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SEPARABLE MEASURES AND THE DUNFORD-PETTIS PROPERTY

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Let X be a complete regular space. We denote by $C_b(X)$ the Banach space of all real-valued bounded continuous functions on X endowed with the supremum-norm.

In this paper we give a characterisation of weakly compact operators defined from $C_b(X)$ into a Banach space E which are β_∞-continuous, where β_∞ is a locally convex topology on $C_b(X)$ introduced by Wheeler. We also prove that $(C_b(X), \beta_\infty)$ has the strict Dunford-Pettis property and, if X is a σ-compact space, $(C_b(X), \beta_\infty)$, has the Dunford-Pettis property.

The concepts introduced here arise from the theory of strict topologies for a completely regular space due to Sentilles [8] and several authors. The first three locally convex topologies considered by them on $C_b(X)$ were β_0, β and β_1 which relate to the theory of measure on topological spaces developed in, for example, Varadarajan [10]. Such topological vector spaces have as duals the spaces $M_t(X)$, $M_\tau(X)$ and $M_\sigma(X)$ of tight, τ-additives and σ-additives measures respectively.

Another important strict topology defined on $C_b(X)$ is β_∞ introduced by Wheeler [11]. The dual of $(C_b(X), \beta_\infty)$ is $M_\infty(X)$, the space of all μ-additive measures.

1. PRELIMINARY RESULTS AND NOTATIONS

Let E be a Hausdorff locally convex space. E is said to have the D-P (respectively the strict D-P) property if for any Banach space E and every linear continuous mapping $T\colon E \to F$ for which $T(B)$ is relatively weakly compact in F for every bounded set $B \subset E$, $T(C)$ is relatively compact (respectively $\{Tx_n\}_n$ is Cauchy) in F for any absolutely convex weakly compact subset C (respectively weak Cauchy sequence $\{x_n\}_n$) in E, [2, Definition 1 and 3]. The following theorem was proved by Khurana [3].

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Theorem 1.1. *If E is a locally convex space with the strict D-P property and E', the topological dual of E, has a σ-compact dense subset in the $\sigma(E', E)$-topology, then E has the D-P property.*

Let $\mathcal{K}$ denote the class of all bounded and equicontinuous subsets of $C_b(X)$. β_∞ is defined to be the finest locally convex topology on $C_b(X)$ agreeing with the pointwise topology on each $H \in \mathcal{K}$. It is known [4] that $\beta \leqslant \beta_\infty \leqslant \beta_\sigma$ and then $M_\infty(X)$ is contained in $M_\sigma(X)$. It is not difficult to see that a σ-additive measure μ is μ-additive if and only if $\mu_{|H}$ is pointwise continuous on each $H \in \mathcal{K}$. Thus every Dirac measure δ_x, $x \in X$, is a μ-additive measure and the span $M_d(X)$ of $\{\delta_x : x \in X\}$ is contained in $M_\infty(X)$. Pachl [6] proved that $M_\infty(X)$ is the completion of $(M_d(X), \mathcal{K}\text{-top})$, where the $\mathcal{K}$-top is the topology of uniform convergence on each $H \in \mathcal{K}$. Since X is, in particular, a Hausdorff uniform space and such uniformity is generated by the family of pseudometrics $\{d_H : H \in \mathcal{K}\}$, where

$$d_H(x, y) = \sup\{|f(x) - f(y)| : f \in H\} = \sup\{|\delta_x(f) - \delta_y(f)| : f \in H\},$$

we have that X is a topological subspace of $(M_\infty(X), \mathcal{K}\text{-top})$.

Wheeler and Sentilles [9] proved that a σ-additive measure μ is μ-additive if and only if $\sum_{\alpha \in A} \mu(f_\alpha) = \mu(1)$ for any partition of unity $\{f_\alpha : \alpha \in A\}$ in $C_b(X)$. After that, Koumoullis [5] exhibited a family U of compact subsets of $\beta X - X$, where βX denotes the Stone-Cech compactification of X, and proved that a σ-additive measure μ is μ-additive if and only if $|\overline{\mu}|(K) = 0$ for all $K \in U$, where $\overline{\mu}$ is the corresponding regular Borel measure on βX (via the isometry of $C_b(X)$ and $C(\beta X)$).

In [1] we proved that any bounded linear operator defined on $C_b(X)$ into a Banach space E is represented by a finitely additive vector measure defined on $Ba(X)$, where $Ba(X)$ is the Baire σ-algebra in X.

For every bounded linear operator $T : C_b(X) \to E$ we denote $\overline{T}$ the corresponding linear operator defined on $C(\beta X)$ to E by $\overline{T}(\overline{f}) = T(f)$, where $\overline{f}$ is the unique extension of f to βX. It is clear that, for each $x' \in E'$, $\overline{x' \circ T} = x' \circ \overline{T}$ and if m is the associated finitely additive vector measure of T, then $\overline{x' \circ m} = x' \circ \overline{m}$ where $\overline{m}$ is the associated finitely additive vector measure of $\overline{T}$. We will write $\|m\|(A)$ for the supremum of the set $\{|x' \circ m|(A) : \|x'\| \leqslant 1\}$.

2. Characterisation of a weakly compact β_∞-continuous operator

Theorem 2.1. *Let T be a weakly compact operator defined from $C_b(X)$ to a Banach space E. The following statements are equivalent:*

(a) *T is β_∞-continuous.*

(b) *$T_{|H}$ is pointwise continuous for all $H \in \mathcal{K}$.*

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(c) $\|m\|(K) = 0$ *for all* $K \in U$.

(d) *For every partition of unity* $\{f_\alpha\}_{\alpha \in A}$ *in* X *and for every* $\varepsilon > 0$, *there exists a finite subset* F *of* A *such that* $|x' \circ m| \left(1 - \sum_{\alpha \in F} f_\alpha\right) < \varepsilon$ *uniformly in* x', $\|x'\| \leqslant 1$.

PROOF: (a) $\Rightarrow$ (b) Since β_∞ coincides with the pointwise topology on each H in $\mathcal{K}$, the statement follows.

(b) $\Rightarrow$ (c) Since $x' \circ T_{|H}$ is pointwise continuous for all $x' \in E'$, we have that $x' \circ T \in M_\infty(X)$ and then $|x' \circ \overline{m}|(K) = 0$ for all $K \in U$ (see [5, Theorem 3.2]). Thus $\|\overline{m}\|(K) = 0$.

(c) $\Rightarrow$ (d) Let $\{f_\alpha\}_\alpha$ be a partition of unity for X and $0 < \varepsilon < 1$. For every finite subset F of A, we define $Z_F = \{x \in \beta X : \sum_{\alpha \in F} \overline{f}_\alpha(x) < 1 - \varepsilon\}$ and let K be the intersection of all Z_F. Then K is a compact subset of $\beta X - X$ and belongs to U. See [5, Lemma 3.1].

We claim that $\|m\|(K) = \inf\{\|\overline{m}\|(Z_F) : F \text{ is a finite subset of } A\}$. In fact, since $K \in U$, for all finite subsets F of A, we have $\|\overline{m}\|(K) \leqslant \inf \|\overline{m}\|(Z_F)$ ($\|m\|$ is a non-decreasing set function). For each $x' \in E'$, $|x' \circ \overline{m}|(K) = \inf |x' \circ \overline{m}|(Z_F)$ and therefore $\|\overline{m}\|(K) = \sup_{\|x'\| \leqslant 1} |x' \circ \overline{m}|(K) = \sup_{\|x'\| \leqslant 1} \inf_F |x' \circ \overline{m}|(Z_F)$. Thus, to prove the claim is to prove that $\inf_F \sup_{\|x'\| \leqslant 1} |x' \circ \overline{m}|(Z_F) \leqslant \sup_{\|x'\| \leqslant 1} \inf_F |x' \circ \overline{m}|(Z_F)$.

Take a finite subset F of A and $\varepsilon > 0$. Then there is $x'_\varepsilon \in E'$ such that $\sup_{\|x'\| \leqslant 1} |x' \circ \overline{m}|(Z_F) < |x'_\varepsilon \circ \overline{m}|(Z_F) + \varepsilon$. That implies

$$
\begin{aligned}
\inf_F \|\overline{m}\|(Z_F) &= \inf_F \sup_{\|x'\| \leqslant 1} |x' \circ \overline{m}|(Z_F) \\
&< \inf_F |x'_\varepsilon \circ \overline{m}|(Z_F) + \varepsilon \\
&< \sup_{\|x'\| \leqslant 1} \inf_F |x' \circ \overline{m}|(Z_F) + \varepsilon \\
&< \|\overline{m}\|(K) + \varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, we have $\inf_F \|\overline{m}\|(Z_F) \leqslant \|\overline{m}\|(K)$.

Now, take $x' \in E'$ with $\|x'\| \leqslant 1$ and $\varepsilon > 0$; by above, there is a finite subset F

of A such that $\|m\|(Z_F) < \|\overline{m}\|(K) + \varepsilon$. Thus

$$\begin{aligned}
|x' \circ m|(1) - \sum_{\alpha\in F} |x' \circ m|(f_\alpha) = |x' \circ m|\left(1 - \sum_{\alpha\in F} f_\alpha\right) &= |x' \circ \overline{m}|\left(1 - \sum_{\alpha\in F} \overline{f}_\alpha\right) \\
&= \int_{Z_F}\left(1 - \sum_{\alpha\in F} \overline{f}_\alpha\right) d\,|x' \circ \overline{m}| \\
&\quad + \int_{\beta X - X}\left(1 - \sum_{\alpha\in F} \overline{f}_\alpha\right) d\,|x' \circ \overline{m}| \\
&\leqslant |x' \circ \overline{m}|(Z_F) + \varepsilon\,|x' \circ \overline{m}|(\beta X) \\
&\leqslant \|\overline{m}\|(Z_F) + \varepsilon\,\|\overline{m}\|(\beta X) \\
&< \varepsilon(1 + \|\overline{m}\|(\beta X)).
\end{aligned}$$

Therefore $|x' \circ m|\left(1 - \sum_{\alpha\in F} f_\alpha\right) < \varepsilon$ uniformly in x', $\|x'\| \leqslant 1$.

(d) $\Rightarrow$ (a) Since $\{|x' \circ m| : \|x'\| \leqslant 1\}$ is bounded, we have that for every partition of unity $\{f_\alpha\}_{\alpha\in A}$ and $\varepsilon > 0$, there exists a finite subset F of A such that $|x' \circ m|\left(1 - \sum_{\alpha\in F} f_\alpha\right) < \varepsilon$ uniformly in x', $\|x'\| \leqslant 1$, which implies that $\{|x' \circ m| : \|x'\| \leqslant 1\}$ is β_∞-equicontinuous [5, Proposition 3.6] and T is β_∞-continuous. ∎

3. The strict Dunford-Pettis and the Dunford-Pettis property in $(C_b(X), \beta_\infty)$

Theorem 3.1. *$(C_b(X), \beta_\infty)$ has the strict Dunford-Pettis property.*

Proof: Let T be a weakly compact β_∞-continuous operator from $C_b(X)$ into a Banach space E. Since $\beta_\infty \leqslant \beta_\sigma$ (see Khurana [4]), the associated vector measure m is σ-additive and so it admits a control positive real-valued measure μ [1]. Take a weakly Cauchy sequence $\{f_n\}_n$ in $C_b(X)$. Then $\{f_n\}_n$ is β_∞-bounded which implies $\{f_n\}_n$ is norm bounded (see for example Khurana [4]); thus, there is $L > 0$ such that $\|f_n\| \leqslant L/2$ for all $n \in \mathbb{N}$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu(F) < \delta$ implies $\|m\|(F) < \varepsilon/2L$. On the other hand, for each $x \in X$, $\{f_n(x)\}_n$ is a Cauchy sequence in $\mathbb{R}$ which implies, by Egoroff's Theorem, that there exists F_δ in $Ba(X)$ such that $\{f_n\}_n$ is uniformly Cauchy on $X - F_\delta$ and $\mu(F_\delta) < \delta$.

Choose $n_0 \in \mathbb{N}$ so that, for $n, m \geqslant n_0$, we get

$$\sup\{\|f_n(x) - f_m(x)\| : x \in X - F_\delta\} < \varepsilon/2M, \text{ with } M = \|m\|(X).$$

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Thus, if $n, m \geqslant n_0$, then

$$\begin{aligned}\|Tf_n - Tf_m\| &\leqslant \left\|\int_{X-F_\delta}(f_n - f_m)dm\right\| + \left\|\int_{F_\delta}(f_n - f_m)dm\right\| \\ &\leqslant (\sup\{\|f_n(x) - f_m(x)\| : x \in X\})M + L\,\|m\|\,(F_\delta) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon,\end{aligned}$$

so $\{T(f_n)\}_n$ converges in E and the theorem is proved. ∎

Theorem 3.2. *$(C_b(X), \beta_\infty)$ possesses the Dunford-Pettis property, if X is σ-compact.*

Proof: Let $\{K_n\}_n$ be an increasing sequence of compact subsets of X such that $\cup K_n$ is dense in X. Let L_n denote $\overline{\text{absco}(K_n)}$ in M_∞. Since K_n are compact in $M_\infty(X)$ with the $\mathcal{K}$-topology and the space $(M_\infty(X), \mathcal{K}\text{-top})$ is complete, we have that L_n are compact subsets of $M_\infty(X)$. Moreover $\{L_n\}_n$ is an increasing sequence.

We claim that $\cup L_n$ is $\mathcal{K}$-dense in $M_\infty(X)$. In fact, take $\mu \in M_\infty(X)$ and a balanced neighbourhood V of μ. Since $M_d(X)$ is $\mathcal{K}$-dense in $M_\infty(X)$, V contains some element of $M_d(X)$, say $v = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$, with $x_1, x_2, \ldots, x_n \in X$. Suppose that $0 < \alpha = \sum_{i=1}^{n} |\alpha_i| \leqslant 1$ (if $\alpha = 0$, $V \cap \cup L_n \neq \emptyset$ and we are done) and take neighbourhoods $W_1, W_2, \ldots, W_n$ of $\delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_n}$ respectively such that $\alpha_1 W_1 + \alpha_2 W_2 + \ldots + \alpha_n W_n \subset V$. Since $W_i \cap X$ is a neighbourhood of x_i in X, we have that there exists $\delta_{y_i} \in K_{n_i}$, such that $\delta_{y_i} \in W_i \cap X$. Thus $\sum_{i=1}^{n} \alpha_i \delta_{y_i} \in \sum_{i=1}^{n} \alpha_i W_i \subset V$ and $\sum_{i=1}^{n} \alpha_i \delta_{y_i} \in L_N$, where $N = \max\{n_i : i = 1, \ldots, n\}$ and therefore $V \cap (\cup L_n) \neq \emptyset$.

Suppose now that $\alpha > 1$; hence $\alpha \sum_{i=1}^{n} (\alpha_i/\alpha)\delta_{x_i} \in V$ and then $\sum_{i=1}^{n} (\alpha_i/\alpha)\delta_{x_i} \in 1/\alpha V \subset V$. Applying the above argument to $1/\alpha V$, we get $\emptyset \neq 1/\alpha V \cap (\cup L_n) \subset V \cap (\cup L_n)$. Thus, for every $\mu \in M_\infty(X)$ and every neighbourhood V of μ, $V \cap (\cup L_n) \neq \emptyset$.

Therefore, $(M_\infty(X), \mathcal{K}\text{-top})$ has a σ-compact dense which implies that $(M_\infty(X), \sigma(M_\infty(X), C_b(X)))$ has a σ-compact dense subset. The conclusion of the Theorems follows from Theorems 1.1 and 3.1. ∎

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