# MINIMAL COCKCROFT SUBGROUPS

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**Statement of results.** Consider any group $G$. A $[G,2]$-*complex* is a connected 2-dimensional CW-complex with fundamental group $G$. If $X$ is a $[G,2]$-complex and $L$ is a subgroup of $G$, let $X_L$ denote the covering complex of $X$ corresponding to the subgroup $L$. We say that a $[G,2]$-complex is *L-Cockcroft* if the Hurewicz map $h_L:\pi_2(X)\to H_2(X_L)$ is trivial. In case $L=G$ we call $X$ *Cockcroft.* There are interesting classes of 2-complexes that have the Cockcroft property. A $[G,2]$-complex $X$ is *aspherical* if $\pi_2(X)=0$. It was observed in [**4**] that a subcomplex of an aspherical 2-complex is Cockcroft. The Cockcroft property is of interest to group theorists as well. Let $X$ be a $[G,2]$-complex modelled on a presentation $\langle S;R\rangle$ of the group $G$. If it can be shown that $X$ is Cockcroft, then it follows from Hopf's theorem (see [**2**, p. 31]) that $H_2(G)$ is isomorphic to $H_2(X)$. In particular $H_2(G)$ is free abelian. For a survey on the Cockcroft property see Dyer [**5**]. A collection $\{G_\alpha:\alpha\in\Omega\}$ of subgroups of a group $G$ that is totally ordered by inclusion is called a *chain of subgroups of G*. Defining $\beta\leq\alpha$ if and only if $G_\alpha\leq G_\beta$ makes $\Omega$ into a totally ordered set. The main result of this paper is the following theorem.

THEOREM 1. *Let* $\{G_\alpha:\alpha\in\Omega\}$ *be a chain of subgroups of a group G. A* $[G,2]$-*complex X that is* $G_\alpha$-*Cockcroft for all* $\alpha\in\Omega$ *is also* $\left(\bigcap_{\alpha\in\Omega}G_\alpha\right)$-*Cockcroft.*

Theorem 1 together with Zorn's lemma give the next result.

COROLLARY 1. *Let X be a Cockcroft* $[G,2]$-*complex. Then G contains a minimal subgroup L such that X is L-Cockcroft.*

It is a longstanding open question raised by J. H. C. Whitehead [**9**] whether a subcomplex of an aspherical complex is aspherical. Suppose $X$ is a subcomplex of an aspherical 2-complex $Y$ and denote by $K$ the kernel of the map $\pi_1(X)-\pi_1(Y)$ induced by inclusion. J. F. Adams [**1**] showed that if $X$ is not aspherical then $K$ contains a nontrivial perfect subgroup. He studied a certain system of coverings $\{X_{K_\alpha}\}_{\alpha\in\Omega}$ of $X_K$, where $\{K_\alpha\}_{\alpha\in\Omega}$ is the set of characteristic subgroups of $K$ such that the quotients $K/K_\alpha$ are $C$-conservative for any abelian group $C$. A group $G$ is $C$-conservative if the functor $C\otimes_{CG}$-detects monomorphisms between projective $CG$-modules; i.e. if $\Psi:P\to Q$ is a homomorphism between projective $CG$-modules and $C\otimes_{CG}\Psi:C\otimes_{CG}P\to C\otimes_{CG}Q$ is injective, then $\Psi$ is injective (see also Howie [**8**]). Adams observed that $N$, the intersection of all groups $K_\alpha$, is perfect and that $H_2(X_N)=0$. If one assumes $X$ to be non-aspherical, then the second homology of the universal covering of $X$ is non-trivial. Thus $X_N$ is different from the universal covering and therefore $N$ is non-trivial (see also Howie [**6**] and [**7**]).

The proof of Theorem 1 relies on a lemma that deals with arbitrary systems of coverings $\{X_{G_\alpha}\}_{\alpha\in\Omega}$ of a $[G,2]$-complex $X$. We show that $H_2(X_N)$ embeds in $\varprojlim H_2(X_{G_\alpha})$, where $N$ is the intersection of all the $G_\alpha$. We use this result also to characterize non-asphericity of a 2-complex $X$ with $H_2(X)=0$ by the existence of a certain minimal subgroup of $\pi_1(X)$.

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THEOREM 2. *Let $X$ be a $[G,2]$-complex with $H_2(X)=0$. The following statements are equivalent*:

(i) *$X$ is non-aspherical*;

(ii) *there exists a non-trivial minimal subgroup $L$ of $G$ such that $H_2(X_L)=0$.*

*Furthermore, if $X$ is non-aspherical, then no group $L$ as in* (ii) *can have a nontrivial $\mathbb{Z}$-conservative quotient*; *in particular $L_{ab}$ is torsion.*

Assume now that $X$ is a subcomplex of an aspherical 2-complex $Y$. As before let $K$ denote the kernel of the homomorphism $\pi_1(X)\to\pi_1(Y)$ induced by the inclusion map. The covering complex $X_K$ of $X$ can be viewed as a subcomplex of the universal covering complex $\tilde{Y}$ of $Y$. Since $X_K$ and $\tilde{Y}$ are 2-complexes, the map $H_2(X_K)\to H_2(\tilde{Y})$ induced by inclusion is injective. Since $H_2(\tilde{Y})=\pi_2(\tilde{Y})=0$ it follows that $H_2(X_K)=0$. Theorem 2 applied to the complex $X_K$ together with the fact that $X$ is non-aspherical if and only if $X_K$ is non-aspherical, yield the following result.

COROLLARY 2. *Let $X$ be a $[G,2]$-complex that is a subcomplex of an aspherical 2-complex $Y$. Let $K$ be the kernel of the homomorphism $\pi_1(X)\to\pi_1(Y)$ induced by inclusion. The following statements are equivalent*:

(i) *$X$ is non-aspherical*;

(ii) *there exists a nontrivial minimal subgroup $L$ of $K$ such that $H_2(X_L)=0$.*

*Furthermore, if $X$ is non-aspherical, then no group $L$ as in* (ii) *can have a non-trivial $\mathbb{Z}$-conservative quotient*; *in particular $L_{ab}$ is torsion.*

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**Proof of results.** Let $X$ be a $[G,2]$-complex and let $\{G_\alpha:\alpha\in\Omega\}$ be a chain of subgroups of $G$. Denote by $\tilde{X}$ the universal covering complex of $X$ and by $p$ the covering projection

$$p:\tilde{X}\to X.$$

The preimage $p^{-1}(c)$ of each open cell $c$ in $X$ consists of open cells $\tilde{c}_g$, $g\in G$, such that

$$p|_{\tilde{c}_g}:\tilde{c}_g\to c$$

is a homeomorphism. For each $G_\alpha$, the orbit complex $\tilde{X}/G_\alpha$, denoted by $X_\alpha$, is the covering complex $X_{G_\alpha}$ with covering projection

$$p_\alpha:\tilde{X}\to X_\alpha.$$

Denote by $N$ the intersection $\bigcap_{\alpha\in\Omega} G_\alpha$ and by $p_N$ the covering projection

$$p_N:\tilde{X}\to X_N.$$

Let $p_{\alpha N}$ be the covering projection

$$p_{\alpha N}:X_N\to X_\alpha$$

and let $p_{\beta\alpha}$ be the covering projection

$$p_{\beta\alpha}:X_\alpha\to X_\beta$$

for $\alpha\geq\beta$. The cells in $X_N$ and in $X_\alpha$ are just $N$ and $G_\alpha$ orbits of cells in $\tilde{X}$. So if $N*\tilde{c}=\{n*\tilde{c}:n\in N,\ \tilde{c}\text{ an open cell of }\tilde{X}\}$ is an open cell of $X_N$, then $p_{\alpha N}$ sends this open cell homeomorphically onto the open cell $G_\alpha * \tilde{c}$ of $X_{G_\alpha}$ and $p_{\beta\alpha}$ sends the open cell $G_\alpha * \tilde{c}$ of $X_\alpha$ homeomorphically onto the open cell $G_\beta * \tilde{c}$ of $X_\beta$ for $\alpha \geq \beta$. Now $(C_2(X_\alpha), p_{\alpha\beta_*})_{\alpha,\beta\in\Omega}$ is an inverse system of Abelian groups with inverse limit $\varprojlim C_2(X_\alpha)$.

LEMMA 1. *$\varprojlim p_{\alpha N_*}: C_2(X_N) \to \varprojlim C_2(X_\alpha)$ is injective and yields an injection from $H_2(X_N)$ to $\varprojlim H_2(X_\alpha)$ when restricted to $H_2(X_N)$; in particular, if all the $H_2(X_\alpha)$ are trivial, then $H_2(X_N)$ is trivial.*

*Proof.* First we show that if $c_1 = N * \tilde{c}_1$ and $c_2 = N * \tilde{c}_2$ are two different open cells in $X_N$, then there exists an element $\beta \in \Omega$ such that $p_{\beta N}(c_1)$ and $p_{\beta N}(c_2)$ are two different open cells in $X_\beta$. Suppose not. Then

$$G_\alpha * \tilde{c}_1 = G_\alpha * \tilde{c}_2$$

for all $\alpha \in \Omega$. So, in particular,

$$\tilde{c}_1 \in G_\alpha * \tilde{c}_2$$

for all $\alpha \in \Omega$. Then for each $\alpha \in \Omega$ there exists a $g_\alpha$ in $G_\alpha$ such that

$$\tilde{c}_1 = g_\alpha * \tilde{c}_2.$$

Fix an element $\gamma \in \Omega$; then $g_\alpha * \tilde{c}_2 = \tilde{c}_1 = g_\gamma * \tilde{c}_2$ for all $\alpha \in \Omega$; hence $g_\gamma^{-1} g_\alpha * \tilde{c}_2 = \tilde{c}_2$ for all $\alpha \in \Omega$. Since $G$ acts freely on the set of open cells of $\tilde{X}$ this says that $g_\gamma^{-1} g_\alpha = 1$; thus $g_\gamma = g_\alpha \in G_\alpha$ for all $\alpha \in \Omega$ and therefore $g_\gamma$ is an element of the intersection $N$. Since

$$\tilde{c}_1 = g_\gamma * \tilde{c}_2,$$

we have $c_1 = N * \tilde{c}_1 = N * \tilde{c}_2 = c_2$, which contradicts our assumption that $c_1$ and $c_2$ are different cells. Suppose now that

$$z = \sum_{k=1}^{m} n_k c_k,$$

is a nontrivial element of $C_2(X_N)$, so that the integers $n_k$ are nonzero and the cells $c_k$ are different 2-cells of $X_N$. If $m = 1$, then

$$p_{\alpha N_*}(z) = n_1 p_{\alpha N}(c_1) \neq 0$$

for all $\alpha \in \Omega$. If $m > 1$ then for every pair $\{i, j\}$, $i, j \in \{1, \ldots, m\}$, we can find an element $\beta(i,j) \in \Omega$ such that $p_{\beta(i,j)N}(c_i)$ and $p_{\beta(i,j)N}(c_j)$ are two different 2-cells of $X_{\beta(i,j)}$. Let $\beta$ be the largest element among the finitely many $\beta(i,j)$. Then $p_{\beta N}(c_i)$ and $p_{\beta N}(c_j)$ are different cells for any pair $(i,j)$, $i, j \in \{1, \ldots, m\}$, so

$$p_{\beta N_*}(z) = \sum_{k=1}^{m} n_k p_{\beta N}(c_k) \neq 0.$$

This shows that

$$\varprojlim p_{\alpha N_*}(z) \neq 0.$$

LEMMA 2. $(\varprojlim p_{\alpha N_*}) \circ h_N = \varprojlim h_\alpha$.

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*Proof.* From the commutative diagram

$$\begin{array}{ccccccc} \pi_2(X) & \xrightarrow{p_{\#}^{-1}} & \pi_2(\tilde{X}) & \xrightarrow{h} & H_2(\tilde{X}) & \longrightarrow & C_2(\tilde{X}) \\ & \searrow & & & \downarrow{\scriptstyle p_{N_*}} & & \downarrow \\ & & \overset{h_N}{\searrow} & & H_2(X_L) & \longrightarrow & C_2(X_L) \end{array}$$

we see that for every $\alpha \in \Omega$,

$$p_{\alpha N_*} \circ h_N = p_{\alpha N_*} \circ p_{N_*} \circ h \circ p_{\#}^{-1} = p_{\alpha_*} \circ h \circ p_{\#}^{-1} = h_\alpha.$$

Hence $(\varprojlim p_{\alpha N_*}) \circ h_N = \varprojlim h_\alpha$.

*Proof of Theorem* 1. Since $X$ is $G_\alpha$-Cockcroft for every $\alpha \in \Omega$, each $h_\alpha$ is the zero map. Hence $\varprojlim h_\alpha$ is the zero map. Lemma 2 and the fact that, by Lemma 1, $\varprojlim p_{\alpha N_*}$ is injective show that $h_N$ is the zero map as well. So $X$ is $N$-Cockcroft.

*Proof of Theorem* 2. Only the direction (i) $\Rightarrow$ (ii) requires a proof. If $\{G_\alpha : \alpha \in \Omega\}$ is a chain of subgroups of $G$ such that $H_2(X_\alpha) = 0$ for all $\alpha$, then $H_2(X_N) = 0$ by Lemma 1; as before $X_\alpha$ is the 2-complex $\tilde{X}_{G_\alpha}$ and $N$ is the intersection of all the $G_\alpha$. The existence of a minimal subgroup $L$ such that $H_2(X_L) = 0$ now follows from Zorn's Lemma. If $L/K$ were a non-trivial $\mathbb{Z}$-conservative quotient of $L$, then $K$ would be a proper subgroup of $L$ with $H_2(X_K) = 0$ by definition of $\mathbb{Z}$-conservative. This contradicts minimality of $L$.

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