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LATTICE-ORDERED MODULES OF QUOTIENTS

STUART A. STEINBERG

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Abstract

Let Q be the ring of quotients of the f-ring R with respect to a positive hereditary torsion theory and suppose Q is a right f-ring. It is shown that if the finitely-generated right ideals of R are principal, then Q is an f-ring. Also, if Q_R is injective, Q is an f-ring if and only if its Jacobson radical is convex. Moreover, a class of po-rings is introduced (which includes the classes of commutative po-rings and right convex f-rings) over which $Q(M)$ is an f-module for each f-module M.

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1. Introduction

F. W. Anderson (1965) has given necessary and sufficient conditions for the maximal right quotient ring of an f-ring R to be an f-ring extension of R, and Steinberg (1972c) has shown that one of Anderson's conditions describes when the injective hull of a nonsingular f-module M is an f-module extension of M. In his thesis, Georgoudis (1972) considered the analogous questions for the ring (module) of quotients Q of an f-ring (f-module) R with respect to a hereditary torsion theory, and similar characterizations were given. Also, see Bigard (1973), Zaharoff (1977) and Steinberg (1978). In this note one of the questions we are concerned with is whether or not Q is an f-ring extension of R provided Q_R is an f-module extension of R_R. If the finitely generated right ideals of R are principal this is the case, and if Q_R is injective it is the case precisely when the Jacobson radical of Q is a convex ideal.

Throughout, all rings R will have an identity element and all modules will be right unital modules. A hereditary torsion theory for R is completely determined by its associated topology $\mathcal{G}$ of *dense* right ideals which consists of all right ideals D of R such that R/D is torsion, or by its associated radical t, where $t(M)$ consists of all elements of the module M which are annihilated by an element of $\mathcal{G}$. The module of quotients of M is given by

$$Q(M) = Q(M/t(M)) = [\bigcup_{D\in\mathscr{S}} \mathrm{Hom}_R(D, M/t(M))]/\sim,$$

where two R-homomorphisms f and g are equivalent via the equivalence relation $\sim$ if they agree on some $D\in\mathscr{S}$. ψ: $M\to Q(M)$ will denote the canonical map. $Q(M)$ is characterized by the conditions: (i) $Q(M)$ is an essential extension of $\psi(M)$, (ii) $Q(M)/\psi(M)$ is torsion and (iii) $Q(M)$ is *$\mathscr{S}$-injective* in the sense that for each $D\in\mathscr{S}$ every $f\in\mathrm{Hom}_R(D, Q(M))$ may be extended to an element of $\mathrm{Hom}_R(R, Q(M))$. For $q\in Q(M)$ we will let

$$D_q = \{d\in R\colon qd\in\psi(M)\}\in\mathscr{S}.$$

For more details, see Stenstrom (1971) and Goldman (1969).

Whenever $\mathscr{S}$ is a topology for a po-ring R we will require $\mathscr{S}$ to be *positive* in the sense that $D\in\mathscr{S}$ implies $D^+R\in\mathscr{S}$. ($X^+ = \{x\in X\colon x\geqslant 0\}$ denotes the positive cone of the po-set X.) Bigard (1973), p. 5-04 has noted that if R is a directed po-ring in which the square of every element is positive and if $\frac{1}{2}\in R$, then each topology for R is positive. We note that a unital po-ring which is 2-semiclosed (that is, $2x\in R^+$ implies $x\in R^+$) and has squares positive is necessarily directed. For if $a\in R$, then $0\leqslant(1-a)^2 = 1-2a+a^2$ yields $2a\leqslant 1+a^2\leqslant 2(1+a^2)$. So $1+a^2$ is an upper bound for $\{0, a\}$. Moreover, the following might be noteworthy.

LEMMA 1. *Let R be a ring which contains $\frac{1}{2}$, and let $\mathscr{S}$ be a topology for R. If $D\in\mathscr{S}$ and if A is the right ideal of R generated by the squares of elements of D, then $A\in\mathscr{S}$.*

PROOF. We will show that $2D^3R\subseteq A$, for any right ideal D. If $a, b\in D$, then

$$ab+ba = (a+b)^2-a^2-b^2\equiv 0 \pmod{A}.$$

So for each $x\in R$, $abx\equiv -bax$. If $a, b, c\in D$ and $x\in R$, then

$$2abcx = abcx+abcx\equiv -bcax-bacx\equiv -bcax+bcax = 0.$$

The module M over the po-ring R is called a *po-module* (*l-module*) if M is a partially-ordered (lattice-ordered) group and $M^+R^+\subseteq M^+$. If the l-module M is embeddable (both as a module and as a lattice) in a product of totally ordered R-modules, then M is called an *f-module*. If it satisfies the condition that $xr\wedge y = 0$ ($xr\wedge yr = 0$) whenever $x\wedge y = 0$ in M and $r\in R^+$, then M is called a *p-module* (*d-module*). An f-module is a p-module and a p-module is a d-module, and if R is directed a d-module is an f-module. An *f-ring* is an l-ring R that is embeddable in a product of totally ordered rings. Equivalently, R is an f-ring if it is a right and left f-ring, that is, R_R and ${}_RR$ are both f-modules. For x in the f-module M, if we let

$$x^+ = x\vee 0,\quad x^- = (-x)\vee 0\quad\text{and}\quad |x| = x^+\vee x^-,$$

then

$$x = x^+ - x^-, \quad |x| = x^+ + x^- \quad \text{and} \quad x^+ \wedge x^- = 0.$$

If $\mathscr{T}$ is a positive topology then the torsion submodule of each po-module M is a convex submodule (that is, $0 \leqslant x \leqslant y$ and $y \in t(M)$ implies $x \in t(M)$). So $M/t(M) \cong \psi(M)$ is a po-module over $R/t(R) \cong \psi(R)$. When M is an f-module $t(M)$ is a convex l-submodule, so $\psi(M)$ is a d-module over $\psi(R)$. Define a positive cone in $Q(M)$ by

$$Q(M)^+ = \{q \in Q(M) \colon \exists\, D \in \mathscr{T} \text{ with } qD^+ \subseteq \psi(M)^+\}.$$

Then $Q(M)$, with this positive cone, is a po-module over the po-ring $Q(R)$. Moreover, if M is a d-module, then $Q(M)^+$ is the largest positive cone P of $Q(M)$ for which $Q(M)$ is a po-module over R and $P \cap \psi(M) = \psi(M)^+$. For if P is such a positive cone, then it is easy to see that $P \subseteq Q(M)^+$; and if $p \in Q(M)^+ \cap \psi(M)$, then $0 = (pd)^- = p^- d$ for each $d \in D^+$, where $pD^+ \subseteq \psi(M)^+$ and $D \in \mathscr{T}$. Thus $p^- = 0$, $p \in \psi(M)^+$, and $Q(M)^+ \cap \psi(M) = \psi(M)^+$. In a similar manner it can be shown that if $Q(M)$ is an f-module extension of $\psi(M)$, then the positive cone of $Q(M)$ must be $Q(M)^+$.

Since $\psi \colon R \to Q(R)$ is an order preserving ring homomorphism it is clear that a po-module over $Q(R)$ is an f-module or d-module or p-module over R provided it is such an l-module over $Q(R)$. Conversely, if $Q(M)$ is a d-module over R, then it is a d-module over $Q(R)$, and hence an f-module over $Q(R)^+ - Q(R)^+$, the largest directed subring of $Q(R)$.

If, for the f-module M_R, $Q(M)$ is an R-f-module extension of $\psi(M)$, then M will be called a *Df-module*, and if $Q(R)$ is an f-ring extension of the f-ring R, then R will be called a *Df-ring*. Georgoudis' (1972) theorem is then

THEOREM 2. *Let $\mathscr{T}$ be a positive topology for the directed po-ring R. The following statements are equivalent for the f-module M_R.*

(a) *M is a $\mathscr{T}f$-module.*

(b) *If $q \in Q(M)$ and $d_1, d_2 \in D_q^+$, then $(qd_1)^+ \wedge (qd_2)^- = 0$.*

(c) *If $q \in Q(M)$ there exists $D \in \mathscr{T}$ such that $qD \subseteq \psi(M)$ and $(qd_1)^+ \wedge (qd_2)^- = 0$ for $d_1, d_2 \in D^+$.*

Moreover, the f-ring R is a $\mathscr{T}f$-ring if and only if R_R is a $\mathscr{T}f$-module and satisfies:

(d) *If $q \in Q(R)^+$ and $d_1, d_2 \in D_q^+$ with $d_1 \wedge d_2 = 0$, then $qd_1 \wedge \psi(d_2) = 0$.*

If N is a submodule of the module M, then the closure of N in M (relative to the topology $\mathscr{T}$) is defined by $Cl(N)/N = t(M/N)$. If N is an l-submodule (convex submodule) of the d-module M, then $Cl(N)$ is an l-submodule (convex submodule) of M. Consequently if N is an l-submodule (convex submodule) of the $\mathscr{T}f$-module M, then $Q(N) = Cl_{Q(M)}(\psi(N))$ is an f-module extension of $\psi(N)$ (convex submodule of $Q(M)$). So the class of $\mathscr{T}f$-modules is closed under l-submodules. This fact and the fact that the direct product of a family of torsion-free $\mathscr{D}f$-modules is a $\mathscr{D}f$-module yields that it is also closed under direct sums. We also note that if Q is a right exact functor and N is a convex l-submodule of the $\mathscr{D}f$-module M, then M/N is a $\mathscr{D}f$-module. For $Q(M/N) \cong Q(M)/Q(N)$ as modules, and since

$$\psi(M/N) \cong [M/N]/t(M/N) \cong M/Cl_M(N),$$

$Q(M/N)$ is an f-module extension of $\psi(M/N)$.

2. Finiteness conditions

A module M is called *$\mathscr{D}$-closed* if $Q(M) = M$. A topology $\mathscr{D}$ is *perfect* if $Q(M) = M \bigotimes_R Q(R)$ for each M_R. Other characterizations of a perfect topology are given in Stenstrom (1971), Theorem 13.1. Part of one of these is that a directed union of $\mathscr{D}$-closed modules is $\mathscr{D}$-closed. The next theorem, which generalizes Steinberg (1972c) Theorem 3.15, shows that the $\mathscr{D}f$-property is a local property. Recall that the *polar* of a subset X of the f-module M_R is the convex l-submodule

$$X' = \{y \in M : |y| \wedge |x| = 0 \text{ for each } x \in X\}.$$

THEOREM 3. *Let $\mathscr{D}$ be a positive topology for the directed po-ring R, and assume that Q is a right exact functor. Let M be an R-f-module and suppose that either*

(a) *$\psi(M)$ is a subdirect product of totally ordered torsion-free modules and the union of a chain of $\mathscr{D}$-closed convex l-submodules is $\mathscr{D}$-closed, or*

(b) *$\psi(M)$ has the maximum condition on convex l-submodules.*

Then M is a $\mathscr{D}f$-module if and only if each principal convex l-submodule $C_R(g)$ is a $\mathscr{D}f$-module.

PROOF. We first note that (b) implies that $\psi(M)$ is embeddable in a finite product of totally ordered torsion-free modules. For if N is a polar of $\psi(M)$, then $\psi(M)/N$ is torsion-free. Now the intersection of the maximal polars is 0, and if N is a maximal polar, then $\psi(M)/N$ is totally ordered (see Conrad (1970), p. 2.9).

Now suppose that every principal convex l-submodule $C_R(g)$ is a $\mathscr{D}f$-module. If $\varphi(M)$ is a totally ordered torsion-free homomorphic image of M, then $\varphi(M)$ has this same property since Q is right exact. Moreover, since $\psi(M)$ is a subdirect product of the $\varphi(M)$, if each $\varphi(M)$ is a $\mathscr{D}f$-module, then so is M. Thus we may assume that M is totally ordered and torsion-free.

Let $q \in Q(M)$ and for each $d \in D_q^+$ let $Q_d = Q(C_R(qd)) \subseteq Q(M)$. Each Q_d is a $\mathscr{D}$-closed convex (totally ordered) submodule of $Q(M)$, and, since M is totally ordered, $\{Q_d\}$ is a chain. By hypothesis $T = \bigcup_d Q_d$ is $\mathscr{D}$-closed. Clearly, T is an f-module extension of $N = \bigcup_d C_R(qd)$ and in fact $T = Q(N)$. Now $qD_q^+ \subseteq T$, so $q \in T$; and if

$d_1, d_2 \in D_q^+$, then

$$(qd_1)^+ \wedge (qd_2)^- = q^+ d_1 \wedge q^- d_2 = 0.$$

But then M is a $\mathscr{D}f$-module by Theorem 2.

Bigard (1973), Proposition 8, has shown that if $\mathscr{D}$ contains a cofinal subset of finitely generated right ideals (equivalently, each directed union of $\mathscr{D}$-closed modules is $\mathscr{D}$-closed, Stenstrom (1971), Proposition 12.2), then every torsion-free f-module is a subdirect product of totally ordered torsion-free f-modules. This is not true without this finiteness condition (see Steinberg (1972c), Proposition 2.7).

Next we consider sufficient conditions for M to be a $\mathscr{D}f$-module if $Q(M) = M \bigotimes_R Q(R)$. A *prime* subgroup of the l-group M is a convex l-subgroup N such that M/N is totally ordered. In an f-module a minimal prime sub-group is a submodule (Steinberg (1972a), 1.1, or Conrad and Diem (1971), 2.1). If $x \in M$, then $\operatorname{ann} x = \{r \in R\colon xr = 0\}$.

THEOREM 4. *Let $\mathscr{D}$ be a positive topology for the f-ring R, and assume that R_R is a $\mathscr{D}f$-module. Suppose that the po-set of cyclic submodules of $\psi(M)$ is directed, and that for each $x \in \psi(M)$ there is a minimal prime subgroup P of R, such that $\operatorname{ann} x \subseteq P$. Then if $Q(M) = M \bigotimes_R Q(R)$, M is a $\mathscr{D}f$-module.*

PROOF. Let $\{P_\alpha\colon \alpha \in A\}$ be the collection of minimal prime subgroups of R, and let C_α be the convex l-submodule of $\psi(M)$ generated by $\psi(M)P_\alpha$. Take $0 < x \in \bigcap C_\alpha$, and let $\beta \in A$ be such that $\operatorname{ann} x \subseteq P_\beta$. Since $x \in C_\beta$, $x \leqslant zp$ for some $z \in \psi(M)^+$ and $p \in P_\beta^+$. Since P_β is a minimal prime subgroup, by Johnson and Kist (1962), Theorem 6.5, there exists $r \in R^+ \setminus P_\beta$ with $p \wedge r = 0$. But then $pr = 0$ and so $0 \leqslant xr \leqslant zpr = 0$. Thus $r \in P_\beta$, which is impossible. So $\bigcap C_\alpha = 0$, and $\psi(M)$ is a subdirect product of a family of f-modules $\{N_j\}$, where for each N_j there exists an α such that N_j is a totally ordered module over the totally ordered ring $R_\alpha = R/P_\alpha$.

Let $q = \sum x_i \otimes u_i \in M \bigotimes_R Q(R)$, and take $D \in \mathscr{D}$ and $y \in M$ with $u_i D \subseteq R$ and $x_i \otimes 1 = (y \otimes 1) r_i$ for each i. If $d_1, d_2 \in D^+$, then

$$t = (qd_1)^+ \wedge (qd_2)^- = [(y \otimes 1)(\textstyle\sum r_i u_i d_1)]^+ \wedge [(y \otimes 1)(\sum r_i u_i d_2)]^-.$$

If $t > 0$, then in some N_j,

$$(\overline{y \otimes 1})(\textstyle\sum r_i u_i d_1) > 0 > (\overline{y \otimes 1})(\sum r_i u_i d_2).$$

Now N_j is an R_α-module and since R_R is a $\mathscr{D}$-f-module, $\sum r_i u_i d_1$ and $\sum r_i u_i d_2$ are either both positive or both negative in R_α. So the previous inequalities are impossible and $t = 0$.

We note that in Example 3.7 of Steinberg (1972c), there is an example of an f-module which satisfies the hypothesis of this theorem yet which is not a $\mathscr{D}f$-module.

In that example M_R is simple, $\mathscr{D}$ is perfect and R is an l-ring that is not an f-ring, though $Q(R)$ is an l-ring extension of $\psi(R) = R$.

3. Does (a) imply (d) in Theorem 2?

We continue to assume that $\mathscr{D}$ is a positive topology for the directed po-ring R. It is unknown if a unital right f-ring is an f-ring. We suspect that if R_R is a $\mathscr{D}f$-module, then in fact, $Q(R)$ is an f-ring. One reason for this is that in a unital right f-ring $1^- = 0$. Thus, by Birkhoff and Pierce (1956), p. 60, the identity $x^+x^- = 0$ is satisfied, and hence the right f-ring $Q(R)$ is very close to being an f-ring. In particular, by a result of Diem's (1968) it is an f-ring modulo its lower radical. Of course, the following two statements are equivalent:

1. A unital right f-ring is an f-ring.
2. If $\mathscr{D}$ is a positive topology for the right f-ring R and if R_R is a $\mathscr{D}f$-module, then R is a $\mathscr{D}f$-ring.

We consider two special cases of this question.

Lemma 5. *If R is a $\mathscr{D}f$-ring and $q \in Q(R)$, then there exists $D \in \mathscr{D}$ such that D_R is an l-submodule of R_R and $D \subseteq D_q$.*

Proof. Let $q = s - t$ where $s, t \in Q(R)^+$. The sublattice of R generated by $D_s \cap D_t$ is

$$D = \{\bigvee_j \bigwedge_i d_{ij}: d_{ij} \in D_s \cap D_t\},$$

and it is an l-submodule of R_R. Also

$$q(\bigvee\bigwedge d_{ij}) = \bigvee\bigwedge sd_{ij} - \bigvee\bigwedge td_{ij} \in \psi(R), \quad \text{so } D \subseteq D_q.$$

An l-ring is called *right convex* if each of its right ideals is aconvex l-subgroup. Regular f-rings and left injective f-rings are right convex (Steinberg (1973)). Georgoudis (1972) has shown that each f-module is a $\mathscr{D}f$-module if R is an f-ring that is either commutative or right convex. Also, see Anderson (1965) and Steinberg (1973). Henriksen (1977), p. 407, has considered the following condition on an l-ring R:

(*) $$0 \leqslant x \leqslant y^2 \text{ implies } x \in yR.$$

Let us call a po-ring a (**)-ring if it has the commutative property

(**) $$0 \leqslant x \leqslant y \text{ implies } xy \in yR^+.$$

Note that all commutative po-rings, right convex f-rings and left f-rings satisfying Henriksen's condition are (**)-rings.

THEOREM 6. *Let R be a directed $(**)$-po-ring, and let $\mathscr{D}$ be a positive topology for R. Then every f-module M over R is a $\mathscr{D}f$-module. If R is an f-ring, then it is a $\mathscr{D}f$-ring.*

PROOF. Let $q \in Q(M)$ and let $D \in \mathscr{D}$ with $qD \subseteq \psi(M)$. Take $d_1, d_2, x \in D^+$, and let $d = d_1 + d_2 + x$. Since $d_i \leqslant d$ there exists $r_i \in R^+$ with $d_i d = dr_i$. Then

$$0 \leqslant [(qd_1)^- \wedge (qd_2)^-]\, x \leqslant [(qd_1)^+ \wedge (qd_2)^-]\, d$$
$$= (qdr_1)^+ \wedge (qdr_2)^- = (qd)^+ r_1 \wedge (qd)^- r_2 = 0.$$

So M is a $\mathscr{D}f$-module by Theorem 2.

If R is an f-ring and $d_1 \wedge d_2 = 0$, then, using the notation of the previous paragraph (with $0 \leqslant q$),

$$0 \leqslant [qd_1 \wedge \psi(d_2)]\, x \leqslant [(q \vee 1)d_1 \wedge (q \vee 1)d_2]\, d$$
$$= (q \vee 1)dr_1 \wedge (q \vee 1)dr_2 = (q \vee 1)d[r_1 \wedge r_2]$$
$$= (q \vee 1)[d_1 d \wedge d_2 d] = 0.$$

Thus, again by Theorem 2, R is a $\mathscr{D}f$-ring.

Note that the previous argument did not require R to be unital. So if R is a $(**)$-f-ring whose left annihilator is 0, then the maximal right quotient ring of R is an f-ring extension of R. Also if $\mathscr{D}$ contains a cofinal subset $\mathscr{E}$ such that for each $D \in \mathscr{E}$ the po-set $\{dR^+ \mid d \in D^+\}$ is directed above, then a similar argument gives the conclusion of the theorem.

COROLLARY 7. *Suppose that R is an f-ring and for each $q \in Q(R)$ the po-set $\{dR\colon d \in D_q\}$ is directed above. Then R is a $\mathscr{D}f$-ring if and only if each D_q contains a dense l-submodule of R_R.*

PROOF. The condition is necessary by Lemma 5. Conversely, suppose that D is a dense l-submodule of R_R, and let $q \in Q(R)$ with $qD \subseteq \psi(R)$. If $d_1, d_2 \in D^+$, take $d \in D$ with $d_1 R + d_2 R \subseteq dR$. Then $d_i = dr_i$ for some $r_1, r_2 \in R$, and also $d_i = |d|\,|r_i|$ with $|d| \in D$. So $\{dR^+ \mid d \in D\}$ is directed above, and hence R is a $\mathscr{D}f$-ring by the remarks preceding the corollary.

COROLLARY 8. *Suppose that each finitely generated right ideal of the f-ring R is principal. If $\mathscr{D}$ is a positive topology for R, then the following statements are equivalent.*

(a) *R is a $\mathscr{D}f$-ring.*

(b) *R is a $\mathscr{D}f$-module.*

(c) *Each D_q contains a dense l-submodule of R_R.*

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PROOF. It suffices to verify that (b) implies (a), that is, that (a) implies (d) in Theorem 2. So suppose that $q \in Q(R)^+$, that $d_1, d_2 \in D_q^+$ with $d_1 \wedge d_2 = 0$, and that $d_1 R + d_2 R = dR$. Then $d = d_1 a + d_2 b$ and, since $|d_1 a| \wedge |d_2 b| = 0$,

$$|d| = |d_1 a| + |d_2 b| = d_1 |a| + d_2 |b| \in D_q.$$

Since $d_i = dr_i = |d|\,|r_i|$, as in the verification of (d) in Theorem 6,

$$0 \leqslant qd_1 \wedge \psi(d_2) \leqslant (q \vee 1)|d|(|r_1| \wedge |r_2|) = 0.$$

LEMMA 9. *The following statements are equivalent for the right injective right f-ring S.*

(a) *S is an f-ring.*
(b) *If $s \geqslant 1$, then s is invertible.*
(c) *The Jacobson radical $J(S)$ of S is a convex ideal.*
(d) *If A is a nonzero right annihilator of S, then $A^+ \neq 0$.*

PROOF. We know that (a) implies each of (b), (c) and (d). Also, (b) implies (a). For suppose that $x, y, z \in S^+$ and $x \wedge y = 0$. Let $s = z \vee 1$. Since squares are positive in S, $s^{-1} = s(s^{-1})^2 \in S^+$. But then multiplication by s is a lattice homomorphism (Steinberg (1972b), Lemma 1), and so

$$0 \leqslant zx \wedge y \leqslant sx \wedge sy = s(x \wedge y) = 0.$$

Thus S is a left f-ring and hence an f-ring.

(c) implies (b). By Steinberg (1976), Lemma 5, the idempotents of S are central, and since the idempotents of $\bar{S} = S/J(S)$ can be lifted to S, the regular ring $\bar{S}$ is strongly regular by Lambek (1966), p. 102. Since $J(S)$ is convex, $\bar{S}$ is a po-ring. Thus if $s \geqslant 1$ and $\bar{s}\bar{x} = 0$, then $0 = \bar{s}\bar{x}^2 \geqslant \bar{x}^2 \geqslant 0$ yields that $\bar{x} = 0$. So $\bar{s}$, and hence s also, is invertible.

(d) implies (b). If $s \geqslant 1$, then its right annihilator is 0, and, since S is right injective, s has a left inverse which must be two-sided since idempotents are central.

COROLLARY 10. *If R_R is a $\mathscr{D}f$-module and $Q(R)_R$ is injective, then R is a $\mathscr{D}f$-ring if and only if $Q(R)$ satisfies one (and hence all) of the conditions of Lemma 9.*

PROOF. If $Q(R)$ is an injective right R-module, then it is also an injective right $Q(R)$-module by Stenstrom (1971), p. 41.

If R is an f-ring and R_R is a $\mathscr{D}f$-module, then (with some identification) we may assume that

$$R = \{q \in Q(R)\colon \text{multiplication by } |q| \text{ is a lattice homomorphism of } Q\}.$$

So R is a convex l-subring of Q, Q is a two-sided R-f-bimodule, and the largest convex l-subgroup of each D_q is a dense right ideal. Thus $\mathscr{D}$ contains a cofinal subset of right l-ideals. So to show that the right f-ring Q is an f-ring it suffices to do so under these special conditions.

ADDED IN PROOF: With regard to the first paragraph of section 3, a unital right f-ring that is not an f-ring has been constructed.

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University of Toledo
Toledo, Ohio 43606
U.S.A.

Author's current address:
University of Illinois
Urbana, Illinois 61801
U.S.A.