspond to K in G, then $K_1G^* = K_2G^* = H$, say, and $K_1 \cap K_2 = 1$. Also $H/K_1 \cong H/K_2 \cong G^* \cong G$, and hence $H \in R_0\mathfrak{X}$.

Since $\mathfrak{X} \leqq \mathfrak{G}$, we have $H = \langle x_1, \ldots, x_m \rangle$, where m is finite and $x_1, \ldots, x_m$ are suitable elements of H. Also $G^* \in \mathfrak{G}$, and thus $G^* = \langle y_1, \ldots, y_n \rangle$, where n is finite, and $y_1, \ldots, y_n$ are suitable elements of G^*. Now $H = K_1G^*$, and thus we can write $x_i = u_iv_i$ with $u_i \in K_1$ and $v_i \in G^*$ $(i = 1, \ldots, m)$. Hence $H = \langle u_1, \ldots, u_m, y_1, \ldots, y_n \rangle$. Since $K_1 \lhd H$ and $K_1 \cap G^* = 1$, it follows that $K_1 = \langle u_1^{G^*}, \ldots, u_m^{G^*} \rangle$, or equivalently $K_1 = \langle u_1^{G_1}, \ldots, u_m^{G_1} \rangle$. This contradicts the hypothesis that K is not generated by a finite number of conjugacy classes of G, and thus completes the proof of Theorem 1.

3. Proof of Theorem 2. If G is any group, we write (G) for the class consisting of all groups isomorphic to G, together with all groups of order 1.

LEMMA 3.1. *If G is any group, then every central factor of G occurs as a quotient of some group in the class $R_0(G)$.*

Proof. Let $D = G_1 \times G_2$ be the direct product of two copies G_1 and G_2 of G, and let $G^* = \{(a, a) : a \in G\}$ be the diagonal subgroup of D. Suppose that L/M is any central factor of G, and let L_1, M_1 and L_2, M_2 be the subgroups of G_1 and G_2 corresponding to L, M in G. Then $L_1G^* = L_2G^* = R$ say, and L_1 and L_2 are normal subgroups of R with $R/L_1 \cong R/L_2 \cong G^* \cong G$. Also $L_1 \cap L_2 = 1$, so that $R \in R_0(G)$.

Since L/M is a central factor of G, we have $[L_i, R] \leqq M_i$, $i = 1, 2$. This implies that $M_1G^* = M_2G^* = S$, say, is normal in R; further,

$$R/S \cong L_1/M_1 \cong L_2/M_2.$$

Thus L/M is a homomorphic image of R and Lemma 3.1 is proved.

LEMMA 3.2. *If $\mathfrak{X}$ is any Q-closed class, then $C_{\mathfrak{X}} = R_0C_{\mathfrak{X}}$.*

Proof. Let G/H and G/K be $C_{\mathfrak{X}}$-groups, with $H \cap K = 1$, and suppose that L/M is a central factor of G, and that $L/M \in \mathfrak{X}$. Then

$$L/M(L \cap H) = L/(L \cap MH) \cong LH/MH,$$

so that LH/MH is a central factor of G/H, and also an $\mathfrak{X}$-group. Therefore $L = M(L \cap H)$.

Again $M(L \cap H)/M \cong (L \cap H)/(M \cap H) \cong (L \cap H)K/(M \cap H)K$; this latter group is a central factor of G/K and at the same time a member of $\mathfrak{X}$. Hence $M(L \cap H) = M$, and hence $L = M(L \cap H) = M$. Thus G has no non-trivial central factors in $\mathfrak{X}$.

To prove the lemma, it is now sufficient to show that $G \in \mathfrak{X}^{-Q}$. Suppose that $N \lhd G$ and $G/N \in \mathfrak{X}$. Then G/NH and G/NK are both $\mathfrak{X}$-groups, and hence $NH = NK = G$. Therefore $[G, H] = [NK, H] = [N, H]$, and thus $H/[N, H]$ is a central factor of G. Since $[N, H] \leqq N \cap H$, this implies that $H/(N \cap H)$ is also a central factor of G; but $G/N \cong H/(H \cap N)$, and it follows that G/N is a central factor of G, and at the same time an $\mathfrak{X}$-group.

We have already shown that G has no non-trivial central factors in $\mathfrak{X}$, and we can therefore deduce that $G = N$. The proof of Lemma 3.2 is now complete.

These lemmas enable us to prove Theorem 2. By Lemma 3.2, $C_{\mathfrak{X}} = R_0C_{\mathfrak{X}}$. On the other hand, if G has a non-trivial central factor in $\mathfrak{X}$, then Lemma 3.1 shows that we can find a group in $R_0(G)$ which has a non-trivial quotient in $\mathfrak{X}$. Thus $R_0(G) \nleqq \mathfrak{X}^{-Q}$, and Theorem 2 follows.

4. Proof of Theorem 3. The technique used in constructing the group needed for the proof of Theorem 3 comes from [**2**]; as in that paper, we shall abbreviate the maximal condition for subgroups to max.

Let Γ be a soluble group satisfying max-n but not max; for instance, we could take the wreath product of two infinite cyclic groups. (Γ is necessarily finitely generated.) Let C be a cyclic group of order p and consider the standard wreath product $W = C \wr \Gamma$. If F_p is the field of p elements, and R is the group algebra of Γ over F_p, then W may be regarded as the split extension of the additive group of R by Γ, with the elements of Γ acting on R as right multipliers. Then the right ideals of R coincide with the normal subgroups of W contained in R. Hence W satisfies max-n if and only if R satisfies the maximal condition for right ideals.

With every subgroup H of Γ, we associate a right ideal

$$H^* = \langle (h - 1)R: h \in H\rangle$$

of R. If $S = \{s_1, s_2, \ldots\}$ is a transversal to H in Γ, then H^* has

$$\{(h - 1)s: 1 \neq h \in H, s \in S\}$$

as an additive basis; for if

$$\sum_{ij} \lambda_{ij}(h_{ij} - 1)s_i = 0,$$

where $\lambda_{ij} \in F_p$, $h_{ij} \in H$, and $h_{ij} \neq h_{ik}$ whenever $j \neq k$, then the definition of S implies that

$$\sum_{j} \lambda_{ij}(h_{ij} - 1) = 0$$

for all i, and this in turn implies that $\lambda_{ij} = 0$ for all i and j.

Suppose that $H_1 < H_2 \leqq \Gamma$, and let H_1^* and H_2^* be the corresponding right ideals of R. If $h \in H_2$ and $h \notin H_1$, then $h - 1 \in H_2^*$ but $h - 1 \notin H_1^*$ since a transversal to H_1 in H_2 forms part of an additive basis of H_1^*. Thus if Γ does not satisfy max, then R does not satisfy the maximal condition for right ideals, so that W does not satisfy max-n.

Now let p be an odd prime, and write D for the additive group of R. Since D is abelian, it admits an automorphism t of order 2 which maps every element of D to its inverse. We take $T = \langle t\rangle$. Γ may also be considered as a subgroup of the group of automorphisms of D and when this is done, $\Gamma \cap T = 1$.

Moreover, if $\xi \in \Gamma$, then $\xi^{-1}t^{-1}\xi t$ acts as the identity automorphism, and thus Γ and T generate a group isomorphic to the direct product $\Gamma \times T$.

Every subgroup of D is invariant under T. Since W does not satisfy max-n, this implies that the group $G = D\Gamma T$ does not satisfy max-n either. It is also clear that G is soluble. It remains to show that if L/M is a central factor of G, then L/M is finitely generated.

Using the definition of t, we have:

$$L \cap D = [L \cap D, T] \leqq [L, T] \cap [D, T] \leqq M \cap D \leqq L \cap D,$$

and thus $L \cap D = M \cap D$. Since

$$(L \cap DM)/M \cong (L \cap D)M/M \cong (L \cap D)/(M \cap D),$$

it follows that $L \cap DM = M$. Hence

$$L/M = L/(L \cap DM) \cong LD/MD$$

which is a central factor of G/D; but G/D is isomorphic to ΓT which satisfies max-n since Γ satisfies max-n by hypothesis.

Hence L/M is finitely generated, as required.

5. Proof of Theorem 4. The construction which follows is an elaboration of that used to prove [**3**, Theorem 7]. Let H be a Camm group with $H \notin \mathfrak{X}$, and for each positive integer i, take a group H_i isomorphic to H, and an isomorphism from H onto H_i. If $h \in H$, we write h_i for the image of h under this isomorphism. We regard $H_1, H_2, \ldots$ as permutation groups by taking their regular representations, and we define W to be the wreath product $H_1 \wr H_2 \wr \ldots$. Using the notation of [**5**], this is $\mathrm{Wr}\, H_i$, $i \in \Lambda$, where Λ is the set of positive integers in their natural order.

Let K be another isomorphic copy of H, and consider the standard complete wreath product $W \wr^{-} K$. Our notation for this will be as follows. We write E for the group of all unrestricted functions from K into W, with the multiplication of two elements $f, g \in E$ given by $(fg)(x) = f(x)g(x)$, $x \in K$. Then $W \wr^{-} K$ is a split extension of E by K, where the automorphism of E induced by an element $y \in K$ is defined by

$$f^y(x) = f(xy^{-1}), \qquad f \in E, x \in K.$$

Since K is a Camm group, it contains an infinite cyclic subgroup $\langle t \rangle$; we fix t for the rest of the argument. Corresponding to each element $h \in H$, we define a function $\bar{h} \in E$ by

$$\bar{h}(x) = \begin{cases} h_i & \text{if } x = t^{2^i} \ (i \geqq 1), \\ 1 & \text{otherwise,} \end{cases}$$

and we take

$$\bar{H} = \{\bar{h} \colon h \in H\}.$$

It is easy to verify that the map $h \rightarrow \bar{h}$ $(h \in H)$ is an isomorphism from H onto $\bar{H}$. We shall show that the subgroup G of $W \wr^{-} K$ generated by $\bar{H}$

and K satisfies the conditions of Theorem 4. We note immediately that $G \in \mathfrak{G}$, since $\bar{H}$ and K are both finitely generated.

We take $C = G \cap E$, and we use D to denote the subgroup of E formed by the *restricted* functions from K to W; in other words, $f \in D$ if and only if $\{x: f(x) \neq 1\}$ is finite.

LEMMA 5.1 $D \leqq C$.

Proof. If $u \in W$, we can define a function $f_u \in D$ by

$$f_u(x) = \begin{cases} u & \text{if } x = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then the map $u \to f_u$ $(u \in W)$ is an isomorphism from W into D, and we identify W with its image under this isomorphism. Clearly, D is generated by the conjugates W^y $(y \in K)$ of W, and it is therefore sufficient to show that $W \leqq C$.

Suppose that $h \in H$ and $n \geqq 1$, and let $f = t^{2^n}\bar{h}t^{-2^n}$ and $g = t^{2^{n+1}}\bar{h}t^{-2^{n+1}}$. Then $f \in C$ and $g \in C$ and we have:

$$f(x) = \begin{cases} h_i & \text{if } x = t^{2^i - 2^n} \ (i \geqq 1), \\ 1 & \text{otherwise,} \end{cases}$$

$$g(x) = \begin{cases} h_j & \text{if } x = t^{2^j - 2^{n+1}} \ (j \geqq 1), \\ 1 & \text{otherwise.} \end{cases}$$

We look for elements $x \in K$ for which $f(x) \neq 1 \neq g(x)$. Any such element arises from positive integers i and j such that $2^i - 2^n = 2^j - 2^{n+1}$, or equivalently $2^j - 2^i = 2^{n+1} - 2^n$. The only solution of this equation is given by $i = n$ and $j = n + 1$, which corresponds to the case when $x = 1$. It follows that the values of the function $[f, g]$ are given by

$$[f, g](x) = \begin{cases} [h_n, h_{n+1}] & \text{if } x = 1, \\ 1 & \text{otherwise.} \end{cases}$$

This is the function which we have identified with the element $[h_n, h_{n+1}]$ of W, and thus $h_n^{-1}h_n^{h_{n+1}} = [h_n, h_{n+1}] \in C$ for all $h \in H$ and all $n \geqq 1$.

Suppose that $h, k \in H$ and $n \geqq 1$. Then the commutator

$$[h_n^{-1}h_n^{h_{n+1}}, k_n^{-1}k_n^{k_{n+1}}]$$

lies in C, since each of its components does. These components also lie in the subgroup $H_n \wr H_{n+1}$ of W. Now the normal closure of H_n in $H_n \wr H_{n+1}$ is the direct product of the conjugates H_n^z $(z \in H_{n+1})$. Thus h_n^{-1} and k_n^{-1} are in the same direct factor, but if h and k are distinct and non-trivial, then $h_n^{h_{n+1}}$ and $k_n^{k_{n+1}}$ lie in different direct factors. Hence

$$[h_n^{-1}h_n^{h_{n+1}}, k_n^{-1}k_n^{k_{n+1}}] = [h_n^{-1}, k_n^{-1}] = [h^{-1}, k^{-1}]_n,$$

where h and k are arbitrary distinct non-trivial elements of H. Therefore

$H_n = H_n' \leqq C$. This holds for all positive integers n, and hence $W \leqq C$, as required.

Using Lemma 5.1, we can show that G does not satisfy max-n. For each $i \geqq 1$ we define B_i and $\bar{B}_i$ to be the normal closures of $H_1 \wr H_2 \wr \ldots \wr H_i$ in W and $W \wr^- K$, respectively. It is clear from the result in [**5**] that $B_1 < B_2 < \ldots$ is a strictly ascending chain of normal subgroups of W. Moreover, if $i \geqq 1$, then $\bar{B}_i$ is the direct product of the groups B_i^y $(y \in K)$, and thus $\bar{B}_1 < \bar{B}_2 < \ldots$ is a strictly ascending chain of normal subgroups of $W \wr^- K$. Since each group $\bar{B}_i$ is contained in D, it follows from Lemma 5.1 that G does not satisfy max-n.

Lemma 5.2. *$G/D \cong H \wr K$.*

Proof. Since $D\bar{H}/D \cong \bar{H} \cong H$, it is sufficient to show that the normal closure of $D\bar{H}/D$ in G/D is the direct product of the groups $D\bar{H}^y/D$ $(y \in K)$; but H is simple, and thus we can do this by supposing that $y \in K$, $y \neq 1$, and proving that $D\bar{H}/D$ commutes elementwise with $D\bar{H}^y/D$. Thus what we must verify is that $[\bar{h}, \bar{k}^y] \in D$ whenever $h, k \in H$.

Now all the elements $x \in K$ with $\bar{h}(x) \neq 1$ lie in $\langle t \rangle$, whereas the elements with $\bar{k}^y(x) \neq 1$ lie in the coset $\langle t \rangle y$. If $y \notin \langle t \rangle$, this implies that $[\bar{h}, \bar{k}^y] = 1$. If $y \in \langle t \rangle$, then an argument like that used in the proof of Lemma 5.1 shows that there is at most one element $x \in K$ such that $\bar{h}(x) \neq 1 \neq \bar{k}^y(x)$. This implies that $[\bar{h}, \bar{k}^y] \in D$, and thus proves Lemma 5.2.

Lemma 5.3. *If $\mathfrak{X}$ is $\{Q, S_n\}$-closed and $H \notin \mathfrak{X}$, then W is $\mathfrak{X}$ pluperfect.*

Proof. Suppose that U is a subnormal subgroup of W, and that U/V is a non-trivial quotient of U lying in $\mathfrak{X}$. Since

$$W = \bigcup_{i=1}^{\infty} B_i,$$

it follows that if $B_0 = 1$, then

$$V = (U \cap B_0)V \leqq (U \cap B_1)V \leqq \ldots$$

is an ascending chain whose union is U. Since $V < U$, there must be some $i \geqq 0$ such that $(U \cap B_i)V < (U \cap B_{i+1})V$. Then $(U \cap B_{i+1})V/(U \cap B_i)V$ is a quotient of a normal subgroup of U/V, and hence is a non-trivial $\mathfrak{X}$-group. However, it is isomorphic to $(B_{i+1} \cap U)B_i/(B_{i+1} \cap V)B_i$ which is a quotient of a subnormal subgroup of B_{i+1}/B_i. This latter group is a direct product of conjugates of B_iH_{i+1}/B_i, which are all isomorphic to the simple group H. Hence $(U \cap B_{i+1})V/(U \cap B_i)V$ is isomorphic to a direct product of copies of H. Since it is also a non-trivial $\mathfrak{X}$-group, while $H \notin \mathfrak{X}$ by hypothesis, this gives a contradiction, which is sufficient to prove Lemma 5.3.

We are now ready to complete the proof of Theorem 4; the only fact left to prove is that G is $\mathfrak{X}$ pluperfect. We suppose, by way of contradiction, that

L is a subnormal subgroup of G, and that L/M is a non-trivial quotient of L lying in $\mathfrak{X}$. Then

$$(*) \qquad M \leqq (L \cap D)M \leqq (L \cap C)M \leqq L \quad \text{and} \quad M < L$$

so that at least one of these inclusions is proper.

Assume first that $M < (L \cap D)M$, and suppose that the elements of the countable group K are $x_1, x_2, \ldots$. For each $i \geqq 1$, we define

$$N_i = \langle W^{x_1}, \ldots, W^{x_i} \rangle,$$

and we take N_0 to be 1. Then $1 = N_0 \leqq N_1 \leqq \ldots$ is an ascending chain of subgroups of D such that $\bigcup_{i=1}^{\infty} N_i = D$.

Hence $M = (L \cap N_0)M \leqq (L \cap N_1)M \leqq \ldots$ is an ascending chain of subgroups of L whose union is $(L \cap D)M$. We can therefore find a number i such that $(L \cap N_i)M < (L \cap N_{i+1})M$. Then $(L \cap N_{i+1})M/(L \cap N_i)M$ is a quotient of a subnormal subgroup of L/M, and hence is a non-trivial $\mathfrak{X}$-group. On the other hand, it is isomorphic to $(N_{i+1} \cap L)N_i/(N_{i+1} \cap M)N_i$ which is a quotient of a subnormal subgroup of N_{i+1}/N_i. This latter group is isomorphic to W, and hence we have a contradiction to Lemma 5.3.

Suppose next that $(L \cap D)M < (L \cap C)M$. Then $(L \cap C)M/(L \cap D)M$ is a non-trivial quotient of a normal subgroup of L/M and is therefore an $\mathfrak{X}$-group. Moreover,

$$(L \cap C)M/(L \cap D)M \cong (C \cap L)D/(C \cap M)D$$

which is a quotient of a subnormal subgroup of C/D; but Lemma 5.2 implies that C/D is a direct product of copies of H, so that we obtain a contradiction as before.

The final possibility allowed by $(*)$ is that $L \cap CM = (L \cap C)M < L$. Then $L/(L \cap CM)$ is a non-trivial $\mathfrak{X}$-group, and is also isomorphic to CL/CM which is a quotient of a subnormal subgroup of G/C. Since G/C is isomorphic to the simple group K, we conclude that $L/(L \cap CM) \cong K$. But $K \cong H$, and $H \notin \mathfrak{X}$ by hypothesis. This contradiction completes the proof of Theorem 4.

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