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Symmetric multiple zeta functions*

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Abstract. **In this study, we introduce multiple zeta functions with structures similar to those of symmetric functions such as the Schur P-, Schur Q-, symplectic and orthogonal functions in representation theory. Their basic properties, such as the domain of absolute convergence, are first considered. Then, by restricting ourselves to the truncated multiple zeta functions, we derive the Pfaffian expression of the Schur Q-multiple zeta functions, the sum formula for Schur P- and Schur Q-multiple zeta functions, the determinant expressions of symplectic and orthogonal Schur multiple zeta functions by making an assumption on variables. Finally, we generalize those to the quasi-symmetric functions.**

1 Introduction

The well-known Hall–Littlewood symmetric functions $P_\lambda(\boldsymbol{x};t)$ are a family of symmetric functions that depend on a parameter t:
For $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ being a partition, that is, $\lambda_i \in \mathbb{Z}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$ and $\boldsymbol{x} = (x_1, x_2, \cdots, x_r)$ being variables,

$$P_\lambda(\boldsymbol{x};t) = \frac{1}{v_\lambda(t)} \sum_{\sigma \in \mathfrak{S}_r} \sigma \left(\boldsymbol{x}^\lambda \prod_{1 \leq i < j \leq r} \frac{x_i - t x_j}{x_i - x_j} \right), \tag{1.1}$$

where $v_\lambda(t) = \prod_{j \geq 0} \prod_{k=1}^{m_j} \frac{1-t^k}{1-t}$ with $m_j = \#\{i \mid 1 \leq i \leq r, \lambda_i = j\}$, $\mathfrak{S}_r$ is the symmetric group of degree r, and $\boldsymbol{x}^\lambda = x_1^{\lambda_1} \ldots x_r^{\lambda_r}$. When $t = 0$, the function is the Schur polynomial which we denote by $s_\lambda(\boldsymbol{x}) = P_\lambda(\boldsymbol{x};0)$. Schur polynomials are irreducible general linear characters and can be written combinatorially by means of a semi-standard Young tableau. Mainly in representation theory, much research has been studied on this function since its introduction. One of them is the determinant formula called the Jacobi–Trudi identity, which is proved by the method of lattice path model known as the Lindström–Gessel–Viennot lattice path procedure. When $t = -1$ in (1.1), the function is known as the Schur P-function or the Q-function, expressed as $P_\lambda(\boldsymbol{x}) = P_\lambda(\boldsymbol{x};-1)$ or $Q_\lambda(\boldsymbol{x}) = 2^r P_\lambda(\boldsymbol{x};-1)$, respectively, which was introduced by Schur ([22]). We note that the Schur Q-function was originally defined via certain pfaffian expressions in his analysis of projective representations of symmetric groups. The

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tableau description of Schur Q-functions was introduced by Stembridge ([23]) for using the theory of shifted tableaux developed by Worley ([25]) and Sagan ([20]), and the combinatorial structure of this function was revealed. In his paper [24], Stembridge showed that the tableau definition agrees with Schur's pfaffian expressions by a generalization of the Lindström–Gessel–Viennot lattice path procedure. In parallel with the above theory, symplectic and orthogonal Schur functions, which are irreducible symplectic and orthogonal characters and can also be defined combinatorially, have been developed. It is well-known that a similar discussion such as a determinant formula holds by using an analogue of the Lindström–Gessel–Viennot lattice path procedure. (see Hamel-Goulden [10], Hamel [9] and Foley-King [4]).

The Schur multiple zeta function introduced by Nakasuji, Phuksuwan, and Yamasaki ([17]), is a generalization of both the multiple zeta and zeta-star functions of the Euler-Zagier type with a combinatorial structure similar to a Schur polynomial. Because this function has combinatorial and analytic features, the characteristics of both of these features have been investigated in recent years. Nakasuji, Phuksuwan, and Yamasaki [17] obtained some determinant formulas such as the Jacobi–Trudi, Giambelli, and dual Cauchy formulas for Schur multiple zeta functions by using the Lindström–Gessel–Viennot lattice path procedure and the properties of Young tableaux. These type formulas gave a new type of identities among the multiple zeta-functions of the Euler-Zagier type. Therefore, it is natural to ask whether we can define multiple zeta functions with structures similar to those of symmetric functions such as the Schur P- or Q-functions, symplectic or orthogonal functions. In this study, we focus on this point.

Remark 1.1 The term *symmetric multiple zeta function* has been previously defined by Kaneko and Zagier in [12, 13]. While their definition differs from the one we propose in this paper, we adopt the same terminology to highlight the symmetric structures inherent in our definition.

In Section 2, for $\boldsymbol{s} = (s_{ij}) \in ST(\lambda, \mathbb{C})$ being the set of all shifted tableaux of shape λ over $\mathbb{C}$, we introduce the *Schur P-multiple zeta functions* and the *Schur Q-multiple zeta functions* of shape λ as the following series

$$\zeta_\lambda^P(\boldsymbol{s}) = \sum_{M \in PSST(\lambda)} \frac{1}{M^{\boldsymbol{s}}}, \quad \text{and} \quad \zeta_\lambda^Q(\boldsymbol{s}) = \sum_{M \in QSST(\lambda)} \frac{1}{M^{\boldsymbol{s}}},$$

respectively, where $PSST(\lambda)$ and $QSST(\lambda)$ are the sets of semi-standard marked shifted tableaux of shape λ satisfying certain conditions (see the detail in Section 2), and discuss their basic properties such as the domain of convergence. In Section 3, we consider the pfaffian expression of the (truncated) Schur Q-multiple zeta functions by following Stembridge's way ([23]). Here, the truncated Schur Q-multiple zeta function is

$$\zeta_\lambda^{Q,N}(\boldsymbol{s}) = \sum_{M \in QSST_N(\lambda)} \frac{1}{M^{\boldsymbol{s}}}$$

for a fixed positive integer $N \in \mathbb{N}$, where $QSST_N(\lambda)$ are the sets of all $(m_{ij}) \in QSST(\lambda)$ such that $m_{ij} \leq N$ for all i, j. In Section 4, we demonstrate that the pfaffian expression, obtained in Section 3, can be easily generalized to the skew type. In Section

5, after reviewing the outside decomposition of the shifted Young diagram according to Hamel-Goulden [10], we apply it to our skew type Schur Q-multiple zeta functions and derive the pfaffian expressions associated with that decomposition. In Section 6, we discuss the sum formula for our Schur P- and Q-multiple zeta functions. Sections 7, 8, and 9 are devoted to discussions of symplectic and orthogonal Schur multiple zeta functions, which are defined as follows. For a positive integer N and $\boldsymbol{s} = (s_{ij}) \in T(\lambda, \mathbb{C})$ being the set of all Young tableaux of shape λ over $\mathbb{C}$, we define the *symplectic Schur multiple zeta functions* and the *orthogonal Schur multiple zeta functions* of shape λ as the following series

$$\zeta_\lambda^{\mathrm{sp},N}(\boldsymbol{s}) = \sum_{M \in SP_N(\lambda)} \frac{1}{M^{\boldsymbol{s}}}, \quad \text{and} \quad \zeta_\lambda^{\mathrm{so},N}(\boldsymbol{s}) = \sum_{M \in SO_N(\lambda)} \frac{1}{M^{\boldsymbol{s}}},$$

respectively, where $SP_N(\lambda)$ and $SO_N(\lambda)$ are the sets of all symplectic tableaux and so-tableaux of shape λ (see the detail in Section 7 and 8). We construct directed graphs corresponding to these functions as analogous to the original symplectic and orthogonal Schur functions attributed to Hamel ([9]) and provide the determinant expressions in a manner similar to that of Hamel where we apply the Stembridge Theorem [24]. Further, we provide their decomposition into a sum of truncated multiple zeta or zeta-star functions. Lastly, in Section 10, we study the extension of all of these functions to quasi-symmetric functions. We derive the pfaffian expressions for Schur Q-type quasi-symmetric functions and determinant expressions for symplectic type and orthogonal type quasi-symmetric functions.

2 Basic properties of the Schur P- and Q-multiple zeta functions

We first review the basic terminology to define Schur P- and Q-multiple zeta functions. A partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is termed strict, if $\lambda_1 > \lambda_2 > \cdots > \lambda_r \geq 0$. Then, we associate the strict partition λ with *the shifted diagram*

$$SD(\lambda) = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq r, i \leq j \leq \lambda_i + i - 1\}$$

depicted as a collection of square boxes where the i-th row has λ_i boxes. We say that $(i, j) \in SD(\lambda)$ is a corner of λ if $(i+1, j) \notin SD(\lambda)$ and $(i, j+1) \notin SD(\lambda)$ and denote by $SC(\lambda) \subset SD(\lambda)$ the set of all corners of λ; for example, $SC((4, 2, 1)) = \{(1, 4), (3, 3)\}$. For a strict partition λ, a shifted tableau (t_{ij}) of shape λ over a set X is a filling of $SD(\lambda)$ with $t_{ij} \in X$ into the (i, j) box of $SD(\lambda)$. We denote by $ST(\lambda, X)$ the set of all shifted tableaux of shape λ over X.

***Definition* 2.1** (semi-standard marked shifted tableau) Let $\mathbb{N}'$ be the set $\{1', 1, 2', 2, \ldots\}$ with the total ordering $1' < 1 < 2' < 2 < \cdots$. Then, a *semi-standard marked shifted tableau* $\boldsymbol{t} = (t_{ij}) \in ST(\lambda, \mathbb{N}')$ is obtained by numbering all the boxes of $SD(\lambda)$ with numbers from $\mathbb{N}'$ such that

PST1 the entries of $\boldsymbol{t}$ are weakly increasing along each column and row of $\boldsymbol{t}$,
PST2 for each $i = 1, 2, \ldots$, there is at most one i' per row,
PST3 for each $i = 1, 2, \ldots$, there is at most one i per column,
PST4 there is no i' on the main diagonal.

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We denote by $PSST(\lambda)$ the set of semi-standard marked shifted tableaux of shape λ. Similarly, we denote by $QSST(\lambda)$ the set of semi-standard marked shifted tableaux of shape λ without the diagonal condition PST4.

***Definition* 2.2** (Schur P-multiple zeta functions) For a given set $\boldsymbol{s} = (s_{ij}) \in ST(\lambda, \mathbb{C})$ of variables, the *Schur P-multiple zeta functions* of shape λ are defined as

$$\zeta_\lambda^P(\boldsymbol{s}) = \sum_{M \in PSST(\lambda)} \frac{1}{M^{\boldsymbol{s}}}, \tag{2.1}$$

where $M^{\boldsymbol{s}} = \prod_{(i,j) \in SD(\lambda)} |m_{ij}|^{s_{ij}}$ for $M = (m_{ij}) \in PSST(\lambda)$ and $|i| = |i'| = i$.

For example, when $\lambda = (6, 5, 3, 1)$,

$$M = \begin{array}{cccccc} \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2'} & \boxed{3'} & \boxed{4'} \\ & \boxed{2} & \boxed{2} & \boxed{3} & \boxed{4'} & \boxed{5'} \\ & & \boxed{3} & \boxed{4'} & \boxed{4} & \\ & & & \boxed{5} & & \end{array} \in PSST(\lambda),$$

and

$$\frac{1}{M^{\boldsymbol{s}}} = \frac{1}{1^{s_{11}} 1^{s_{12}} 1^{s_{13}} 2^{s_{14}} 3^{s_{15}} 4^{s_{16}} 2^{s_{22}} 2^{s_{23}} 3^{s_{24}} 4^{s_{25}} 5^{s_{26}} 3^{s_{33}} 4^{s_{34}} 4^{s_{35}} 5^{s_{44}}}.$$

Similarly, We define the Schur Q-multiple zeta functions.

***Definition* 2.3** (Schur Q-multiple zeta functions) For a given set $\boldsymbol{s} = (s_{ij}) \in ST(\lambda, \mathbb{C})$ of variables, the *Schur Q-multiple zeta functions* of shape λ are defined to be

$$\zeta_\lambda^Q(\boldsymbol{s}) = \sum_{M \in QSST(\lambda)} \frac{1}{M^{\boldsymbol{s}}}, \tag{2.2}$$

where $M^{\boldsymbol{s}} = \prod_{(i,j) \in SD(\lambda)} |m_{ij}|^{s_{ij}}$ for $M = (m_{ij}) \in QSST(\lambda)$ and $|i| = |i'| = i$.

For a strict partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, by the definitions of ζ_λ^P and ζ_λ^Q, we can allow the main diagonal entries in tableaux $M \in QSST(\lambda)$ to be marked and obtain

$$\zeta_\lambda^Q(\boldsymbol{s}) = 2^r \zeta_\lambda^P(\boldsymbol{s}). \tag{2.3}$$

As in the Introduction, we define the truncated P- and Q-multiple zeta functions:

For a fixed positive integer $N \in \mathbb{N}$, let $PSST_N(\lambda)$ and $QSST_N(\lambda)$ be the sets of all $(m_{ij}) \in PSST(\lambda)$ and $QSST(\lambda)$ such that $m_{ij} \le N$ for all i, j. Then, we define

$$\zeta_\lambda^{P,N}(\boldsymbol{s}) = \sum_{M \in PSST_N(\lambda)} \frac{1}{M^{\boldsymbol{s}}}, \text{ and } \zeta_\lambda^{Q,N}(\boldsymbol{s}) = \sum_{M \in QSST_N(\lambda)} \frac{1}{M^{\boldsymbol{s}}}.$$

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In this section, we prove some basic properties of the Schur P- and Q-multiple zeta functions. We first consider the domain of absolute convergence of the series (2.1) and (2.2).

***Lemma* 2.1** *Let*

$$W_\lambda^Q = \left\{ \boldsymbol{s} = (s_{ij}) \in ST(\lambda, \mathbb{C}) \;\middle|\; \begin{array}{l} Re(s_{ij}) \geq 1 \text{ for all } (i,j) \in SD(\lambda) \setminus SC(\lambda) \\ Re(s_{ij}) > 1 \text{ for all } (i,j) \in SC(\lambda) \end{array} \right\}.$$

Then, the series (2.1) *and* (2.2) *converge absolutely if* $\boldsymbol{s} \in W_\lambda^Q$.

Proof By (2.3), it suffices to consider ζ_λ^Q. Let λ be a strict partition and $SC(\lambda) = \{(i_1, j_1), \dots, (i_k, j_k)\}$ where $i_1 < \cdots < i_k$ and $j_1 > \cdots > j_k$. Let $i_0 = 0$. Because

$$\left| \sum_{\substack{M \in QSST(\lambda) \\ m_{ij} \leq N}} \frac{1}{M^{\boldsymbol{s}}} \right| \leq \prod_{\ell=1}^{k} \sum_{\substack{M \in QSST(\lambda_\ell) \\ m_{ij} \leq N}} \prod_{(i,j) \in SD(\lambda_\ell)} \frac{1}{|m_{ij}|^{\mathrm{Re}(t_{ij,\ell})}},$$

where $\lambda_\ell = (j_\ell - i_{\ell-1}, j_\ell - i_{\ell-1} - 1, \dots, j_\ell - i_\ell + 1)$ and $t_{ij,\ell} = s_{i+i_{\ell-1}, j+i_{\ell-1}}$, we prove that for $\lambda = (\lambda_1, \dots, \lambda_r) := (\lambda_1, \lambda_1 - 1, \dots, \lambda_1 - r + 1)$,

$$\sum_{\substack{M \in QSST(\lambda) \\ m_{ij} \leq N}} \prod_{(i,j) \in SD(\lambda)} \frac{1}{|m_{ij}|^{\mathrm{Re}(s_{ij})}} \tag{2.4}$$

converges absolutely for $\boldsymbol{s} \in W_\lambda^Q$ as $N \to \infty$. Rearranging the order of summation, we have

$$\sum_{\substack{M \in QSST(\lambda) \\ m_{ij} \leq N}} \prod_{(i,j) \in SD(\lambda)} \frac{1}{|m_{ij}|^{\mathrm{Re}(s_{ij})}} = \sum_{N_1=1}^{N} \left(\sum_{\substack{(m_{ij}) \in QSST(\lambda) \\ m_{r\lambda_r} = N_1}} \prod_{\substack{(i,j) \in SD(\lambda) \\ (i,j) \neq (r,\lambda_r)}} \frac{1}{|m_{ij}|^{\mathrm{Re}(s_{ij})}} \right) \frac{1}{N_1^{\mathrm{Re}(s_{r\lambda_r})}}.$$

By extending the region of summation and product, it holds that

$$\sum_{\substack{M \in QSST(\lambda) \\ m_{ij} \leq N}} \prod_{(i,j) \in SD(\lambda)} \frac{1}{|m_{ij}|^{\mathrm{Re}(s_{ij})}} \leq 2^{r\lambda_r} \sum_{N_1=1}^{N} \left(\prod_{\substack{i=1 \\ (i,j) \neq (r,\lambda_r)}}^{r} \prod_{j=1}^{\lambda_r} \sum_{m_{ij}=1}^{N_1} \frac{1}{m_{ij}} \right) \frac{1}{N_1^{\mathrm{Re}(s_{r\lambda_r})}}.$$

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Because for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 1$ such that

$$\sum_{m_{ij}=1}^{N} \frac{1}{m_{ij}} < \frac{C_\varepsilon}{2} N^\varepsilon,$$

we can estimate that

$$\sum_{\substack{M \in QSST(\lambda) \\ m_{ij} \leq N}} \prod_{(i,j) \in SD(\lambda)} \frac{1}{|m_{ij}|^{\mathrm{Re}(s_{ij})}} \leq C_\varepsilon^{r\lambda_r} \sum_{N_1=1}^{N} \frac{N_1^{\varepsilon r \lambda_r}}{N_1^{\mathrm{Re}(s_{r\lambda_r})}}.$$

We can choose a sufficiently small ε such that $\mathrm{Re}(s_{r\lambda_r}) - \varepsilon r \lambda_r > 1$. Thus, (2.4) converges absolutely and we obtain the lemma. ∎

We next show that a Schur Q-multiple zeta function can be written as a linear combination of the multiple zeta (star) functions as well as the Schur multiple zeta functions. Indeed, for a strict partition λ of n, let $\mathcal{SF}(\lambda)$ be the set of all bijections $f : SD(\lambda) \to \{1, 2, \ldots, n\}$ satisfying the following two conditions:

(i) for all i, $f((i, j_1)) < f((i, j_2))$ if and only if $j_1 < j_2$,
(ii) for all j, $f((i_1, j)) < f((i_2, j))$ if and only if $i_1 < i_2$.

For $\boldsymbol{s} = (s_{ij}) \in ST(\lambda, \mathbb{C})$, put

$$V(\boldsymbol{s}) = \left\{ \left(s_{f^{-1}(1)}, s_{f^{-1}(2)}, \ldots, s_{f^{-1}(n)}\right) \in \mathbb{C}^n \;\middle|\; f \in \mathcal{SF}(\lambda) \right\}.$$

We write $\boldsymbol{t} \preceq_s \boldsymbol{s}$ for $\boldsymbol{t} = (t_1, t_2, \ldots, t_m) \in \mathbb{C}^m$ if there exists $(v_1, v_2, \ldots, v_n) \in V(\boldsymbol{s})$ satisfying the following: for all $1 \leq k \leq m$, there exist $1 \leq h_k \leq m$ and $l_k \geq 0$ such that

(i) $t_k = v_{h_k} + v_{h_k+1} + \cdots + v_{h_k+l_k}$,
(ii) there are no $(i_1, i_2; j_1, j_2)$ with $i_1 < i_2$ and $j_1 < j_2$ such that $\{s_{i_1 j_1}, s_{i_1 j_2}, s_{i_2 j_2}\} \subset \{v_{h_k}, v_{h_k+1}, \ldots, v_{h_k+l_k}\}$, and
(iii) $\bigsqcup_{k=1}^{m} \{h_k, h_k+1, \ldots, h_k+l_k\} = \{1, 2, \ldots, n\}$ (disjoint union).

Here, we note that since $|m_{ij}| = |m'_{ij}| = m_{ij}$ for any positive integer m_{ij} in (2.2), the definition of ζ_λ^Q, we have

$$\zeta_\lambda^Q(\boldsymbol{s}) = \sum_{\boldsymbol{t} \preceq_s \boldsymbol{s}} 2^{m(\boldsymbol{t})} \zeta(\boldsymbol{t}), \tag{2.5}$$

where $m(\boldsymbol{t})$ is a positive integer that depends on the way in which the comma , is changed to the plus + sign. Moreover, by an Inclusion-Exclusion principle, one can also obtain its "dual" expression

$$\zeta_\lambda^Q(\boldsymbol{s}) = \sum_{\boldsymbol{t} \preceq_s \boldsymbol{s}} (-1)^{n-\mathrm{dep}(\boldsymbol{t})} 2^{m(\boldsymbol{t})} \zeta^\star(\boldsymbol{t}), \tag{2.6}$$

where dep is the number of variables. Combining (2.5) and (2.6) with identity (2.3), we can decompose the Schur P-multiple zeta function into a linear combination of multiple zeta (star) functions defined by

$$\zeta(s_1,\dots,s_r)=\sum_{1\le n_1<\cdots<n_r}\frac{1}{n_1^{s_1}\cdots n_r^{s_r}},\quad \zeta^{\star}(s_1,\dots,s_r)=\sum_{1\le n_1\le\cdots\le n_r}\frac{1}{n_1^{s_1}\cdots n_r^{s_r}}.$$

Example 2.2 For $\boldsymbol{s}=(s_{ij})\in ST((3,1),\mathbb{C})$, we have

$$V(\boldsymbol{s})=\{(s_{11},s_{12},s_{13},s_{22}),(s_{11},s_{12},s_{22},s_{13})\}.$$

One can confirm that $\boldsymbol{t}\preceq_s \boldsymbol{s}$ if and only if $\boldsymbol{t}$ is one of the following:

$$\begin{aligned}&(s_{11},s_{12},s_{13},s_{22}),(s_{11}+s_{12},s_{13},s_{22}),(s_{11},s_{12}+s_{13},s_{22}),(s_{11},s_{12},s_{13}+s_{22}),\\&(s_{11}+s_{12}+s_{13},s_{22}),(s_{11}+s_{12},s_{13}+s_{22}),(s_{11},s_{12}+s_{13}+s_{22}),\\&(s_{11},s_{12},s_{22},s_{13}),(s_{11}+s_{12},s_{22},s_{13}),(s_{11},s_{12}+s_{22},s_{13}).\end{aligned}$$

This shows that when $\boldsymbol{s}\in W^{Q}_{(3,1)}$

$$\begin{aligned}&\zeta^{Q}_{(3,1)}\left(\begin{array}{|c|c|c|}\hline s_{11}&s_{12}&s_{13}\\\hline \multicolumn{1}{c|}{}&s_{22}&\multicolumn{1}{c}{}\\\cline{2-2}\end{array}\right)\\
&=16\zeta(s_{11},s_{12},s_{13},s_{22})+8\zeta(s_{11}+s_{12},s_{13},s_{22})+8\zeta(s_{11},s_{12}+s_{13},s_{22})\\
&\quad+16\zeta(s_{11},s_{12},s_{13}+s_{22})+4\zeta(s_{11}+s_{12}+s_{13},s_{22})\\
&\quad+8\zeta(s_{11}+s_{12},s_{13}+s_{22})+4\zeta(s_{11},s_{12}+s_{13}+s_{22})\\
&\quad+16\zeta(s_{11},s_{12},s_{22},s_{13})+8\zeta(s_{11}+s_{12},s_{22},s_{13})+8\zeta(s_{11},s_{12}+s_{22},s_{13})\\
&=16\zeta^{\star}(s_{11},s_{12},s_{13},s_{22})-8\zeta^{\star}(s_{11}+s_{12},s_{13},s_{22})-8\zeta^{\star}(s_{11},s_{12}+s_{13},s_{22})\\
&\quad-16\zeta^{\star}(s_{11},s_{12},s_{13}+s_{22})+4\zeta^{\star}(s_{11}+s_{12}+s_{13},s_{22})\\
&\quad+8\zeta^{\star}(s_{11}+s_{12},s_{13}+s_{22})+4\zeta^{\star}(s_{11},s_{12}+s_{13}+s_{22})\\
&\quad+16\zeta^{\star}(s_{11},s_{12},s_{22},s_{13})-8\zeta^{\star}(s_{11}+s_{12},s_{22},s_{13})-8\zeta^{\star}(s_{11},s_{12}+s_{22},s_{13}).\end{aligned}$$

Example 2.3 It holds that

$$\begin{aligned}\zeta^{Q}_{(r)}\left(\begin{array}{|c|c|c|}\hline s_{11}&\cdots&s_{1r}\\\hline\end{array}\right)&=\sum_{\boldsymbol{\ell}}2^{\mathrm{dep}(\boldsymbol{\ell})}\zeta(\boldsymbol{\ell}),\\
\zeta^{Q}_{(r)}\left(\begin{array}{|c|c|c|}\hline s_{11}&\cdots&s_{1r}\\\hline\end{array}\right)&=\sum_{\boldsymbol{\ell}}(-1)^{r-\mathrm{dep}(\boldsymbol{\ell})}2^{\mathrm{dep}(\boldsymbol{\ell})}\zeta^{\star}(\boldsymbol{\ell}),\end{aligned}$$

where $\boldsymbol{\ell}$ runs over all indices of the form $\boldsymbol{\ell}=(s_{11}\square s_{12}\square\cdots\square s_{1r})$ in which each $\square$ is filled by a comma , or a plus + sign.

By (2.3), the Schur P-multiple zeta functions can be similarly decomposed into a linear combination of multiple zeta (star) functions.

We next provide a short observation for a relation between Schur Q-multiple zeta values and the Two-One formula conjectured by Ohno and Zudilin [19], and proved by Zhao [28].

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***Theorem* 2.4** (*Two-One formula* [19, 28]) *For a non-negative integer k, we denote $\mu_{2k+1} = (1, \{2\}^k)$. Then for any admissible index $\boldsymbol{k} = (k_1, \dots, k_r)$ with odd entries $k_1, \dots, k_r$, the following identities are valid:*

$$\zeta^\star(\mu_{k_1}, \dots, \mu_{k_r}) = \sum_{\boldsymbol{\ell} \preceq \boldsymbol{k}} 2^{\mathrm{dep}(\boldsymbol{\ell})} \zeta(\boldsymbol{\ell}),$$
$$= \sum_{\boldsymbol{\ell} \preceq \boldsymbol{k}} (-1)^{r-\mathrm{dep}(\boldsymbol{\ell})} 2^{\mathrm{dep}(\boldsymbol{\ell})} \zeta^\star(\boldsymbol{\ell}),$$

where the sum $\sum_{\boldsymbol{\ell} \preceq \boldsymbol{k}}$ extends over all indices of the form $\boldsymbol{\ell} = (k_1 \square k_2 \square \cdots \square k_r)$ in which each $\square$ is filled by the comma , or the plus + sign.

Combining Theorem 2.4 with Lemma 2.3, we have the following theorem.

***Theorem* 2.5** *For r-tuple $(k_1, \dots, k_r)$ of positive odd integers with $k_r \geq 3$,*

$$\zeta^Q_{(r)}\left(\boxed{k_1 \,|\, k_2 \,|\, \cdots \,|\, k_r}\right) = \zeta^\star(\mu_{k_1}, \dots, \mu_{k_r})$$
$$= \zeta^\star(1, \{2\}^{\frac{k_1-1}{2}}, \dots, 1, \{2\}^{\frac{k_r-1}{2}}).$$

This theorem contributes a non-trivial identity between a single Schur Q-multiple zeta value and multiple zeta star value.

***Corollary* 2.6** *For a positive integer $k \geq 4$*

$$\zeta^Q_{(k-2)}\left(\boxed{1 \,|\, \cdots \,|\, 1 \,|\, 3}\right) = \frac{k-1}{2} \zeta^Q_{(1)}\left(\boxed{k}\right).$$

Proof By Theorem 2.5 and the sum formula (Theorem 6.1), we have

$$\zeta^Q_{(k-2)}\left(\boxed{1 \,|\, \cdots \,|\, 1 \,|\, 3}\right) = \zeta^\star(\{1\}^{k-2}, 2) = (k-1)\zeta(k).$$

By (2.5), we have $\zeta^Q_{(1)}\left(\boxed{k}\right) = 2\zeta(k)$ and

$$\zeta^Q_{(k-2)}\left(\boxed{1 \,|\, \cdots \,|\, 1 \,|\, 3}\right) = \frac{k-1}{2} \zeta^Q_{(1)}\left(\boxed{k}\right).$$

■

3 Pfaffian expression of the Schur Q-multiple zeta functions

The original Schur Q-polynomial is known to have a pfaffian expression [15]. In this section, we provide a pfaffian expression of the Schur Q-multiple zeta function by following the Stembridge approach [24]. We first recall the definition of a pfaffian. Let $\mathfrak{S}_n$ be the symmetric group of degree n. Then, for a given square matrix $A = (a_{ij})_{1 \leq i, j \leq n}$,

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the determinant $\det(A)$ is defined by

$$\det(A) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)},$$

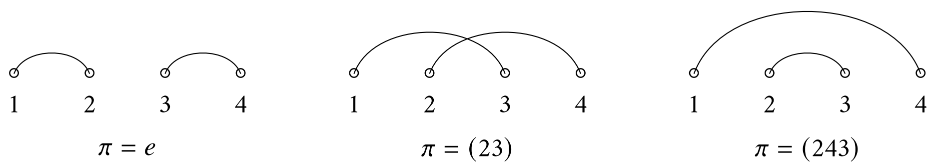
where $\operatorname{sgn}(\sigma)$ is the signature of σ.

We derive the pfaffian by defining a set $\mathfrak{F}_{2n}$, a subset of the symmetric group $\mathfrak{S}_{2n}$ of an even degree,

$$\mathfrak{F}_{2n} = \left\{ \pi \in \mathfrak{S}_{2n} \;\middle|\; \begin{array}{c} \pi(1) < \pi(3) < \cdots < \pi(2n-1), \\ \pi(1) < \pi(2), \pi(3) < \pi(4), \ldots, \pi(2n-1) < \pi(2n) \end{array} \right\}.$$

For an ordered $2n$-tuple $\boldsymbol{v} = (v_1, \ldots, v_{2n})$ of vertices, we say that a set of edges $\pi = \{((v_i, v_j), \ldots, (v_k, v_l))\}$ on $\boldsymbol{v}$ is a 1-factor if each v_i is incident with exactly one edge.

Example 3.1 The following are 1-factors of $\{1, 2, 3, 4\}$.



By convention, we always list the edges of a 1-factor π in the form (v_i, v_j) with $i < j$. It is known that a bijection can be constructed from $\mathfrak{F}_{2n}$ to the set of 1-factors by $\pi \mapsto \{(v_{\pi(1)}, v_{\pi(2)}), \ldots, (v_{\pi(2n-1)}, v_{\pi(2n)})\}$, and

$$|\mathfrak{F}_{2n}| = \frac{(2n)!}{2^n n!}.$$

Then, for a given $2n \times 2n$ upper triangular or anti-symmetric matrix $A = (a_{ij})_{1 \le i,j \le 2n}$, the pfaffian $\operatorname{pf}(A)$ of A is defined by

$$\operatorname{pf}(A) = \sum_{\pi \in \mathfrak{F}_{2n}} \operatorname{sgn}(\pi) \prod_{i=1}^{n} a_{\pi(2i-1),\pi(2i)}.$$

Let $D = (V, E)$ be a directed graph with vertices V and edges E, with the assignment of a direction to each edge with no directed cycles. Multiple edges are allowed. We denote by $u \to v$ an edge directed from u to v. For any pair of vertices u, v, we denote by $\mathscr{P}(u, v)$ the set of directed D-paths from u to v on D. If $u = u$, then $\mathscr{P}(u, u)$ is a set of a single path of length zero.

Let I and J be ordered sets of vertices of D. Then I is said to be D-compatible with J if, whenever $u < u'$ in I and $v > v'$ in J, every path $P \in \mathscr{P}(u, v)$ intersects every path $Q \in \mathscr{P}(u', v')$. Here, if two paths have a common vertex, we say that they intersect.

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For any vertex $u \in V$ and subset $I \subset V$, let $\mathscr{P}(u; I)$ denote the set of directed paths from u to any $v \in I$, and let

$$Q_I(u) = \sum_{P \in \mathscr{P}(u;I)} w(P),$$

where w is a particular weight function defined on edges. For any r-tuple $\boldsymbol{u} = (u_1, \ldots, u_r)$ of vertices, let $\mathscr{P}(\boldsymbol{u}; I)$ be the set of r-tuples of paths $P_i \in \mathscr{P}(u_i; I)$. The weight function w is extended to tuples of paths by

$$w(P_1, \ldots, P_r) = \prod_{i=1}^{r} w(P_i).$$

Then we define

$$Q_I(u_1, \ldots, u_r) = \sum_{(P_1,\ldots,P_r) \in \mathscr{P}(\boldsymbol{u};I)} w(P_1, \ldots, P_r).$$

Theorem* 3.2 *([24, Theorem 3.1]) *Let $\boldsymbol{u} = (u_1, \ldots, u_r)$ be an r-tuple of vertices in a directed acyclic graph D, and assume that r is even. If $I \subset V$ is a totally ordered subset of the vertices such that u is D-compatible with I, then*

$$Q_I(\boldsymbol{u}) = \mathrm{pf}(Q_I(u_i, u_j))_{1 \le i < j \le r}.$$

Remark 3.3 ([24]) In the case of r being odd, we may adjoin a phantom vertex u_{r+1} to V, with no incident edges, and include u_{r+1} in I. We order all other vertices of I before u_{r+1} and replace r by $r + 1$.

Stembridge constructed a directed graph D corresponding to the Schur Q-functions [24]. Moreover, Stembridge applied Theorem 3.2 to obtain the following pfaffian expression of the Schur Q-polynomial.

Theorem* 3.4 *([24, Theorem 6.1]) *Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a strict partition of even length. Then*

$$Q_\lambda = \mathrm{pf}(Q_{(\lambda_i, \lambda_j)})_{1 \le i < j \le r}.$$

Following the Stembridge approach, we construct a directed graph D corresponding to the Schur Q-multiple zeta functions. We begin with the vertex set of pairs of non-negative integers, and direct an edge $u \to v$ whenever $u - v = (1, 0), (0, 1)$, or $(1, 1)$. Subsequently, we delete the edges $u \to v$ that contain points whose first coordinates are both zero, as well as those of which the second coordinates are both zero. Finally, we divide each of the vertices $(0, j)$ with $j > 1$ into two vertices, say $(0, j)$ and $(0, j + 1)'$, such that the edge $(1, j + 1) \to (0, j)$ is redirected to $(0, j + 1)'$, whereas the edge $(1, j) \to (0, j)$ remains intact. Fix a positive integer N and a partition λ, and let $\boldsymbol{u} = (u_1, \ldots, u_r)$ be the r-tuple of vertices with $u_i = (\lambda_i, N)$. Without loss of generality, we may assume that r is even (if r is odd, set $\lambda_{r+1} = 0$ and $u_{r+1} = (0, N + 1)'$, and replace r with $r + 1$). Let $I_N = \{(0, 0), (0, 1), (0, 2)', (0, 2), \ldots, (0, N)', (0, N), (0, N + 1)'\}$.

For any vertex $u \in V$, let $\mathscr{P}_0(u; I)$ be the set of non-intersecting path $P \in \mathscr{P}(u; I)$. For any r-tuple $\boldsymbol{u} = (u_1, \ldots, u_r)$ of vertices, let $\mathscr{P}_0(\boldsymbol{u}; I)$ be the set of non-intersecting r-tuples of paths $P_i \in \mathscr{P}_0(u_i; I)$. Then, an element in $QSST_N(\lambda)$ can be identified with a tuple of non-intersecting paths in $\mathscr{P}_0(\boldsymbol{u}; I_N)$, and $\boldsymbol{u}$ is D-compatible with I_N.

Let $v_i(P) = (v_{i,j}(P))_{j \geq 0}$ be the sequence of vertices representing a path $P \in \mathscr{P}_0(u_i; I_N)$ and let ℓ^i_{xy} be the edge $v_{i,j}(= (x, y)) \to v_{i,j+1}$. If $v_{i,j}(P) - v_{i,j+1}(P) = (1, 0)$ or $(1, 1)$, we assign the weight $w(\ell^i_{xy}) = y^{-s_{i,x+i-1}}$. If $v_{i,j}(P) - v_{i,j+1}(P) = (0, 1)$, we assign the weight $w(\ell^i_{xy}) = 1$. Here, we put $(1, y) - (0, y)' = (1, 1)$ for any positive integer y. Then, we define

$$w(P_i) = \prod_{\ell^i_{xy}} w(\ell^i_{xy})$$

(see Example 3.5), and for $(P_1, \ldots, P_r) \in \mathscr{P}(\boldsymbol{u}; I_N)$,

$$w(P_1, \ldots, P_r) = \prod_{i=1}^{r} w(P_i).$$

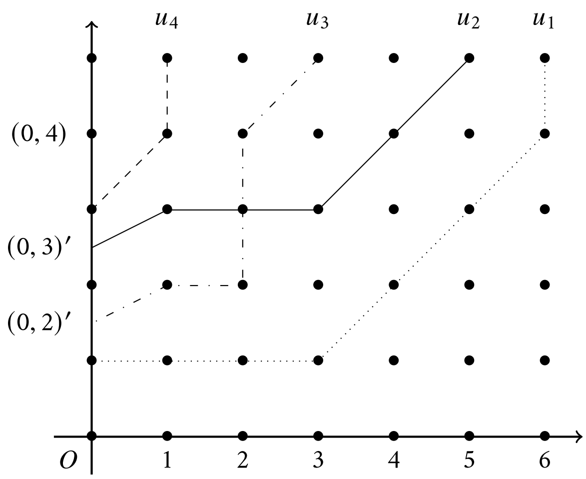
Then, according to the above discussion, we find that

$$\zeta^{Q,N}_\lambda(\boldsymbol{s}) = \sum_{(P_1,\ldots,P_r) \in \mathscr{P}_0(\boldsymbol{u}; I_N)} w(P_1, \ldots, P_r).$$

For a set X, we define

$$ST^{\mathrm{diag}}(\lambda, X) = \{(t_{ij}) \in W^Q_\lambda \mid t_{ij} = t_{1k} \text{ if } j - i = k - 1 \text{ for any } k\}.$$

Example 3.5 Let $\lambda = (6, 5, 3, 1)$ and $N = 5$. Then, Figure 1 is a 4-tuple of paths $(P_1, P_2, P_3, P_4) \in \mathscr{P}(\{u_1, u_2\}; I_N) \oplus \mathscr{P}(\{u_3, u_4\}; I_N)$. Let $(s_{ij}) \in ST(\lambda, \mathbb{C})$. The



1	1	1	2′	3′	4′
	3′	3	3	4′	5′
		2′	2	5′	
			4′		

Figure 1: (P_1, P_2, P_3, P_4) satisfying the condition in Example 3.5.

weights $w(P_i)$ are

$$w(P_1) = \frac{1}{1^{s_{11}} 1^{s_{12}} 1^{s_{13}} 2^{s_{14}} 3^{s_{15}} 4^{s_{16}}}, \qquad w(P_2) = \frac{1}{3^{s_{22}} 3^{s_{23}} 3^{s_{24}} 4^{s_{25}} 5^{s_{26}}}$$
$$w(P_3) = \frac{1}{2^{s_{33}} 2^{s_{34}} 5^{s_{35}}}, \qquad w(P_4) = \frac{1}{4^{s_{44}}}.$$

***Theorem* 3.6 (*Pfaffian expression of the Schur Q-multiple zeta functions*)** *Let r be an even positive integer. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a strict partition with $\lambda_i \geq 0$. Then for $\boldsymbol{s} \in ST^{\mathrm{diag}}(\lambda, \mathbb{C})$,*

$$\zeta_\lambda^Q(\boldsymbol{s}) = \mathrm{pf}(M_\lambda),$$

where $M_\lambda = (a_{ij})$ is an $r \times r$ upper triangular matrix with

$$a_{ij} = \begin{cases} \zeta^Q_{(\lambda_i,\lambda_j)}(\boldsymbol{s}_{(\lambda_i,\lambda_j)}) & \text{for } i < j, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\boldsymbol{s}_{(\lambda_i,\lambda_j)} = \begin{array}{|c|c|c|c|c|c|} \hline s_{ii} & \cdots & \cdots & \cdots & \cdots & s_{it_i} \\ \hline \multicolumn{1}{c|}{} & s_{jj} & \cdots & s_{jt_j} & \multicolumn{2}{c}{} \\ \cline{2-4} \end{array},$$

where $t_i = i + \lambda_i - 1$.

Proof We can prove this by following Stembridge's method (see [24, Theorem 3.1]). Indeed, a similar discussion is proceeded in terms of appropriate weight corresponding to multiple zeta functions:

By the definition of pfaffian,

$$\mathrm{pf}(M_\lambda) = \sum_{\pi \in \mathfrak{F}_n} \mathrm{sgn}(\pi) \prod_{(i,j)\in\pi} \zeta^Q_{(\lambda_i,\lambda_j)}(\boldsymbol{s}_{(\lambda_i,\lambda_j)}). \tag{3.1}$$

It suffices to show that there exists a sign-reversing summand for each summand resulting from $(P_1, \dots, P_r)$ with at least one pair of intersecting paths.

We consider the right-most intersection point (p, q) appearing in paths $(P_1, \dots, P_r)$ for a 1-factor π. For the sake of simplicity, we can assume that for the 1-factor π the two paths P_1 and P_2 intersect at (p, q) (Figure 2). Then, the paths $(P_1, \dots, P_r)$ give rise to

$$S(\pi) = \mathrm{sgn}(\pi) \prod_{j=1}^{t_1} a_{1j}^{-s_{1j}} \prod_{j=2}^{t_2} a_{2j}^{-s_{2j}} \prod_{i=3}^{r} \left(\prod_{j=i}^{t_i} a_{ij}^{-s_{ij}} \right),$$

where a_{ij} is the y-coordinate of the corresponding element of $v^w_{ij}(P_i)$. On the other hand, we consider the r-tuple of paths $(\overline{P}_1, \overline{P}_2, P_3, \dots, P_r)$. Here, $\overline{P}_i$ follows P_i until it meets the first intersection point (p, q), whereupon it follows the other path P_j to the end (Figure 3).

Let $\overline{\pi}$ be the 1-factor obtained by interchanging 1 and 2. Here, it is necessary to verify that for each 1-factor $(i, j) \in \overline{\pi}$, the paths P_i and P_j do not intersect. It suffices to consider the cases involving the modified paths $\overline{P}_i$ and $\overline{P}_j$. The definition of v implies that points of intersection other than v do not exist on the right-hand side of v. Hence,

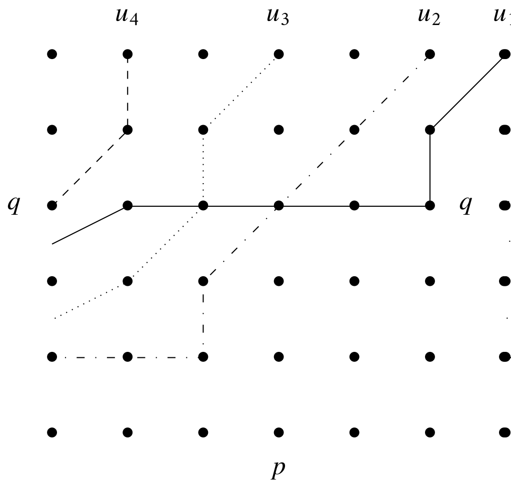


Figure 2: $(P_1, \ldots, P_r)$.

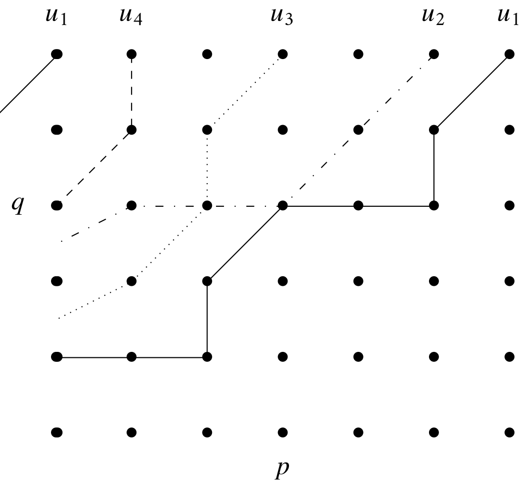


Figure 3: $(\overline{P}_1, \overline{P}_2, P_3, \ldots, P_r)$.

the path P_k will intersect P_1 (resp. P_2) if and only if P_k intersects $\overline{P}_2$ (resp. $\overline{P}_1$). Thus, we confirm that $\overline{\pi}$ appears in (3.1) and the paths $(\overline{P}_1, \overline{P}_2, \ldots, P_r)$ yield

$$S(\overline{\pi}) = \operatorname{sgn}(\overline{\pi}) \prod_{j=2}^{p+1} a_{2j}^{-s_{2j}} \prod_{j=p+1}^{t_1} a_{1j}^{-s_{1j}} \prod_{j=1}^{p} a_{1j}^{-s_{1j}} \prod_{j=p+2}^{t_2} a_{2j}^{-s_{2j}} \prod_{i=3}^{r} \left(\prod_{j=i}^{t_i} a_{ij}^{-s_{ij}} \right).$$

As $\operatorname{sgn}(\pi)\operatorname{sgn}(\overline{\pi}) = -1$ and $s_{1j} = s_{2(j+1)}$, one can confirm that

$$S(\pi) + S(\overline{\pi}) = 0,$$

and this proves the assertion. ■

Example 3.7 Let $\lambda = (3, 2, 1, 0)$. Then, if $(a_{j-i}) = (s_{ij}) \in ST^{\text{diag}}(\lambda, \mathbb{C})$,

$$\zeta_\lambda^Q(\boldsymbol{s}) = \operatorname{pf}\begin{pmatrix} 0 & \zeta^Q_{(3,2)}\left(\begin{array}{|c|c|c|}\hline a_0 & a_1 & a_2 \\ \hline \multicolumn{1}{c|}{} & a_0 & a_1 \\ \cline{2-3}\end{array}\right) & \zeta^Q_{(3,1)}\left(\begin{array}{|c|c|c|}\hline a_0 & a_1 & a_2 \\ \hline \multicolumn{1}{c|}{} & a_0 & \multicolumn{1}{|c}{} \\ \cline{2-2}\end{array}\right) & \zeta^Q_{(3)}\left(\begin{array}{|c|c|c|}\hline a_0 & a_1 & a_2 \\ \hline\end{array}\right) \\ 0 & 0 & \zeta^Q_{(2,1)}\left(\begin{array}{|c|c|}\hline a_0 & a_1 \\ \hline \multicolumn{1}{c|}{} & a_0 \\ \cline{2-2}\end{array}\right) & \zeta^Q_{(2)}\left(\begin{array}{|c|c|}\hline a_0 & a_1 \\ \hline\end{array}\right) \\ 0 & 0 & 0 & \zeta^Q_{(1)}\left(\begin{array}{|c|}\hline a_0 \\ \hline\end{array}\right) \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \zeta^Q_{(3,2)}\left(\begin{array}{|c|c|c|}\hline a_0 & a_1 & a_2 \\ \hline \multicolumn{1}{c|}{} & a_0 & a_1 \\ \cline{2-3}\end{array}\right) \zeta^Q_{(1)}\left(\begin{array}{|c|}\hline a_0 \\ \hline\end{array}\right) - \zeta^Q_{(3,1)}\left(\begin{array}{|c|c|c|}\hline a_0 & a_1 & a_2 \\ \hline \multicolumn{1}{c|}{} & a_0 & \multicolumn{1}{|c}{} \\ \cline{2-2}\end{array}\right) \zeta^Q_{(2)}\left(\begin{array}{|c|c|}\hline a_0 & a_1 \\ \hline\end{array}\right)$$

$$+ \zeta^Q_{(3)}\left(\begin{array}{|c|c|c|}\hline a_0 & a_1 & a_2 \\ \hline\end{array}\right) \zeta^Q_{(2,1)}\left(\begin{array}{|c|c|}\hline a_0 & a_1 \\ \hline \multicolumn{1}{c|}{} & a_0 \\ \cline{2-2}\end{array}\right).$$

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As in [18], we can consider an extension of Theorem 3.6. In preparation, we define

$$\sum_{\mathrm{diag}} = \sum_{\sigma_1 \in S_1} \cdots \sum_{\sigma_{\lambda_1} \in S_{\lambda_1}} \sigma_1 \cdots \sigma_{\lambda_1} \tag{3.2}$$

for S_j being the set of permutations of the elements of $I(j) = \{(k,l) \in SD(\lambda) \mid l - k = j\}$. The sum $\sum_{\mathrm{diag}}$ signifies the sum taken over all permutations of all elements on each diagonal $I(j)$ for all j. We now give an example of (3.2).

Example 3.8 For $\lambda = (3,2)$,

$$\begin{aligned} I(0) &= \{(k,\ell) \in D(\lambda) \mid \ell - k = 0\} = \{(1,1),(2,2)\},\\ I(1) &= \{(k,\ell) \in D(\lambda) \mid \ell - k = 1\} = \{(1,2),(2,3)\},\\ I(2) &= \{(k,\ell) \in D(\lambda) \mid \ell - k = 0\} = \{(1,3)\}. \end{aligned}$$

This leads to

$$S_0 \cong S_1 \cong \mathfrak{S}_2 = \{\mathrm{id}, \sigma_1\},\ S_2 \cong \mathfrak{S}_1 = \{\mathrm{id}\},$$

where σ_1 implies the substitution of the first and second components of $I(j)$ for any j. Therefore,

$$\begin{aligned} \sum_{diag} \zeta_\lambda^Q\left(\begin{array}{|c|c|c|}\hline a & b & c \\ \hline \multicolumn{1}{c|}{} & d & e \\ \cline{2-3}\end{array}\right) = &\\ \zeta_\lambda^Q\left(\begin{array}{|c|c|c|}\hline a & b & c \\ \hline \multicolumn{1}{c|}{} & d & e \\ \cline{2-3}\end{array}\right) & \qquad (\mathrm{id}, \mathrm{id}, \mathrm{id}) \in S_0 \times S_1 \times S_2 \\ + \zeta_\lambda^Q\left(\begin{array}{|c|c|c|}\hline d & b & c \\ \hline \multicolumn{1}{c|}{} & a & e \\ \cline{2-3}\end{array}\right) & \qquad (\sigma_1, \mathrm{id}, \mathrm{id}) \in S_0 \times S_1 \times S_2 \\ + \zeta_\lambda^Q\left(\begin{array}{|c|c|c|}\hline a & e & c \\ \hline \multicolumn{1}{c|}{} & d & b \\ \cline{2-3}\end{array}\right) & \qquad (\mathrm{id}, \sigma_1, \mathrm{id}) \in S_0 \times S_1 \times S_2 \\ + \zeta_\lambda^Q\left(\begin{array}{|c|c|c|}\hline d & e & c \\ \hline \multicolumn{1}{c|}{} & a & b \\ \cline{2-3}\end{array}\right) & \qquad (\sigma_1, \sigma_1, \mathrm{id}) \in S_0 \times S_1 \times S_2. \end{aligned}$$

Also, we define a set $W_{\lambda,H}^Q$ by

$$W_{\lambda,H}^Q = \left\{ \boldsymbol{s} = (s_{ij}) \in ST(\lambda,\mathbb{C}) \ \middle|\ \begin{array}{l} \mathrm{Re}(s_{ij}) \geq 1 \text{ for all } (i,j) \in SD(\lambda) \setminus H(\lambda) \\ \mathrm{Re}(s_{ij}) > 1 \text{ for all } (i,j) \in H(\lambda) \end{array} \right\},$$

where $H(\lambda) = \{(i,j) \in SD(\lambda) \mid i - j \in \{k - \lambda_k \mid 1 \leq k \leq r\}\}$. Following the proof of Theorem 3.6 and [18, Lemma 3.1], the following theorem can be proved.

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***Theorem* 3.9** *For any strict partition $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\boldsymbol{s} \in W^Q_{\lambda,H}$, we have*

$$\sum_{\text{diag}} \zeta^Q_\lambda(\boldsymbol{s}) = \sum_{\text{diag}} \mathrm{pf}(M_\lambda),$$

where M_λ is defined as in Theorem 3.6.

4 Pfaffian expression of the skew type Schur Q-multiple zeta functions

For the strict partitions λ, μ, we write $\mu \leq \lambda$ if $SD(\mu) \subset SD(\lambda)$. For $\mu \leq \lambda$, the *skew shifted diagram* of λ/μ is defined as $SD(\lambda/\mu) = SD(\lambda) \setminus SD(\mu)$. We use the same notations $ST(\lambda/\mu, X)$, $ST^{\text{diag}}(\lambda/\mu, X)$ for a set X, and $PSST(\lambda/\mu)$ as in the previous sections.

***Definition* 4.1** (skew Schur P- and skew Q-multiple zeta functions) Let $\boldsymbol{s} = (s_{ij}) \in ST(\lambda/\mu, \mathbb{C})$. We define skew Schur P- and skew Q-multiple zeta functions associated with λ/μ by

$$\zeta^P_{\lambda/\mu}(\boldsymbol{s}) = \sum_{M \in PSST(\lambda/\mu)} \frac{1}{M^{\boldsymbol{s}}}, \tag{4.1}$$

and

$$\zeta^Q_{\lambda/\mu}(\boldsymbol{s}) = \sum_{M \in QSST(\lambda/\mu)} \frac{1}{M^{\boldsymbol{s}}}. \tag{4.2}$$

Let D be the directed graph defined above and I_N be the same as in the previous section for a fixed positive integer N. We define two sequences of vertices $\boldsymbol{u} = (u_1, \dots, u_r)$ and $\boldsymbol{v} = (v_1, \dots, v_s)$ by $u_i = (\lambda_i, N)$ and $v_i = (\mu_i, 0)$. We define $\boldsymbol{v} \oplus I_N$ by the union of $\boldsymbol{v}$ and I_N, ordered such that each v_i precedes each $v \in I_N$. Then the shifted Young tableaux of shape λ/μ with maximal entry N can be identified with the non-intersecting paths $(P_1, \dots, P_r)$ in $\mathscr{P}_0(\boldsymbol{u}, \boldsymbol{v} \oplus I_N)$, and $\boldsymbol{u}$ is D-compatible with $\boldsymbol{v} \oplus I_N$ such that $P_i \in \mathscr{P}(u_i, v_i)$ for $1 \leq i \leq s$ and $P_i \in \mathscr{P}(u_i, I_N)$ for $s < i \leq r$. The weights of paths are defined in the same way as in Section 3.

Example 4.1 Let $\lambda = (6, 5, 3, 1)$ and $\mu = (3, 1)$. Then Figure 4 is a 4-tuple of non-intersecting paths $(P_1, P_2, P_3, P_4) \in \mathscr{P}_0(\boldsymbol{u}, \boldsymbol{v} \oplus I_5)$.

Let $(s_{ij}) \in ST(\lambda/\mu, \mathbb{C})$. The weights $w(P_i)$ are

$$w(P_1) = \frac{1}{1^{s_{14}} 3^{s_{15}} 4^{s_{16}}}, \qquad w(P_2) = \frac{1}{1^{s_{23}} 2^{s_{24}} 4^{s_{25}} 5^{s_{26}}},$$
$$w(P_3) = \frac{1}{2^{s_{33}} 3^{s_{34}} 5^{s_{35}}}, \qquad w(P_4) = \frac{1}{4^{s_{44}}}.$$

Then, we find that

$$\zeta^{Q,N}_{\lambda/\mu}(\boldsymbol{s}) = \sum_{(P_1,\dots,P_r) \in \mathscr{P}_0(\boldsymbol{u};\boldsymbol{v} \oplus I_N)} w(P_1, \dots, P_r).$$

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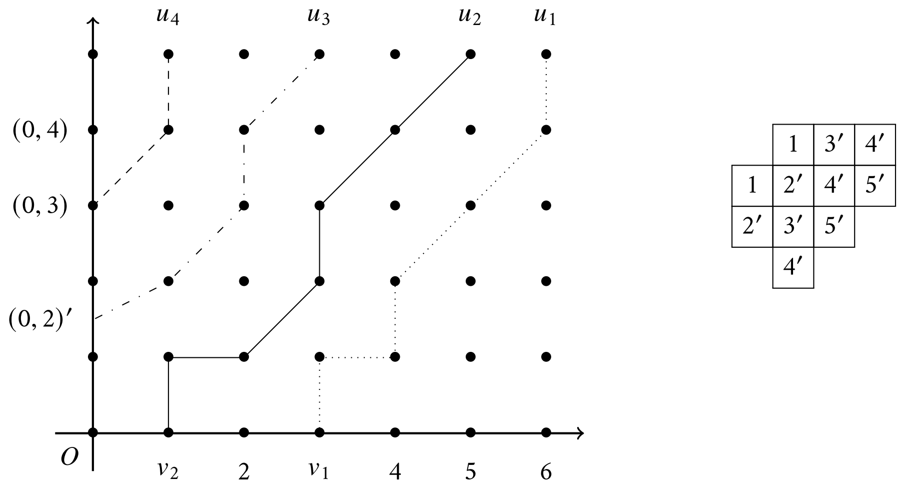

Figure 4: (P_1, P_2, P_3, P_4) satisfying the condition in Example 4.1.

As we proceed a similar discussion as in Theorem 3.6 for the skew Schur Q-multiple zeta functions, in another word, applying the Stembridge method in [24, Theorem 3.2] to our case, we have the following result.

Theorem* 4.2** (Pfaffian expression of the skew Schur Q-multiple zeta functions***) *Let $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\mu = (\mu_1, \ldots, \mu_s)$ be strict partitions into with $\lambda_i \geq 0$ and $2|r+s$. Then for $\boldsymbol{s} \in ST^{\mathrm{diag}}(\lambda/\mu, \mathbb{C})$,*

$$\zeta^{Q}_{\lambda/\mu}(\boldsymbol{s}) = \mathrm{pf}\begin{pmatrix} M_\lambda & H_{\lambda,\mu} \\ 0 & 0 \end{pmatrix},$$

where $M_\lambda = (a_{ij})$ is an $r \times r$ upper triangular matrix with

$$a_{ij} = \zeta^{Q}_{(\lambda_i,\lambda_j)}(\boldsymbol{s}_{(\lambda_i,\lambda_j)}),$$

$$\boldsymbol{s}_{(\lambda_i,\lambda_j)} = \begin{array}{|c|c|c|c|c|} \hline s_{i,i} & \cdots & \cdots & \cdots & s_{i,t_i} \\ \hline \multicolumn{1}{c|}{} & s_{j,j} & \cdots & s_{j,t_j} & \multicolumn{1}{c}{} \\ \cline{2-4} \end{array},$$

where $t_i = i + \lambda_i - 1$ and $H_{\lambda,\mu} = (b_{ij})$ is an $r \times s$ matrix with

$$b_{ij} = \zeta^{Q}_{(\lambda_i - \mu_{s-j+1})}(s_{i,i+j+\mu_s-1}, \ldots, s_{i,t_i}).$$

Remark 4.3 In [24, Theorem 3.2], one may find $-H_{\lambda,\mu}$ in the lower left part of the matrix. Pfaffian can be computed for upper triangular or anti-symmetric matrices. For simplicity, we focus on upper triangular matrices, as the symmetry conditions automatically enforce the full structure of the skew-symmetric matrix.

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Example 4.4 Let $\lambda = (3, 2, 1)$ and $\mu = (2)$. Then, if $(a_{j-i}) = (s_{ij}) \in ST^{\text{diag}}(\lambda/\mu, \mathbb{C})$

$$\zeta^Q_{\lambda/\mu}(\boldsymbol{s}) = \mathrm{pf}\begin{pmatrix} 0 & \zeta^Q_{(3,2)}\left(\begin{array}{|c|c|c|}\hline a_0 & a_1 & a_2 \\ \hline \multicolumn{1}{c|}{} & a_0 & a_1 \\ \cline{2-3}\end{array}\right) & \zeta^Q_{(3,1)}\left(\begin{array}{|c|c|c|}\hline a_0 & a_1 & a_2 \\ \hline \multicolumn{1}{c|}{} & a_0 & \multicolumn{1}{|c}{} \\ \cline{2-2}\end{array}\right) & \zeta^Q_{(1)}\left(\begin{array}{|c|}\hline a_2 \\ \hline\end{array}\right) \\ 0 & 0 & \zeta^Q_{(2,1)}\left(\begin{array}{|c|c|}\hline a_0 & a_1 \\ \hline \multicolumn{1}{c|}{} & a_0 \\ \cline{2-2}\end{array}\right) & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= -\zeta^Q_{(3,1)}\left(\begin{array}{|c|c|c|}\hline a_0 & a_1 & a_2 \\ \hline \multicolumn{1}{c|}{} & a_0 & \multicolumn{1}{|c}{} \\ \cline{2-2}\end{array}\right) + \zeta^Q_{(1)}\left(\begin{array}{|c|}\hline a_2 \\ \hline\end{array}\right)\zeta^Q_{(2,1)}\left(\begin{array}{|c|c|}\hline a_0 & a_1 \\ \hline \multicolumn{1}{c|}{} & a_0 \\ \cline{2-2}\end{array}\right).$$

5 Outside decomposition

Hamel and Goulden proved a general determinant formula which expressed a Schur function as a determinant of skew Schur functions whose shapes are *strips* ([10], see also [3]). Subsequently, Hamel proved expressions of Schur Q-functions as determinants or pfaffians associated with the *outside decomposition* of shifted Young diagrams into *strips* ([8]). In their study of the multiple zeta function, Bachmann and Charlton proved general Jacobi–Trudi formulas for Schur multiple zeta functions for each *outside decomposition*. In fact, they proved the Jacobi–Trudi formula for more general functions ([1]).

We first review the basic terminology of an *outside decomposition* given by Hamel and Goulden ([10]). For each box α of skew (shifted) diagram of λ/μ, we define the *content* of α as the quantity $j - i$ where α lies in row i and in column j of the skew (shifted) diagram (conveniently referred to as (i, j)). A strip in a skew-shaped diagram is a skew (shifted) diagram with an edgewise connected set of boxes that contains no 2×2 block of boxes. In other words, a strip has at most one box on each of its diagonals. We say that the starting box of a strip is the box that is bottommost and leftmost in the strip and the ending box of a strip is the box which is topmost and rightmost in the strip.

***Definition* 5.1** (Outside decomposition) Suppose $(\theta_1, \ldots, \theta_r)$ are disjoint strips in a skew (shifted) diagram of λ/μ and each strip has a starting box on the left or bottom perimeter of the diagram and an ending box on the right or top perimeter of the diagram. Then if the union of these strips is the skew shape diagram of λ/μ, we say the totally ordered set $(\theta_1, \ldots, \theta_r)$ is an outside decomposition of λ/μ.

Example 5.1 ($\lambda = (5, 4, 2, 1)$) We provide two examples of an outside decomposition $(\theta_1, \ldots, \theta_5)$ of λ.

We now define an operation $\theta_i \# \theta_j$ of strips θ_i and θ_j in the same skew diagram. They are part of an outside decomposition. The following procedure is well-defined by [9, Property 2.4].

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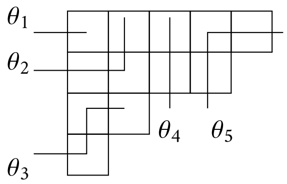

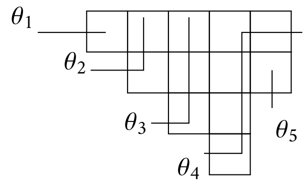

Case.1 Suppose θ_i and θ_j have some boxes with the same content. Slide θ_i along top-left-to-bottom-right diagonals so that the box of content k of θ_i is superimposed on the box of content k of θ_j for all $k \in \mathbb{Z}$ such that both θ_i and θ_j admit a box of content k. We define $\theta_i \# \theta_j$ to be the diagram obtained from this superposition by taking all boxes between the ending box of θ_i and the starting box of θ_j inclusive.

Case.2 Suppose θ_i and θ_j are two disconnected pieces and thus do not have any boxes of the same content. The starting box of one will be to the right and/or above the ending box of the other. To bridge the gap between θ_i and θ_j, insert boxes from the ending box of the one to the starting box of the other such that these inserted boxes follow the approached-from-the-left or approached-from-below arrangement as do other boxes of the same content in the outside decomposition. If there exists content such that the diagram does not include a box with that content (and therefore no determination of the direction from which the box is approached), then arbitrarily choose from which direction boxes of this content should be approached, fix this choice for all boxes of the same content in that particular diagram, and bridge the gap between θ_i and θ_j accordingly. Define $\theta_i \# \theta_j$ as in Case 1 with the following additional conventions: if the ending box of θ_i is connected via an edge to the starting box of θ_j, and occurs below or to the left of it, then $\theta_i \# \theta_j = \emptyset$; if the ending box of θ_i is not edge connected but occurs below or to the left of the starting box of θ_j, $\theta_i \# \theta_j$ is undefined.

If $\boldsymbol{s} = (s_{ij})$ satisfies $s_{ij} = s_{k\ell}$ with $i - j = k - \ell$, then we may define operation $\boldsymbol{s}_{\lambda_i} \# \boldsymbol{s}_{\lambda_j}$ of $\boldsymbol{s}_{\lambda_i}$ and $\boldsymbol{s}_{\lambda_j}$ in the same manner with the operation $\theta_i \# \theta_j$. We note that because $\boldsymbol{s} = (s_{ij})$ have constant entries on the diagonals, this procedure is well-defined.

Example 5.2 For the outside decomposition of the Young diagram λ in the figure on the left in Example 5.1, for example, in $\theta_4 \# \theta_1$, $\theta_1 = \boxed{0}$ moved below $\theta_4 = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}$. The approached-from-below arrangement gives $\theta_4 \# \theta_1 = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 0 \\ \hline\end{array}$. Similarly, we have

$$\theta_1 \# \theta_2 = \begin{array}{|c|c|}\hline -1 & 0 \\ \hline\end{array}, \ \theta_2 \# \theta_1 = \begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline\end{array}, \ \theta_1 \# \theta_4 = \emptyset,$$

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$$\theta_1\#\theta_5 \text{ is undefined, and } \theta_5\#\theta_1 = \begin{array}{ccc} & 3 & 4 \\ & 2 & \\ 0 & 1 & \end{array},$$

where the numbers indicate contents.

Example 5.3 Let

$$\boldsymbol{s} = \begin{array}{ccccc} a_0 & a_1 & a_2 & a_3 & a_4 \\ & a_0 & a_1 & a_2 & a_3 \\ & & a_0 & a_1 & \\ & & & a_0 & \end{array}.$$

For the outside decomposition of the shifted Young diagram λ in Example 5.1 (the figure on the right in the example),

$$\theta_1\#\theta_2 = \boxed{a_0},\ \theta_2\#\theta_1 = \begin{array}{c} a_1 \\ a_0 \end{array},\ \theta_1\#\theta_4 = \boxed{a_0},\ \theta_4\#\theta_1 = \begin{array}{cc} a_3 & a_4 \\ a_2 & \\ a_1 & \\ a_0 & \end{array},$$

$$\theta_1\#\theta_5 \text{ is undefined, and } \theta_5\#\theta_1 = \begin{array}{c} a_3 \\ a_2 \\ a_1 \\ a_0 \end{array}.$$

Hamel ([8]) generalized the classical pfaffian expression of the Schur Q-function involving outside decompositions. To explain the result, we extend the strips of our outside decomposition to the main diagonal, let ρ be a strip consisting of a single box of content 0 so that $\rho = \boxed{0}$, where the number indicates the content. This allows us to define $\overline{\theta_i} = \theta_i\#\rho$. Let $(\overline{\theta}_p, \overline{\theta}_q)$ be formed by juxtaposing $\overline{\theta}_p$ and $\overline{\theta}_q$ with their boxes of content 0 lying on the main diagonal with that of $\overline{\theta}_p$ immediately above and to the left of $\overline{\theta}_q$.

Example 5.4 The $\overline{\theta}_p$ and $(\overline{\theta}_p, \overline{\theta}_q)$ of the shifted Young diagram λ in Example 5.1 (the figure on the right in the example) are $\overline{\theta_p} = \theta_p$ for $1 \leq p \leq 4$ and

$$(\overline{\theta}_1, \overline{\theta}_2) = \begin{array}{cc} 0 & 1 \\ & 0 \end{array},\ (\overline{\theta}_2, \overline{\theta}_1) = \begin{array}{cc} 1 & \\ 0 & \\ & 0 \end{array},\ (\overline{\theta}_2, \overline{\theta}_4) = \begin{array}{ccc} & 3 & 4 \\ 1 & 2 & \\ 0 & 1 & \\ & 0 & \end{array},\ (\overline{\theta}_4, \overline{\theta}_2) = \begin{array}{cc} 3 & 4 \\ 2 & \\ 1 & \\ 0 & 1 \\ & 0 \end{array},$$

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$$\overline{\theta}_5 = \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline 0 \\ \hline \end{array}, \ (\overline{\theta}_4, \overline{\theta}_5) = \begin{array}{cc} \hline \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{1} \\ \hline & \multicolumn{1}{|c|}{0} \\ \cline{2-2} \end{array}, \ (\overline{\theta}_5, \overline{\theta}_4) = \begin{array}{ccc} \cline{1-1} \multicolumn{1}{|c|}{3} & & \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \\ \cline{1-2} \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{1} & \\ \cline{1-2} & \multicolumn{1}{|c|}{0} & \\ \cline{2-2} \end{array},$$

where the numbers indicate contents.

Proceeding the discussion in terms of Schur Q-multiple zeta function following the method in [8, Theorem 1.4] (cf. [4, Theorem 4.3]), we have the theorem below.

***Theorem* 5.5** *Let λ and μ be strict partitions with $\mu \leq \lambda$. Let $\theta = (\theta_1, \theta_2, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_r)$ be an outside decomposition of $SD(\lambda/\mu)$, where θ_p includes a box on the main diagonal of $SD(\lambda/\mu)$ for $1 \leq p \leq k$ and θ_p does not for $k+1 \leq p \leq r$. If k is odd, we replace θ by $(\emptyset, \theta_1, \ldots, \theta_r)$. Then, for $\boldsymbol{s} \in ST^{\mathrm{diag}}(\lambda/\mu, \mathbb{C})$, the Schur Q-multiple zeta functions satisfy the identity*

$$\zeta^{Q}_{\lambda/\mu}(\boldsymbol{s}) = \mathrm{pf}\begin{pmatrix} \zeta^{Q}_{(\overline{\theta}_p, \overline{\theta}_q)}(\boldsymbol{s}_{(\overline{\theta}_p, \overline{\theta}_q)}) & \zeta^{Q}_{\theta_i \# \theta_{r+k+1-j}}(\boldsymbol{s}_{\theta_i \# \theta_{r+k+1-j}}) \\ -{}^{t}(\zeta^{Q}_{\theta_i \# \theta_{r+k+1-j}}(\boldsymbol{s}_{\theta_i \# \theta_{r+k+1-j}})) & 0 \end{pmatrix}$$

with $1 \leq p, q \leq k$ and $k+1 \leq j \leq r$. Here, if $(\overline{\theta}_p, \overline{\theta}_q)$ is not a shifted tableau, then we replace

$$\zeta^{Q}_{(\overline{\theta}_p, \overline{\theta}_q)}(\boldsymbol{s}_{(\overline{\theta}_p, \overline{\theta}_q)}) = -\zeta^{Q}_{(\overline{\theta}_q, \overline{\theta}_p)}(\boldsymbol{s}_{(\overline{\theta}_q, \overline{\theta}_p)}),$$

and further we put $\zeta^{Q}_{(\overline{\theta}_p, \overline{\theta}_p)}(\boldsymbol{s}_{(\overline{\theta}_p, \overline{\theta}_q)}) = 0$.

6 Sum formula

Multiple zeta values of the Euler-Zagier type are well known to satisfy a large number of linear relations among these multiple zeta values, such as the sum formula and duality formula. The following is the sum formula for multiple zeta values of the Euler-Zagier type.

Theorem* 6.1 *(Granville [7], Zagier) *For positive integers k and r with $k > r$, we have*

$$\sum_{\substack{k_1 + \cdots + k_r = k \\ k_1, \ldots, k_{r-1} \geq 1, k_r \geq 2}} \zeta(k_1, \ldots, k_r) = \zeta(k),$$

$$\sum_{\substack{k_1 + \cdots + k_r = k \\ k_1, \ldots, k_{r-1} \geq 1, k_r \geq 2}} \zeta^{\star}(k_1, \ldots, k_r) = \binom{k-1}{r-1}\zeta(k).$$

As in the classical case, we prove the sum formula for a special case of Schur P- and Q-multiple zeta values.

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Theorem 6.2 *For positive integers k and r with $k > r$, we have*

$$\sum_{\substack{k_1+\cdots+k_r=k \\ k_1,\ldots,k_{r-1}\geq 1, k_r\geq 2}} \zeta^{Q}_{(r)}\left(\begin{array}{|c|c|c|}\hline k_1 & \cdots & k_r \\ \hline\end{array}\right) = \sum_{i=1}^{r} 2^{i}\binom{k-i-1}{r-i}\zeta(k)$$

and

$$\sum_{\substack{k_1+\cdots+k_r=k \\ k_1,\ldots,k_{r-1}\geq 1, k_r\geq 2}} \zeta^{P}_{(r)}\left(\begin{array}{|c|c|c|}\hline k_1 & \cdots & k_r \\ \hline\end{array}\right) = \sum_{i=1}^{r} 2^{i-1}\binom{k-i-1}{r-i}\zeta(k).$$

Proof For $\boldsymbol{k} = \begin{array}{|c|c|c|}\hline k_1 & \cdots & k_r \\ \hline\end{array}$, let $|\boldsymbol{k}| = k_1 + \cdots + k_r$ and $\mathrm{dep}(\boldsymbol{k}) = r$. By (2.3), it suffices to show the first identity. Example 2.3 leads to

$$\begin{aligned}\sum_{\substack{k_1+\cdots+k_r=k \\ k_1,\ldots,k_{r-1}\geq 1, k_r\geq 2}} \zeta^{Q}_{(r)}(\boldsymbol{k}) &= \sum_{\substack{k_1+\cdots+k_r=k \\ k_1,\ldots,k_{r-1}\geq 1, k_r\geq 2}} \sum_{\boldsymbol{\ell}\preceq_s \boldsymbol{k}} 2^{\mathrm{dep}(\boldsymbol{\ell})}\zeta(\boldsymbol{\ell}) \\ &= \sum_{i=1}^{r} 2^{i} \sum_{\substack{k_1+\cdots+k_r=k \\ k_1,\ldots,k_{r-1}\geq 1, k_r\geq 2}} \sum_{\substack{\boldsymbol{\ell}\preceq_s \boldsymbol{k} \\ \mathrm{dep}(\boldsymbol{\ell})=i}} \zeta(\boldsymbol{\ell}).\end{aligned}$$

For fixed $\boldsymbol{\ell}$ with $|\boldsymbol{\ell}| = k$ and $\mathrm{dep}(\boldsymbol{\ell}) = i$, we count the number of $\boldsymbol{k}$ with $\boldsymbol{\ell} \preceq_s \boldsymbol{k}$ with $\boldsymbol{k} \in ST((r), \mathbb{Z})$. Because $\boldsymbol{k}$ has to be admissible, it suffices to choose $r - i$ new division points of $\boldsymbol{\ell}$ out of $(k - 1) - (i - 1) - 1$ possibilities. Therefore,

$$\#\{\boldsymbol{k} \in ST((r), \mathbb{C}) \mid \boldsymbol{\ell} \preceq_s \boldsymbol{k}\} = \binom{k-i-1}{r-i}$$

and we have

$$\sum_{\substack{k_1+\cdots+k_r=k \\ k_1,\ldots,k_{r-1}\geq 1, k_r\geq 2}} \zeta^{Q}_{(r)}(\boldsymbol{k}) = \sum_{i=1}^{r} 2^{i}\binom{k-i-1}{r-i} \sum_{\substack{|\boldsymbol{\ell}|=k \\ \mathrm{dep}(\boldsymbol{\ell})=i}} \zeta(\boldsymbol{\ell}).$$

The sum formula for multiple zeta values of Euler-Zagier type leads to

$$\sum_{\substack{k_1+\cdots+k_r=k \\ k_1,\ldots,k_{r-1}\geq 1, k_r\geq 2}} \zeta^{Q}_{(r)}(\boldsymbol{k}) = \sum_{i=1}^{r} 2^{i}\binom{k-i-1}{r-i}\zeta(k).$$

This proves the first identity. Dividing both sides by 2, we can confirm that the second identity holds. This completes the proof of the theorem. ■

Example 6.3 For $(k, r) = (5, 3)$, we have

$$\sum_{\substack{k_1+k_2+k_3=5 \\ k_1,k_2\geq 1, k_3\geq 2}} \zeta^{Q}_{(3)}\left(\begin{array}{|c|c|c|}\hline k_1 & k_2 & k_3 \\ \hline\end{array}\right) = 11\zeta^{Q}_{(1)}\left(\begin{array}{|c|}\hline 5 \\ \hline\end{array}\right) = 22\zeta(5).$$

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We have the following corollaries of Theorem 6.2. The first is the sum formula in Schur P- or Q-multiple zeta values.

Corollary 6.4 *For positive integers k and r with $k > r$, we have*

$$\sum_{\substack{k_1+\cdots+k_r=k \\ k_1,\ldots,k_{r-1}\geq 1, k_r\geq 2}} \zeta^Q_{(r)}\left(\boxed{k_1 \mid \cdots \mid k_r}\right) = \sum_{i=1}^{r} 2^{i-1}\binom{k-i-1}{r-i}\zeta^Q_{(1)}\left(\boxed{k}\right)$$

and

$$\sum_{\substack{k_1+\cdots+k_r=k \\ k_1,\ldots,k_{r-1}\geq 1, k_r\geq 2}} \zeta^P_{(r)}\left(\boxed{k_1 \mid \cdots \mid k_r}\right) = \sum_{i=1}^{r} 2^{i-1}\binom{k-i-1}{r-i}\zeta^P_{(1)}\left(\boxed{k}\right).$$

Remark 6.5 In [26], Yamamoto introduced the interpolation $\zeta^t(\boldsymbol{k})$ of multiple zeta and zeta star functions. Theorem 1.1 in [26] with $t = \frac{1}{2}$ is the same shape with our Theorem 6.2. One can verify that our $\zeta^Q_{(r)}(\boldsymbol{k})$ and $\zeta_{\frac{1}{2}}(\boldsymbol{k})$ in [26] are equal (up to a multiplicative constant).

Remark 6.6 Recently, In [2], Bachmann–Kadota–Suzuki–Yamamoto–Yamasaki obtained a different type of sum formulas for the Schur multiple zeta values for other types of partition.

The next corollary is the duality formula for a certain shape and weight. Before we explain the duality property of the Schur Q-multiple zeta function, we review the original duality formula for multiple zeta functions. We denote a string $\underbrace{1,\ldots,1}_{r}$ of 1's by $\{1\}^r$. Then for an admissible index

$$\boldsymbol{k} = (\{1\}^{a_1-1}, b_1+1, \{1\}^{a_2-1}, b_2+1, \ldots, \{1\}^{a_m-1}, b_m+1)$$

with positive integers $a_1, b_1, a_2, b_2, \cdots, a_m, b_m \in \mathbb{Z}_{\geq 1}$, the following index is referred to as the *dual* index of $\boldsymbol{k}$:

$$\boldsymbol{k}^\dagger = (\{1\}^{b_m-1}, a_m+1, \{1\}^{b_{m-1}-1}, a_{m_1}+1, \ldots, \{1\}^{b_1-1}, a_1+1).$$

The duality formula is the following.

Theorem 6.7 (**Duality formula** *[27]*) *For any admissible index $\boldsymbol{k} = (k_1, \ldots, k_r)$ and its dual index $\boldsymbol{k}^\dagger = (k_1^\dagger., \ldots, k_s^\dagger)$, we have*

$$\zeta(k_1, \ldots, k_r) = \zeta(k_1^\dagger, \ldots, k_s^\dagger).$$

As a special case of Theorem 6.7, it holds that

$$\zeta(\{1\}^{k-2}, 2) = \zeta(k).$$

Taking $\lambda = (k-1)$ and $\boldsymbol{k} = \boxed{1 \mid \cdots \mid 1 \mid 2} \in ST(\lambda, \mathbb{C})$, we have the following formula similar to the above identity.

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***Corollary* 6.8** *For positive integers k, we have*

$$\zeta^{Q}_{(k-1)}\left(\boxed{1}\boxed{\cdots}\boxed{1}\boxed{2}\right) = (2^{k-1}-1)\zeta^{Q}_{(1)}\left(\boxed{k}\right) = (2^{k}-2)\zeta(k)$$

and

$$\zeta^{P}_{(k-1)}\left(\boxed{1}\boxed{\cdots}\boxed{1}\boxed{2}\right) = (2^{k-1}-1)\zeta^{P}_{(1)}\left(\boxed{k}\right) = (2^{k-1}-1)\zeta(k).$$

***Remark* 6.9** We can say that there may hold the duality-like formula for ζ^P and ζ^Q in general.

7 Symplectic Schur Multiple zeta functions

First, we review the basic terminology to define symplectic or orthogonal Schur multiple zeta functions. We identify a partition λ with *the Young diagram*

$$D(\lambda) = \{(i, j) \in \mathbb{Z}^2 \mid 1 \le i \le r, 1 \le j \le \lambda_i\}$$

depicted as a collection of square boxes with the i-th row having λ_i boxes. For a partition λ, a Young tableau (t_{ij}) of shape λ over a set X is a filling of $D(\lambda)$ with $t_{ij} \in X$ into each box (i, j) of $D(\lambda)$. We denote by $T(\lambda, X)$ the set of all Young tableaux of shape λ over X.

Let $[\overline{N}]$ be the set $\{1, \overline{1}, 2, \overline{2}, \dots, N, \overline{N}\}$ with the total ordering $1 < \overline{1} < 2 < \overline{2} < \cdots < N < \overline{N}$. Then, a *symplectic tableau* $\boldsymbol{t} = (t_{ij}) \in T(\lambda, [\overline{N}])$ is obtained by numbering all the boxes of $D(\lambda)$ with letters from $[\overline{N}]$ such that

SP1 the entries of $\boldsymbol{t}$ are weakly increasing along each row of $\boldsymbol{t}$,
SP2 the entries of $\boldsymbol{t}$ are strictly increasing down each column of $\boldsymbol{t}$,
SP3 for non-negative integer i, the boxes of content $-i$ contain entries that are greater than or equal to $i + 1$.

We refer to the third condition SP3 as the symplectic condition. We denote by $SP_N(\lambda)$ the set of symplectic tableaux of shape λ.

***Definition* 7.1** (symplectic Schur multiple zeta functions) For a given set $\boldsymbol{s} = (s_{ij}) \in T(\lambda, \mathbb{C})$ of variables, the *symplectic Schur multiple zeta functions* of shape λ are defined as

$$\zeta^{\mathrm{sp},N}_{\lambda}(\boldsymbol{s}) = \sum_{M \in SP_N(\lambda)} \frac{1}{M^{\boldsymbol{s}}}, \tag{7.1}$$

where $M^{\boldsymbol{s}} = \prod_{(i,j)\in D(\lambda)} |m_{ij}|^{s_{ij}}$ for $M = (m_{ij}) \in SP_N(\lambda)$ and $|i| = i$, $|\bar{i}| = i^{-1}$.

Hamel constructed a directed graph D corresponding to the symplectic Schur functions [9] and applied the Stembridge Theorem ([24]) to obtain the following determinant expression of the symplectic Schur functions.

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***Theorem* 7.1** *([9, Theorem 3.1]) Let λ/μ be a partition of the skew type. Then, for any outside decomposition $(\theta_1, \dots, \theta_r)$ of λ/μ,*

$$\mathrm{sp}_{\lambda/\mu} = \det(\mathrm{sp}_{\theta_i \# \theta_j})_{1 \le i, j \le r}.$$

Following the Hamel approach, we construct a directed graph D corresponding to the symplectic Schur multiple zeta functions. For a fixed positive integer N, we begin with the y-axis labeled by $1, \overline{1}, 2, \overline{2}, \dots, N, \overline{N}$ and direct an edge $u \;\rightarrow\; v$ whenever $v - u = (0,1), (0,-1), (1,0)$, or $(1,-1)$. We add four restrictions: A down-vertical step must not precede an up-vertical step, an up-vertical step must not precede a down-vertical step, a down-vertical step must not precede a horizontal step, and an up-vertical step must not precede a diagonal step. Because of the symplectic condition, we add a left boundary in the form of a "backwards lattice path" from $(0,1)$ to $(0,\overline{1})$ to $(0,2)$ to $(-1,2)$ to $(-1,\overline{2})$ to $(-1,3)$ to $(-2,3)$ to $(-2,\overline{3})$ to $(-2,4)$ to $(-3,4)\dots$. We indicate this left boundary by the dotted line in Figure 5.

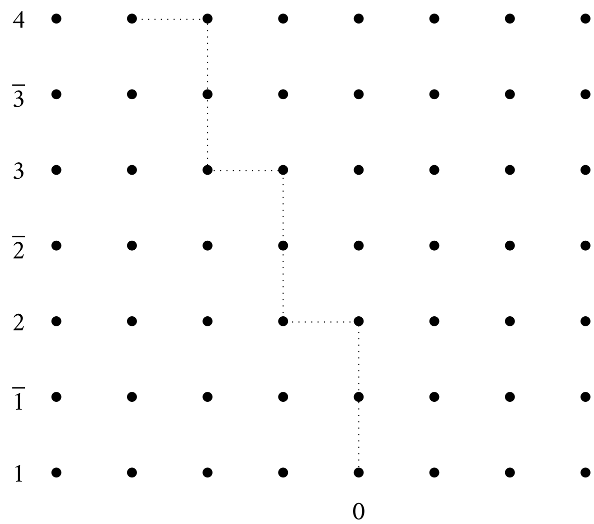

Figure 5: Left boundary given by the symplectic condition.

Hereinafter, we may omit this left boundary for simplicity.

For a fixed outside decomposition $(\theta_1, \dots, \theta_r)$ of λ/μ, we construct a non-intersecting r-tuple of lattice paths that corresponds to a symplectic tableau of shape λ/μ with the outside decomposition $(\theta_1, \dots, \theta_r)$, such that the i-th path corresponds to the i-th strip and begins at B_i and ends at E_i as described next. Fix points $B_i = (t-s, -(t-s)+1)$ if the i-th strip has the starting box (s,t) on the left perimeter of the diagram and if $t - s \le 0$ (i.e., B_i is on the left boundary), or $B_i = (t-s, 1)$ if the i-th strip has the starting box (s,t) on the left perimeter of the diagram and if $t - s > 0$, or $B_i = (t-s, \infty)$ if the i-th strip has the starting box (s,t) on the bottom perimeter of the

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diagram ($B_i = (t - s, \infty)$ if both). Fix points $E_i = (v - u + 1, 1)$ if the i-th strip has the ending box (u, v) on the top perimeter of the diagram, or $E_i = (v - u + 1, \infty)$ if the i-th strip has the ending box (u, v) on the right perimeter of the diagram ($E_i = (v-u+1, \infty)$ if both).

For the j-th strip construct a path starting at B_j (termed the starting point) and ending at E_j (termed the ending point) as follows: if a box containing i (resp. $\bar{i}$) and at coordinates (a, b) in the diagram is approached from the left in the strip, add a horizontal step from $(b-a, i)$ to $(b-a+1, i)$ (resp. $(b-a, \bar{i})$ to $(b-a+1, \bar{i})$); if a box containing i (resp. $\bar{i}$) and at coordinates (a, b) in the diagram is approached from below in the strip, add a diagonal step from $(b-a, \bar{i})$ to $(b-a+1, i)$ (resp. $(b-a, i+1)$ to $(b-a+1, \bar{i})$). We note that the physical locations of the termination points of the steps are independent of the outside decomposition and depend only on the contents of the boxes. In Figure 6, the ending points of the steps are first shown alone and then the complete paths for two different outside decompositions are shown. We note that no two paths can have the same starting and/or ending points, because that would imply two boxes of the same content on the same section of the perimeter. Connect these non-vertical steps with vertical steps. This routine is intended to verify that a unique path exists. In the above setup, Hamel showed that the symplectic tableaux of shape λ/μ can be identified with the non-intersecting paths in $\mathscr{P}_0((B_i); (E_i))$, and (B_i) is D-compatible with (E_i).

We next define the weight of each step. Let $v_i(P) = (v_{i,j}(P))_{j\geq 0}$ be the sequence of vertices representing the path $P \in \mathscr{P}_0(B_i; E_i)$ and let ℓ^i_{xy} be the edge $v_{i,j} (= (x, y)) \to v_{i,j+1}$. If $v_{i,j}(P) - v_{i,j+1}(P) = (1, 0)$ or $(1, -1)$, we assign the weights $w(\ell^i_{xy}) = |y|^{-s_{pq}}$ with $v_{ij} = (x, y)$ and (p, q) being the j-th component of θ_i. If $v_{i,j}(P) - v_{i,j+1}(P) = (0, 1)$ or $(0, -1)$, we assign the weights $w(\ell^i_{xy}) = 1$. Then, we define

$$w(P_i) = \prod_{\ell^i_{xy}} w(\ell^i_{xy}),$$

and for an r-tuple of non-intersecting paths of $(P_1, \dots, P_r)$ with $P_i \in \mathscr{P}(B_i; E_i)$,

$$w(P_1, \dots, P_r) = \prod_{i=1}^{r} w(P_i).$$

Then, owing to the Hamel composition in [9], we find that

$$\zeta^{\mathrm{sp},N}_{\lambda}(\boldsymbol{s}) = \sum_{P_i \in \mathscr{P}(B_i;E_i)} w(P_1, \dots, P_r).$$

Example 7.2 For $\lambda = (5, 3, 3, 1)$, let a 4-tuple of paths $(P_1, P_2, P_3, P_4) \in \mathscr{P}(B_i; E_i)$ be given as in Figure 6. For $(s_{ij}) \in T(\lambda, \mathbb{C})$, the weights $w(P_i)$ are

$$w(P_1) = 1^{s_{11}}, \qquad w(P_2) = \frac{2^{s_{21}}}{3^{s_{22}} 2^{s_{12}}},$$
$$w(P_3) = \frac{3^{s_{32}} 3^{s_{33}}}{4^{s_{41}} 3^{s_{31}} 3^{s_{23}} 2^{s_{13}}}, \qquad w(P_4) = \frac{3^{s_{14}}}{4^{s_{15}}},$$

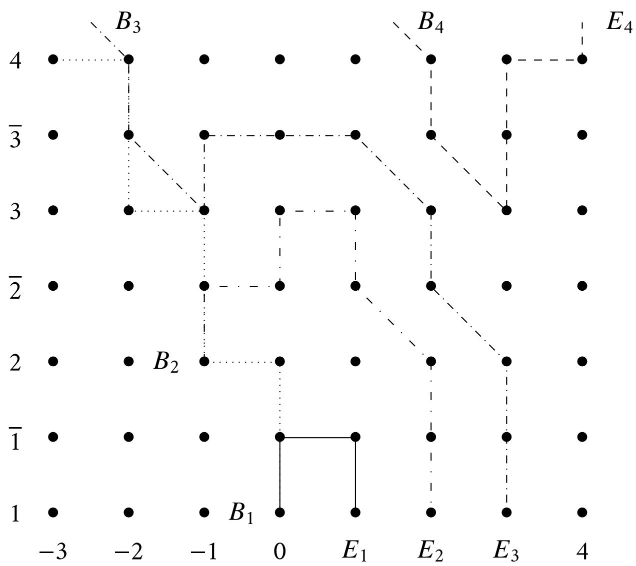

Figure 6: (P_1, P_2, P_3, P_4) satisfying the condition in Example 7.2.

and the corresponding symplectic tableau is

$\overline{1}$	2	2	$\overline{3}$	4
$\overline{2}$	3	3		
3	$\overline{3}$	$\overline{3}$		
4				

.

As the proof in Theorem 3.6, proceeding the discussion in terms of symplectic Schur multiple zeta function following the Hamel method in Theorem 7.1, we have the theorem below.

***Theorem** 7.3* *Let $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\mu = (\mu_1, \ldots, \mu_s)$ be partitions. Then, for $\boldsymbol{s} \in T^{\mathrm{diag}}(\lambda/\mu, \mathbb{C})$ and any outside decomposition $(\theta_1, \ldots, \theta_r)$ of λ/μ,*

$$\zeta^{\mathrm{sp},N}_{\lambda/\mu}(\boldsymbol{s}) = \det(\zeta^{\mathrm{sp},N}_{\theta_i \# \theta_j}(\boldsymbol{s}_{(\lambda_i,\lambda_j)}))_{1 \le i,j \le r},$$

where $\boldsymbol{s}_{(\lambda_i,\lambda_j)} = \boldsymbol{s}_{\lambda_i} \# \boldsymbol{s}_{\lambda_j}$.

***Example** 7.4* Let $\lambda = (3, 2)$ and its outside decomposition (θ_1, θ_2) be depicted as

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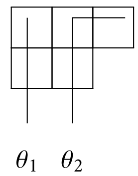

.

Then, if $(a_{j-i}) = (s_{ij}) \in T^{\mathrm{diag}}(\lambda, \mathbb{C})$,

$$\zeta_\lambda^{\mathrm{sp},N}(\boldsymbol{s}) = \det \begin{pmatrix} \zeta_{\theta_1}^{\mathrm{sp},N}\left(\begin{array}{|c|}\hline a_0 \\ \hline a_{-1} \\ \hline\end{array}\right) & \zeta_{\theta_1 \# \theta_2}^{\mathrm{sp},N}\left(\begin{array}{|c|}\hline a_0 \\ \hline\end{array}\right) \\ \zeta_{\theta_2 \# \theta_1}^{\mathrm{sp},N}\left(\begin{array}{|c|c|}\hline a_1 & a_2 \\ \hline a_0 & \\ \cline{1-1} a_{-1} & \\ \cline{1-1}\end{array}\right) & \zeta_{\theta_2}^{\mathrm{sp},N}\left(\begin{array}{|c|c|}\hline a_1 & a_2 \\ \hline a_0 & \\ \cline{1-1}\end{array}\right) \end{pmatrix}$$

$$= \zeta_{\theta_1}^{\mathrm{sp},N}\left(\begin{array}{|c|}\hline a_0 \\ \hline a_{-1} \\ \hline\end{array}\right) \zeta_{\theta_2}^{\mathrm{sp},N}\left(\begin{array}{|c|c|}\hline a_1 & a_2 \\ \hline a_0 & \\ \cline{1-1}\end{array}\right) - \zeta_{\theta_1 \# \theta_2}^{\mathrm{sp},N}\left(\begin{array}{|c|}\hline a_0 \\ \hline\end{array}\right) \zeta_{\theta_2 \# \theta_1}^{\mathrm{sp},N}\left(\begin{array}{|c|c|}\hline a_1 & a_2 \\ \hline a_0 & \\ \cline{1-1} a_{-1} & \\ \cline{1-1}\end{array}\right).$$

Remark 7.5 The function in Example 7.4 satisfies

$$\zeta_{\theta_2 \# \theta_1}^{\mathrm{sp},N}\left(\begin{array}{|c|c|}\hline a_1 & a_2 \\ \hline a_0 & \\ \cline{1-1} a_{-1} & \\ \cline{1-1}\end{array}\right) \neq \zeta_{(2,1,1)}^{\mathrm{sp},N}\left(\begin{array}{|c|c|}\hline a_1 & a_2 \\ \hline a_0 & \\ \cline{1-1} a_{-1} & \\ \cline{1-1}\end{array}\right)$$

in general. We note that for $i = -1, 0, 1, 2$ the contents of each $\boxed{a_i}$ are not the same.

8 Orthogonal Schur multiple zeta functions

Hamel also constructed a directed graph D corresponding to the orthogonal Schur functions [9] and derived the determinant expression of the orthogonal Schur functions. As in Section 7, we construct a directed graph D corresponding to the orthogonal Schur multiple zeta functions. As in Section 7, we prove the results corresponding to the following Hamel result.

We define orthogonal Schur multiple zeta functions. Let $[\overline{N}]^\infty$ be the set $\{1, \overline{1}, 2, \overline{2}, \dots, N, \overline{N}, \infty\}$ with the total ordering $1 < \overline{1} < 2 < \overline{2} < \cdots < N < \overline{N} < \infty$. For a fixed partition λ, a *so-tableau* $\boldsymbol{t} = (t_{ij}) \in T(\lambda, [\overline{N}]^\infty)$ is obtained by numbering all the boxes of $D(\lambda)$ with letters from $[\overline{N}]^\infty$ such that

SO1 the entries of $\boldsymbol{t}$ are weakly increasing along each row of $\boldsymbol{t}$,
SO2 the entries of $\boldsymbol{t}$ are strictly increasing down each column of $\boldsymbol{t}$,
SO3 for non-negative integer i, the boxes of content $-i$ contain entries which are greater than or equal to $i + 1$,
SO4 no two symbols ∞ appear in the same row.

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One may find that the conditions SO1-SO3 are the same as SP1-SP3. We denote by $SO_N(\lambda)$ the set of so-tableaux of shape λ.

***Definition* 8.1** (orthogonal Schur multiple zeta functions) For a given set $\boldsymbol{s} = (s_{ij}) \in T(\lambda, \mathbb{C})$ of variables, the *orthogonal Schur multiple zeta functions* of shape λ are defined as

$$\zeta_{\lambda}^{\mathrm{so},N}(\boldsymbol{s}) = \sum_{M \in SO_N(\lambda)} \frac{1}{M^{\boldsymbol{s}}}, \tag{8.1}$$

where we set $|\infty| = 1$.

We note that the ∞ contributes 1 to the weight of the tableau. Therefore, they are "dummy elements" in a sense.

***Theorem* 8.1** *([9, Theorem 3.2]) Let λ/μ be a partition of the skew type. Then, for any outside decomposition $(\theta_1, \dots, \theta_r)$ of λ/μ,*

$$\mathrm{so}_{\lambda/\mu} = \det(\mathrm{so}_{\theta_i \# \theta_j})_{1 \le i,j \le r}.$$

As in the symplectic Schur multiple zeta functions, we consider the y-axis with $1, \overline{1}, 2, \overline{2}, \dots, N, \overline{N}, \infty$. We define lattice paths with five types of permissible steps. These steps are the four steps in Section 7, and up-diagonal steps from height $\overline{N}$ to height ∞ that increase the x- and y-coordinates by 1, respectively. We distinguish between horizontal steps at integer levels and horizontal steps at ∞. The steps are subject to the same restrictions as in Section 7 plus the following additional restrictions: an up-vertical step must not precede a horizontal step at ∞, and a down-vertical step must not precede an up-diagonal step. We also require that all steps between lines $x = c$ and $x = c + 1$ for all c are either

(1) horizontal at ∞ or down-diagonal, or
(2) horizontal at integer levels or up-diagonal.

Determining whether the steps are of type (1) or (2) depends on the outside decomposition: if boxes of content c are approached from the left, then the steps between $x = c$ and $x = c + 1$ must be of type (2); if the boxes of content c are approached from below, then the steps between $x = c$ and $x = c+1$ must be of type (1). We fix beginning points B_i and ending points E_i as in Section 7 with the adjustment that the y-coordinate of the highest points is $\infty + 1$ instead of ∞. Given $\boldsymbol{s} \in SO(\lambda/\mu, \mathbb{C})$ with an outside decomposition, we can construct an r-tuple of non-intersecting lattice paths. For each strip construct a path as follows: if a box contains i or $\overline{i}$, place a step as in the proof of Section 7. If a box contains ∞, is at coordinates (a, b) in the diagram, and is approached from the left in the strip, add an up-diagonal step from $(a - b, \overline{N})$ to $(a - b + 1, \infty)$; if it is approached from below, add a horizontal step from $(a - b, \infty)$ to $(a - b + 1, \infty)$. We connect these non-vertical paths with vertical paths. The weights of paths are defined in the same way as in Section 7. Note that we put $w(\ell_{xy}^i) = 1$ if $y = \infty$.

Then, owing to the Hamel composition [9], we find that

$$\zeta_{\lambda}^{\mathrm{so},N}(\boldsymbol{s}) = \sum_{P_i \in \mathscr{P}(B_i;E_i)} w(P_1, \dots, P_r).$$

Example 8.2 For $\lambda = (5,3,3,1)$, let $(s_{ij}) \in T(\lambda, \mathbb{C})$. The weights $w(P_i)$ are

$$w(P_1) = 1^{s_{11}}, \qquad w(P_2) = \frac{2^{s_{21}}}{3^{s_{22}}2^{s_{12}}},$$
$$w(P_3) = \frac{3^{s_{32}}3^{s_{33}}}{3^{s_{31}}3^{s_{23}}2^{s_{13}}}, \qquad w(P_4) = 3^{s_{24}}3^{s_{14}},$$

and the corresponding orthogonal tableau is

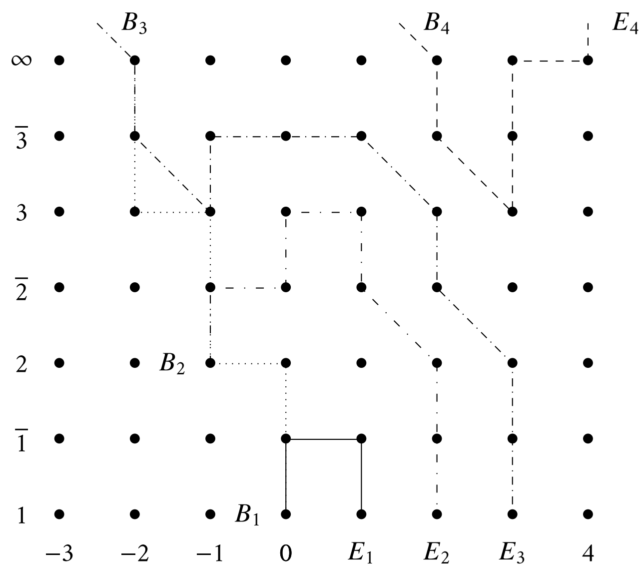

Figure 7: (P_1, P_2, P_3, P_4) satisfying the condition in Example 8.2.

$\overline{1}$	2	2	$\overline{2}$	∞
$\overline{2}$	3	3		
3	$\overline{3}$	$\overline{3}$		
∞				

.

As similar in the previous sections, proceeding the discussion in terms of orthogonal Schur multiple zeta function following the Hamel method in Theorem 8.1, we have the theorem below.

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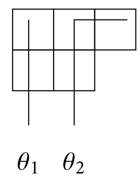
***Theorem* 8.3** *Let $\lambda = (\lambda_1, \dots, \lambda_r)$, $\mu = (\mu_1, \dots, \mu_s)$ be partitions. Then, for $\boldsymbol{s} \in T^{\mathrm{diag}}(\lambda/\mu, \mathbb{C})$ and any outside decomposition $(\theta_1, \dots, \theta_r)$ of λ/μ,*

$$\zeta^{\mathrm{so},N}_{\lambda/\mu}(\boldsymbol{s}) = \det(\zeta^{\mathrm{so},N}_{\theta_i \# \theta_j}(\boldsymbol{s}_{(\lambda_i,\lambda_j)}))_{1\le i,j\le r},$$

where $\boldsymbol{s}_{(\lambda_i,\lambda_j)} = \boldsymbol{s}_{\lambda_i} \# \boldsymbol{s}_{\lambda_j}$.

Example 8.4 Let $\lambda = (3, 2)$ and its outside decomposition (θ_1, θ_2) be depicted as



.

Then, if $(a_{j-i}) = (s_{ij}) \in T^{\mathrm{diag}}(\lambda, \mathbb{C})$, we obtain

$$\zeta^{\mathrm{so},N}_{\lambda}(\boldsymbol{s}) = \det \begin{pmatrix} \zeta^{\mathrm{so},N}_{\theta_1}\left(\begin{array}{|c|}\hline a_0 \\ \hline a_1 \\ \hline \end{array}\right) & \zeta^{\mathrm{so},N}_{\theta_1\#\theta_2}\left(\begin{array}{|c|}\hline a_0 \\ \hline \end{array}\right) \\ \zeta^{\mathrm{so},N}_{\theta_2\#\theta_1}\left(\begin{array}{|c|c|}\hline a_1 & a_2 \\ \hline a_0 & \\ \cline{1-1} a_{-1} & \\ \cline{1-1} \end{array}\right) & \zeta^{\mathrm{so},N}_{\theta_2}\left(\begin{array}{|c|c|}\hline a_1 & a_2 \\ \hline a_0 & \\ \cline{1-1} \end{array}\right) \end{pmatrix}$$

$$= \zeta^{\mathrm{so},N}_{\theta_1}\left(\begin{array}{|c|}\hline a_0 \\ \hline a_1 \\ \hline \end{array}\right) \zeta^{\mathrm{so},N}_{\theta_2}\left(\begin{array}{|c|c|}\hline a_1 & a_2 \\ \hline a_0 & \\ \cline{1-1} \end{array}\right) - \zeta^{\mathrm{so},N}_{\theta_1\#\theta_2}\left(\begin{array}{|c|}\hline a_0 \\ \hline \end{array}\right) \zeta^{\mathrm{so},N}_{\theta_2\#\theta_1}\left(\begin{array}{|c|c|}\hline a_1 & a_2 \\ \hline a_0 & \\ \cline{1-1} a_{-1} & \\ \cline{1-1} \end{array}\right).$$

9 Decomposition of Symplectic and Orthogonal multiple zeta functions

In this section, we express a symplectic and an orthogonal multiple zeta function as a linear combination of the truncated multiple zeta functions. Analogous to the method of the proof of (2.5) and (2.6), by the Inclusion-Exclusion principle, we may find the following decompositions.

***Theorem* 9.1** *For any positive integer N and $\boldsymbol{s} \in T(\lambda, \mathbb{C})$, the function $\zeta^{\square,N}_{\lambda}(\boldsymbol{s})$ for $\square \in \{\mathrm{sp}, \mathrm{so}\}$ can be decomposed as a sum of truncated multiple zeta functions: for a positive integer N,*

$$\zeta^N(s_1, \dots, s_r) = \sum_{1\le n_1 < \cdots < n_r \le N} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}.$$

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Example 9.2 For any positive integer N and $a, b \in \mathbb{C}$, we have

$$\zeta^{\mathrm{sp},N}_{(1)}\left(\boxed{a}\right) = \zeta^N(a) + \zeta^N(-a),$$

$$\zeta^{\mathrm{sp},N}_{(1,1)}\left(\begin{array}{|c|}\hline a\\ \hline b\\ \hline\end{array}\right) = \zeta^N(a,b) + \zeta^N(-a,-b) + \zeta^N(-a,b) + \zeta^N(a,-b) + \zeta^N(a-b) - 1.$$

Example 9.3 For any positive integer N and $a, b \in \mathbb{C}$, we have

$$\zeta^{\mathrm{so},N}_{(1)}\left(\boxed{a}\right) = \zeta^N(a) + \zeta^N(-a) + 1,$$

$$\begin{aligned}\zeta^{\mathrm{so},N}_{(1,1)}\left(\begin{array}{|c|}\hline a\\ \hline b\\ \hline\end{array}\right) =& \zeta^N(a,b) + \zeta^N(-a,-b) + \zeta^N(-a,b) + \zeta^N(a,-b) + \zeta^N(a-b) - 1\\ &+ \zeta^N(a) + \zeta^N(-a).\end{aligned}$$

Note that in the case of $\zeta^{\mathrm{sp},N}_{(\{1\}^r)}$, we have

$$\begin{aligned}\zeta^{\mathrm{sp},N}_{(\{1\}^r)}\left(\begin{array}{|c|}\hline s_1\\ \hline \vdots\\ \hline s_r\\ \hline\end{array}\right) =& \sum_{\mathrm{sign}} \zeta^N(\pm s_1, \ldots, \pm s_r)\\ &+ \sum_{i=1}^{r-1}\sum_{\mathrm{sign}} \zeta^N(\pm s_1, \ldots, \pm s_{i-1}, s_i - s_{i+1}, \pm s_{i+2}, \ldots, \pm s_r)\\ &- \sum_{i=1}^{r-1}\sum_{\mathrm{sign}} \left(\prod_{j=1}^{i-1} j^{\pm s_j}\right) \frac{i^{s_{i+1}}}{i^{s_i}} \zeta^N(\{0\}^i, \pm s_{i+2}, \ldots, \pm s_r)\\ &+ (\cdots),\end{aligned}$$

where $\sum_{\mathrm{sign}}$ means the summation over all cases of plus-minus signs and the last term $(\cdots)$ is caused from ζ^N's whose elements contain at least two different $(s_i - s_{i+1})$'s like $\zeta^N(s_1 - s_2, -s_3, s_4 - s_5, s_6 - s_7, s_8)$.

If we use the decompositions by rows as an outside decomposition of λ/μ, then for any $\boldsymbol{s} \in T^{\mathrm{diag}}(\lambda/\mu, \mathbb{C})$, $\zeta^{\mathrm{sp},N}_{\lambda/\mu}(\boldsymbol{s})$ and $\zeta^{\mathrm{so},N}_{\lambda/\mu}(\boldsymbol{s})$ appear to be decomposed into a sum of $\zeta^{\mathrm{sp},N}_{(\{1\}^r)}$ and $\zeta^{\mathrm{so},N}_{(\{1\}^r)}$, respectively. As in Remark 7.5, we note that the outside decomposition and operation $\theta_i \# \theta_j$ retains the content and two different functions may be associated with the same shape $\lambda = (\{1\}^r)$ and the same variable $\boldsymbol{s} = (s_{ij})$.

Similarly, we attain the following results, in which we decompose the symplectic zeta function into the sum of truncated multiple zeta star functions: for a positive integer N,

$$\zeta^{\star N}(s_1, \ldots, s_r) = \sum_{1 \le n_1 < \cdots < n_r} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}, \ \zeta^{\star}(s_1, \ldots, s_r) = \sum_{1 \le n_1 \le \cdots \le n_r} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}$$

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Theorem 9.4 *For any positive integer N and $s_i \in \mathbb{C}$, we have*

$$\zeta^{\mathrm{sp},N}_{(r)}\left(\boxed{s_1}\boxed{\cdots}\boxed{s_r}\right) = \sum_{\mathrm{sign}}\sum_{\boldsymbol{\ell}}(-1)^{r-\mathrm{dep}(\boldsymbol{\ell})}\zeta^{\star N}(\boldsymbol{\ell}_r),$$

and for $r \geq 2$

$$\zeta^{\mathrm{so},N}_{(r)}\left(\boxed{s_1}\boxed{\cdots}\boxed{s_r}\right) = \sum_{R=r-1}^{r}\sum_{\mathrm{sign}}\sum_{\boldsymbol{\ell}}(-1)^{R-\mathrm{dep}(\boldsymbol{\ell})}\zeta^{\star N}(\boldsymbol{\ell}_R),$$

where $\sum_{\mathrm{sign}}$ means the summation over all cases of plus-minus signs and $\boldsymbol{\ell}$ runs over all indices of the form $\boldsymbol{\ell}_R = (\pm s_1 \square \pm s_2 \square \cdots \square \pm s_R)$ in which each $\square$ is filled by the comma , or the plus sign $+$. If $\square = +$ then $\pm s_j \square \pm s_{j+1}$ is assigned $s_{j+1} - s_j$ and the square is not filled with consecutive plus signs $+$.

Example 9.5 *($r \leq 2$)* For any positive integer N and $a, b \in \mathbb{C}$, we have

$$\begin{aligned}
\zeta^{\mathrm{sp},N}_{(1)}\left(\boxed{a}\right) =& \zeta^{\star N}(a) + \zeta^{\star N}(-a),\\
\zeta^{\mathrm{sp},N}_{(2)}\left(\boxed{a}\boxed{b}\right) =& \zeta^{\star N}(a,b) + \zeta^{\star N}(-a,-b) + \zeta^{\star N}(-a,b) + \zeta^{\star N}(a,-b)\\
&- \zeta^{\star N}(b-a),\\
\zeta^{\mathrm{so},N}_{(1)}\left(\boxed{a}\right) =& \zeta^{\star N}(a) + \zeta^{\star N}(-a) + 1,\\
\zeta^{\mathrm{so},N}_{(2)}\left(\boxed{a}\boxed{b}\right) =& \zeta^{\star N}(a,b) + \zeta^{\star N}(-a,-b) + \zeta^{\star N}(-a,b) + \zeta^{\star N}(a,-b)\\
&- \zeta^{\star N}(b-a) + \zeta^{\star N}(a) + \zeta^{\star N}(-a).
\end{aligned}$$

10 Schur quasi-symmetric functions

We here investigate the quasi-symmetric functions, introduced by Gessel [5], related to symmetric multiple zeta functions defined in this paper. We note that the Schur type quasi-symmetric function was discussed in [17].

10.1 Quasi-symmetric functions

Let $\boldsymbol{t} = (t_1, t_2, \ldots)$ be variables and $\mathfrak{P}$ the subalgebra of $\mathbb{Z}[[t_1, t_2, \ldots]]$ consisting of all formal power series with integer coefficients of bounded degree. We refer to $p = p(\boldsymbol{t}) \in \mathfrak{P}$ as a *quasi-symmetric function* if the coefficient of $t_{k_1}^{\gamma_1} t_{k_2}^{\gamma_2} \cdots t_{k_n}^{\gamma_n}$ of p is the same as that of $t_{h_1}^{\gamma_1} t_{h_2}^{\gamma_2} \cdots t_{h_n}^{\gamma_n}$ of p whenever $k_1 < k_2 < \cdots < k_n$ and $h_1 < h_2 < \cdots < h_n$. The algebra of all quasi-symmetric functions is denoted by Qsym. For a composition $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ of a positive integer, define the *monomial quasi-symmetric function* $M_{\boldsymbol{\gamma}}$ and the *essential quasi-symmetric function* $E_{\boldsymbol{\gamma}}$, respectively, by

$$M_{\boldsymbol{\gamma}} = \sum_{m_1<m_2<\cdots<m_n} t_{m_1}^{\gamma_1} t_{m_2}^{\gamma_2} \cdots t_{m_n}^{\gamma_n}, \quad E_{\boldsymbol{\gamma}} = \sum_{m_1\leq m_2\leq\cdots\leq m_n} t_{m_1}^{\gamma_1} t_{m_2}^{\gamma_2} \cdots t_{m_n}^{\gamma_n}.$$

We know that these respective functions form the integral basis of Qsym. Notice that

$$E_{\boldsymbol{\gamma}} = \sum_{\boldsymbol{\delta} \preceq \boldsymbol{\gamma}} M_{\boldsymbol{\delta}}, \quad M_{\boldsymbol{\gamma}} = \sum_{\boldsymbol{\delta} \preceq \boldsymbol{\gamma}} (-1)^{n-\ell(\boldsymbol{\delta})} E_{\boldsymbol{\delta}}. \tag{10.1}$$

The relation between the multiple zeta values and quasi-symmetric functions was studied by Hoffman [11] (remark that the notations used for the multiple zeta (star) function in [11] are different from ours, which are $\zeta(s_n, s_{n-1}, \dots, s_1)$ and $\zeta^\star(s_n, s_{n-1}, \dots, s_1)$, respectively). Let $\mathfrak{H} = \mathbb{Z}\langle x, y\rangle$ be the noncommutative polynomial algebra over $\mathbb{Z}$. We can define a commutative and associative multiplication $*$, known as the harmonic product, on $\mathfrak{H}$. We refer to $(\mathfrak{H}, *)$ as (integral) harmonic algebra. Let $\mathfrak{H}^1 = \mathbb{Z}1 + y\mathfrak{H}$, which is a subalgebra of $\mathfrak{H}$. Notice that every $w \in \mathfrak{H}^1$ can be written as an integral linear combination of $z_{\gamma_1} z_{\gamma_2} \cdots z_{\gamma_n}$ where $z_\gamma = yx^{\gamma-1}$ for $\gamma \in \mathbb{N}$. For each $N \in \mathbb{N}$, define the homomorphism $\phi_N : \mathfrak{H}^1 \to \mathbb{Z}[t_1, t_2, \dots, t_N]$ by $\phi_N(1) = 1$ and

$$\phi_N(z_{\gamma_1} z_{\gamma_2} \cdots z_{\gamma_n}) = \begin{cases} \displaystyle\sum_{m_1 < m_2 < \cdots < m_n \le N} t_{m_1}^{\gamma_1} t_{m_2}^{\gamma_2} \cdots t_{m_n}^{\gamma_n} & n \le N, \\ 0 & \text{otherwise,} \end{cases}$$

and extend it additively to $\mathfrak{H}^1$. There exists a unique homomorphism $\phi : \mathfrak{H}^1 \to \mathfrak{P}$ such that $\pi_N \phi = \phi_N$, where π_N is the natural projection from $\mathfrak{P}$ to $\mathbb{Z}[t_1, t_2, \dots, t_N]$. We have $\phi(z_{\gamma_1} z_{\gamma_2} \cdots z_{\gamma_n}) = M_{(\gamma_1, \gamma_2, \dots, \gamma_n)}$. Moreover, as described in [11], ϕ is an isomorphism between $\mathfrak{H}^1$ and Qsym.

Let e be the function that sends t_i to $\frac{1}{i}$. Moreover, define $\rho_N : \mathfrak{H}^1 \to \mathbb{R}$ by $\rho_N = e\phi_N$. For a composition $\boldsymbol{\gamma}$, we have

$$\rho_N \phi^{-1}(M_{\boldsymbol{\gamma}}) = \zeta^N(\boldsymbol{\gamma}), \quad \rho_N \phi^{-1}(E_{\boldsymbol{\gamma}}) = \zeta^{\star N}(\boldsymbol{\gamma}).$$

We define the map $\rho : \mathfrak{H}^1 \to \mathbb{R}^{\mathbb{N}}$ by $\rho(w) = (\rho_N(w))_{N \ge 1}$ for $w \in \mathfrak{H}^1$. Notice that if $w \in \mathfrak{H}^0 = \mathbb{Z}1 + y\mathfrak{H}x$, which is a subalgebra of $\mathfrak{H}^1$, then we may understand that $\rho(w) = \lim_{N\to\infty} \rho_N(w) \in \mathbb{R}$. In particular, for a composition $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_n)$ with $\gamma_n \ge 2$, we have

$$\rho\phi^{-1}(M_{\boldsymbol{\gamma}}) = \zeta(\boldsymbol{\gamma}), \quad \rho\phi^{-1}(E_{\boldsymbol{\gamma}}) = \zeta^\star(\boldsymbol{\gamma}). \tag{10.2}$$

10.2 Schur P- and Q-type quasi-symmetric functions

Now, the following *Schur P- and Q-type quasi-symmetric functions* are easily defined. For strict partitions λ and μ, and $\boldsymbol{s} = (s_{ij}) \in ST(\lambda/\mu, \mathbb{C})$, we define Schur P- and Q-type quasi-symmetric functions associated with λ/μ by

$$S^P_{\lambda/\mu}(\boldsymbol{s}) = \sum_{M \in PSST(\lambda/\mu)} \prod_{(i,j) \in SD(\lambda)} t^{s_{ij}}_{|m_{ij}|}, \tag{10.3}$$

and

$$S^Q_{\lambda/\mu}(\boldsymbol{s}) = \sum_{M \in QSST(\lambda/\mu)} \prod_{(i,j) \in SD(\lambda)} t^{s_{ij}}_{|m_{ij}|}. \tag{10.4}$$

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Theorem 10.1 *Let $\lambda = (\lambda_1, \dots, \lambda_r)$, $\mu = (\mu_1, \dots, \mu_s)$ be strict partitions into with $\lambda_i \geq 0$ and $2|r+s$. Then, for $\boldsymbol{s} \in ST^{\mathrm{diag}}(\lambda/\mu, \mathbb{C})$,*

$$S^Q_{\lambda/\mu}(\boldsymbol{s}) = \mathrm{pf}\begin{pmatrix} M_\lambda & H_{\lambda,\mu} \\ 0 & 0 \end{pmatrix},$$

where $M_\lambda = (a_{ij})$ is an $r \times r$ upper triangular matrix with

$$a_{ij} = S^Q_{(\lambda_i,\lambda_j)}(\boldsymbol{s}_{(\lambda_i,\lambda_j)}),$$

$$\boldsymbol{s}_{(\lambda_i,\lambda_j)} = \begin{array}{|c|c|c|c|c|c|}\hline s_{i,i} & \cdots & \cdots & \cdots & \cdots & s_{i,t_i} \\ \hline \multicolumn{1}{c|}{} & s_{j,j} & \cdots & s_{j,t_j} & \multicolumn{2}{c}{} \\ \cline{2-4} \end{array},$$

where $t_i = i + \lambda_i - 1$ and $H_\lambda = (b_{ij})$ is an $r \times s$ matrix with

$$b_{ij} = S^Q_{(\lambda_i - \mu_s - j + 1)}(s_{i(i+j+\mu_s-1)}, \dots, s_{it_i}).$$

Theorem 10.2 *(cf. [4, Theorem 4.3],[8, Theorem 1.4]) Let λ and μ be strict partitions with $\mu \leq \lambda$. Let $\theta = (\theta_1, \theta_2, \dots, \theta_k, \theta_{k+1}, \dots, \theta_r)$ be an outside decomposition of $SD(\lambda/\mu)$, where θ_p includes a box on the main diagonal of $SD(\lambda/\mu)$ for $1 \leq p \leq k$ and θ_p does not for $k+1 \leq p \leq r$. If k is odd, we replace θ by $(\emptyset, \theta_1, \dots, \theta_r)$. Then, the Schur Q-type quasi-symmetric functions satisfy the identity*

$$S^Q_{\lambda/\mu}(\boldsymbol{s}) = \mathrm{pf}\begin{pmatrix} S^Q_{(\overline{\theta}_p,\overline{\theta}_q)}(\boldsymbol{s}_{(\overline{\theta}_p,\overline{\theta}_q)}) & S^Q_{\theta_i \# \theta_{r+k+1-j}}(\boldsymbol{s}_{\theta_i \# \theta_{r+k+1-j}}) \\ -{}^t(S^Q_{\theta_i \# \theta_{r+k+1-j}}(\boldsymbol{s}_{\theta_i \# \theta_{r+k+1-j}})) & 0 \end{pmatrix}$$

with $1 \leq p, q \leq k$ and $k+1 \leq j \leq r$. Here

$$S^Q_{(\overline{\theta}_p,\overline{\theta}_q)}(\boldsymbol{s}_{(\overline{\theta}_p,\overline{\theta}_q)}) = -S^Q_{(\overline{\theta}_q,\overline{\theta}_p)}(\boldsymbol{s}_{(\overline{\theta}_q,\overline{\theta}_p)})$$

and $S^Q_{(\overline{\theta}_p,\overline{\theta}_p)}(\boldsymbol{s}_{(\overline{\theta}_p,\overline{\theta}_q)}) = 0$.

10.3 Symplectic type and Orthogonal type quasi-symmetric functions

Similarly, we define the following *symplectic quasi-symmetric functions* and *orthogonal quasi-symmetric functions*. For partitions λ and μ, and $\boldsymbol{s} = (s_{ij}) \in T(\lambda/\mu, \mathbb{C})$, we define symplectic quasi-symmetric functions and orthogonal quasi-symmetric functions associated with λ/μ by

$$S^{\mathrm{sp},N}_{\lambda/\mu}(\boldsymbol{s}) = \sum_{M \in SP_N(\lambda/\mu)} \prod_{(i,j) \in D(\lambda)} t^{s_{ij}}_{|m_{ij}|}, \tag{10.5}$$

and

$$S^{\mathrm{so},N}_{\lambda/\mu}(\boldsymbol{s}) = \sum_{M \in SO_N(\lambda/\mu)} \prod_{(i,j) \in D(\lambda)} t^{s_{ij}}_{|m_{ij}|}. \tag{10.6}$$

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***Theorem* 10.3** *Let $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\mu = (\mu_1, \ldots, \mu_s)$ be partitions. Then, for $\boldsymbol{s} \in T^{\mathrm{diag}}(\lambda/\mu, \mathbb{C})$ and any outside decomposition $(\theta_1, \ldots, \theta_r)$ of λ/μ,*

$$S^{\mathrm{sp},N}_{\lambda/\mu}(\boldsymbol{s}) = \det(S^{\mathrm{sp},N}_{\theta_i \# \theta_j}(\boldsymbol{s}_{(\lambda_i,\lambda_j)}))_{1 \le i,j \le r},$$

where $\boldsymbol{s}_{(\lambda_i,\lambda_j)} = \boldsymbol{s}_{\lambda_i} \# \boldsymbol{s}_{\lambda_j}$.

***Theorem* 10.4** *Let $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\mu = (\mu_1, \ldots, \mu_s)$ be partitions. Then, for $\boldsymbol{s} \in T^{\mathrm{diag}}(\lambda/\mu, \mathbb{C})$ and any outside decomposition $(\theta_1, \ldots, \theta_r)$ of λ/μ,*

$$S^{\mathrm{so},N}_{\lambda/\mu}(\boldsymbol{s}) = \det(S^{\mathrm{so},N}_{\theta_i \# \theta_j}(\boldsymbol{s}_{(\lambda_i,\lambda_j)}))_{1 \le i,j \le r},$$

where $\boldsymbol{s}_{(\lambda_i,\lambda_j)} = \boldsymbol{s}_{\lambda_i} \# \boldsymbol{s}_{\lambda_j}$.

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