4
Nonperturbative corrections

The effective Lagrangian for heavy quarks has an expansion in powers of $\alpha_s(m_Q)$ and $1/m_Q$. The α_s corrections were discussed in the previous chapter; the $1/m_Q$ corrections are discussed here. By dimensional analysis, these corrections are proportional to Λ_{QCD}/m_Q, necessarily involve the hadronic scale Λ_{QCD}, and are nonperturbative in origin. By using the effective Lagrangian approach, we can systematically include these nonperturbative corrections in computations involving hadrons containing a heavy quark.

4.1 The $1/m_Q$ expansion

The HQET Lagrangian including $1/m_Q$ corrections can be derived from the QCD Lagrangian following the procedure of Sec. 2.6. Substituting Eq. (2.43) into the QCD Lagrangian gives

$$\mathcal{L} = \bar{Q}_v\,(i v\cdot D)\,Q_v - \bar{\mathfrak{Q}}_v(i v\cdot D + 2m_Q)\mathfrak{Q}_v + \bar{Q}_v i \not{D}\mathfrak{Q}_v + \bar{\mathfrak{Q}}_v i \not{D} Q_v, \qquad (4.1)$$

using $\not{v} Q_v = Q_v$ and $\not{v}\mathfrak{Q}_v = -\mathfrak{Q}_v$. It is convenient to project four vectors into components parallel and perpendicular to the velocity v. The perpendicular component of any four-vector X is defined by

$$X_\perp^\mu \equiv X^\mu - X\cdot v v^\mu. \qquad (4.2)$$

The $i\not{D}$ factors in Eq. (4.1) can be replaced by $i\not{D}_\perp$ since $\bar{Q}_v\not{v}\mathfrak{Q}_v = 0$.

The field $\mathfrak{Q}_v$ corresponds to an excitation with mass $2m_Q$, which is the energy required to create a heavy quark–antiquark pair. Here $\mathfrak{Q}_v$ can be integrated out of the theory for physical situations where the use of HQET is justified. This can be done at tree level by solving the $\mathfrak{Q}_v$ equation of motion,

$$(i v\cdot D + 2m_Q)\mathfrak{Q}_v = i\not{D}_\perp Q_v, \qquad (4.3)$$

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and substituting back into the Lagrangian Eq. (4.1), to give

$$\mathcal{L} = \bar{Q}_v\left(iv\cdot D + i\not{D}_\perp \frac{1}{2m_Q + iv\cdot D} i\not{D}_\perp\right)Q_v$$
$$= \bar{Q}_v\left(iv\cdot D - \frac{1}{2m_Q}\not{D}_\perp\not{D}_\perp\right)Q_v + \cdots, \tag{4.4}$$

where the ellipses denote terms of higher order in the $1/m_Q$ expansion. It is convenient to express the term suppressed by $1/m_Q$ as a sum of two terms, one that violates heavy quark spin symmetry and one that doesn't. Specifically,

$$\not{D}_\perp\not{D}_\perp = \gamma_\mu\gamma_\nu D_\perp^\mu D_\perp^\nu = D_\perp^2 + \frac{1}{2}[\gamma_\mu, \gamma_\nu]D_\perp^\mu D_\perp^\nu. \tag{4.5}$$

Using the identity $[D^\mu, D^\nu] = igG^{\mu\nu}$, and the definition $\sigma_{\mu\nu} = i[\gamma_\mu, \gamma_\nu]/2$, this becomes

$$\not{D}_\perp\not{D}_\perp = D_\perp^2 + \frac{g}{2}\sigma_{\mu\nu}G^{\mu\nu}. \tag{4.6}$$

It is not necessary to include any $\perp$ labels on the μ and ν indices of the $\sigma_{\mu\nu}$ term, since $\bar{Q}_v\sigma_{\mu\nu}v^\mu Q_v = 0$. Substituting Eq. (4.6) into Eq. (4.4) gives

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \cdots, \tag{4.7}$$

where $\mathcal{L}_0$ is the lowest order Lagrangian Eq. (2.45), and

$$\mathcal{L}_1 = -\bar{Q}_v\frac{D_\perp^2}{2m_Q}Q_v - g\bar{Q}_v\frac{\sigma_{\mu\nu}G^{\mu\nu}}{4m_Q}Q_v. \tag{4.8}$$

In the nonrelativistic constituent quark model, the term $\bar{Q}_v(D_\perp^2/2m_Q)Q_v$ is the heavy quark kinetic energy $\boldsymbol{p}_Q^2/2m_Q$. It breaks heavy quark flavor symmetry because of the explicit dependence on m_Q, but it does not break heavy quark spin symmetry. The magnetic moment interaction term $-g\bar{Q}_v(\sigma_{\mu\nu}G^{\mu\nu}/4m_Q)Q_v$ breaks both heavy quark spin and flavor symmetries.

Equation (4.8) has been derived at tree level. Including loop corrections changes the Lagrangian to

$$\mathcal{L}_1 = -\bar{Q}_v\frac{D_\perp^2}{2m_Q}Q_v - a(\mu)\,g\bar{Q}_v\frac{\sigma_{\mu\nu}G^{\mu\nu}}{4m_Q}Q_v. \tag{4.9}$$

The tree-level matching calculation Eq. (4.8) implies that

$$a(m_Q) = 1 + \mathcal{O}[\alpha_s(m_Q)]. \tag{4.10}$$

The μ dependence of the magnetic moment operator is canceled by the μ dependence of $a(\mu)$. In the leading logarithmic approximation

$$a(\mu) = \left[\frac{\alpha_s(m_Q)}{\alpha_s(\mu)}\right]^{9/(33-2N_q)}, \tag{4.11}$$

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where N_q is the number of light quark flavors. Loop effects do not change the coefficient of the heavy quark kinetic energy term. In the next section it is shown that this is a consequence of the reparameterization invariance of the effective Lagrangian.

4.2 Reparameterization invariance

The heavy quark momentum p_Q is given by

$$p_Q = m_Q v + k, \tag{4.12}$$

where v is the heavy quark four velocity and k is its residual momentum. This decomposition of p_Q into v and k is not unique. Typically k is of the order of $\Lambda_{\rm QCD}$, which is much smaller than m_Q. A small change in the four velocity of the order of $\Lambda_{\rm QCD}/m_Q$ can be compensated by a change in the residual momentum:

$$\begin{aligned} v &\to v + \varepsilon/m_Q, \\ k &\to k - \varepsilon. \end{aligned} \tag{4.13}$$

Since the four velocity satisfies $v^2 = 1$, the parameter ε must satisfy

$$v \cdot \varepsilon = 0, \tag{4.14}$$

neglecting terms of order $(\varepsilon/m_Q)^2$. In addition to the changes of v and k in Eqs. (4.13), the heavy quark spinor Q_v must also change to preserve the constraint $\not{v} Q_v = Q_v$. Consequently, if

$$Q_v \to Q_v + \delta Q_v, \tag{4.15}$$

δQ_v satisfies

$$\left(\not{v} + \frac{\not{\varepsilon}}{m_Q}\right)(Q_v + \delta Q_v) = Q_v + \delta Q_v. \tag{4.16}$$

At linear order in (ε/m_Q), one finds

$$(1 - \not{v})\delta Q_v = \frac{\not{\varepsilon}}{m_Q} Q_v. \tag{4.17}$$

Therefore a suitable choice for the change in Q_v is

$$\delta Q_v = \frac{\not{\varepsilon}}{2m_Q} Q_v. \tag{4.18}$$

This satisfies $\not{v}\delta Q_v = -\delta Q_v$, since $v \cdot \varepsilon = 0$, so that Eq. (4.17) holds. The solution to Eq. (4.17) is not unique, and we have chosen one that preserves the normalization of the $i v \cdot D$ term. Other choices are equivalent to the above by a simple redefinition of the field.

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In summary, the Lagrange density in Eq. (4.7) must be invariant under the combined changes

$$\begin{aligned} v &\to v + \varepsilon/m_Q, \\ Q_v &\to e^{i\varepsilon\cdot x}\left(1 + \frac{\not{\varepsilon}}{2m_Q}\right)Q_v, \end{aligned} \tag{4.19}$$

where the prefactor $e^{i\varepsilon\cdot x}$ causes a shift in the residual momentum $k \to k - \varepsilon$. Under the transformation in Eq. (4.19),

$$\begin{aligned} \mathcal{L}_0 &\to \mathcal{L}_0 + \frac{1}{m_Q}\bar{Q}_v(i\varepsilon\cdot D)Q_v, \\ \mathcal{L}_1 &\to \mathcal{L}_1 - \frac{1}{m_Q}\bar{Q}_v(i\varepsilon\cdot D)Q_v. \end{aligned} \tag{4.20}$$

Consequently the Lagrangian, $\mathcal{L}_0 + \mathcal{L}_1$, is reparameterization invariant. This would not be the case if the coefficient of the kinetic energy deviated from unity. There can be no corrections to the coefficient of the kinetic energy operator as long as the theory is regularized in a way that preserves reparameterization invariance. Dimensional regularization is such a regulator, since the arguments made in this section hold in n dimensions.

An important feature of reparameterization invariance is that it connects different orders in the $1/m_Q$ expansion, since the transformation Eq. (4.19) explicitly involves m_Q. Thus it can be used to fix the form of some $1/m_Q$ corrections using only information from lower order terms in $1/m_Q$, as was done for the kinetic energy term.

4.3 Masses

Heavy quark symmetry can be used to obtain relations between hadron masses. The hadron mass in the effective theory is $m_H - m_Q$, since the heavy quark mass m_Q has been subtracted from all energies in the field redefinition in Eq. (2.43). At order m_Q, all heavy hadrons containing Q are degenerate, and have the same mass m_Q. At the order of unity, the hadron masses get the contribution

$$\frac{1}{2}\langle H^{(Q)}|\mathcal{H}_0|H^{(Q)}\rangle \equiv \bar{\Lambda}, \tag{4.21}$$

where $\mathcal{H}_0$ is the order $1/m_Q^0$ terms in the HQET Hamiltonian obtained from the Lagrangian term $\bar{Q}_v(iv\cdot D)Q_v$, as well as the terms involving light quarks and gluons. In this section, the hadron states $|H^{(Q)}\rangle$ are in the effective theory with $v = v_r = (1, \mathbf{0})$. The factor $1/2$ arises from the normalization introduced in Sec. 2.7. Here $\bar{\Lambda}$ is a parameter of HQET and has the same value for all particles in a spin-flavor multiplet. The values will be denoted by $\bar{\Lambda}$ for the B, B^*, D,

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and D^* states, $\bar{\Lambda}_\Lambda$ for the Λ_b and Λ_c, and $\bar{\Lambda}_\Sigma$ for the Σ_b, Σ_b^*, Σ_c, and Σ_c^*. In the $SU(3)$ limit, $\bar{\Lambda}$ does not depend on the light quark flavor. If $SU(3)$ breaking is included, $\bar{\Lambda}$ is different for the $B_{u,d}$ and B_s mesons, and will be denoted by $\bar{\Lambda}_{u,d}$ and $\bar{\Lambda}_s$, respectively.

At order $1/m_Q$, there is an additional contribution to the hadron masses given by the expectation value of the $1/m_Q$ correction to the Hamiltonian:

$$\mathcal{H}_1 = -\mathcal{L}_1 = \bar{Q}_v \frac{D_\perp^2}{2m_Q} Q_v + a(\mu) g \bar{Q}_v \frac{\sigma_{\alpha\beta} G^{\alpha\beta}}{4m_Q} Q_v. \tag{4.22}$$

The matrix elements of the two terms in Eq. (4.22) define two nonperturbative parameters, λ_1 and λ_2:

$$\begin{aligned} 2\lambda_1 &= -\langle H^{(Q)} | \bar{Q}_{v_r} D_\perp^2 Q_{v_r} | H^{(Q)} \rangle, \\ 16(\mathbf{S}_Q \cdot \mathbf{S}_\ell)\lambda_2(m_Q) &= a(\mu) \langle H^{(Q)} | \bar{Q}_{v_r} g\sigma_{\alpha\beta} G^{\alpha\beta} Q_{v_r} | H^{(Q)} \rangle. \end{aligned} \tag{4.23}$$

Here λ_1 is independent of m_Q, and λ_2 depends on m_Q through the logarithmic m_Q dependence of $a(\mu)$ in Eq. (4.11); $\lambda_{1,2}$ have the same value for all states in a given spin-flavor multiplet and are expected to be of the order of $\Lambda_{\rm QCD}^2$. The naive expectation that the heavy quark kinetic energy is positive suggests that λ_1 should be negative. The λ_2 matrix element transforms like $\mathbf{S}_Q \cdot \mathbf{S}_\ell$ under the spin symmetry, since that is the transformation property of $\bar{Q}_{v_r} \sigma_{\alpha\beta} G^{\alpha\beta} Q_{v_r}$. Only the two upper components of Q_{v_r} are nonzero, since $\gamma^0 Q_{v_r} = Q_{v_r}$, and $\bar{Q}_{v_r} \sigma_{\alpha\beta} G^{\alpha\beta} Q_{v_r}$ reduces to the matrix element of $\bar{Q}_{v_r} \boldsymbol{\sigma} \cdot \boldsymbol{B} Q_{v_r}$, where $\boldsymbol{B}$ is the chromomagnetic field. The operator $\bar{Q}_{v_r} \boldsymbol{\sigma} Q_{v_r}$ is the heavy quark spin, and the matrix element of $\boldsymbol{B}$ in the hadron must be proportional to the spin of the light degrees of freedom, by rotational invariance and time-reversal invariance, so that the chromomagnetic operator contribution is proportional to $\mathbf{S}_Q \cdot \mathbf{S}_\ell$. Using $\mathbf{S}_Q \cdot \mathbf{S}_\ell = (\mathbf{J}^2 - \mathbf{S}_Q^2 - \mathbf{S}_\ell^2)/2$, one finds that

$$\begin{aligned} m_B &= m_b + \bar{\Lambda} - \frac{\lambda_1}{2m_b} - \frac{3\lambda_2(m_b)}{2m_b}, \\ m_{B^*} &= m_b + \bar{\Lambda} - \frac{\lambda_1}{2m_b} + \frac{\lambda_2(m_b)}{2m_b}, \\ m_{\Lambda_b} &= m_b + \bar{\Lambda}_\Lambda - \frac{\lambda_{\Lambda,1}}{2m_b}, \\ m_{\Sigma_b} &= m_b + \bar{\Lambda}_\Sigma - \frac{\lambda_{\Sigma,1}}{2m_b} - \frac{2\lambda_{\Sigma,2}(m_b)}{m_b}, \\ m_{\Sigma_b^*} &= m_b + \bar{\Lambda}_\Sigma - \frac{\lambda_{\Sigma,1}}{2m_b} + \frac{\lambda_{\Sigma,2}(m_b)}{m_b}, \\ m_D &= m_c + \bar{\Lambda} - \frac{\lambda_1}{2m_c} - \frac{3\lambda_2(m_c)}{2m_c}, \end{aligned} \tag{4.24}$$

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$$
\begin{aligned}
m_{D^*} &= m_c + \bar{\Lambda} - \frac{\lambda_1}{2m_c} + \frac{\lambda_2(m_c)}{2m_c}, \\
m_{\Lambda_c} &= m_c + \bar{\Lambda}_\Lambda - \frac{\lambda_{\Lambda,1}}{2m_c}, \\
m_{\Sigma_c} &= m_c + \bar{\Lambda}_\Sigma - \frac{\lambda_{\Sigma,1}}{2m_c} - \frac{2\lambda_{\Sigma,2}(m_c)}{m_c}, \\
m_{\Sigma_c^*} &= m_c + \bar{\Lambda}_\Sigma - \frac{\lambda_{\Sigma,1}}{2m_c} + \frac{\lambda_{\Sigma,2}(m_c)}{m_c}.
\end{aligned}
$$

The average mass of a heavy quark spin symmetry multiplet, e.g., $(3m_{P^*} + m_P)/4$ for the meson multiplet, does not depend on λ_2. The magnetic interaction λ_2 is responsible for the $B^* - B$ and $D^* - D$ splittings. The observed value of the $B^* - B$ mass difference gives $\lambda_2(m_b) \simeq 0.12\ \text{GeV}^2$.

Equations (4.24) give the meson mass relation

$$0.49\ \text{GeV}^2 \simeq m_{B^*}^2 - m_B^2 \simeq 4\lambda_2 \simeq m_{D^*}^2 - m_D^2 \simeq 0.55\ \text{GeV}^2, \qquad (4.25)$$

up to corrections of order $1/m_{b,c}$, and ignoring the weak m_Q dependence of λ_2. Similarly, one finds that

$$
\begin{aligned}
90 \pm 3\ \text{MeV} = m_{B_s} - m_{B_d} = \bar{\Lambda}_s - \bar{\Lambda}_{u,d} = m_{D_s} - m_{D_d} = 99 \pm 1\ \text{MeV}, \\
345 \pm 9\ \text{MeV} = m_{\Lambda_b} - m_B = \bar{\Lambda}_\Lambda - \bar{\Lambda}_{u,d} = m_{\Lambda_c} - m_D = 416 \pm 1\ \text{MeV}.
\end{aligned}
\qquad (4.26)
$$

The parameters λ_1 and λ_2 are nonperturbative parameters of QCD and have not been computed from first principles. It might appear that very little has been gained by using Eqs. (4.24) for the hadron masses in terms of $\bar{\Lambda}$, λ_1, and λ_2. However, the same hadronic matrix elements also occur in other quantities, such as form factors and decay rates. One can then use the values of $\bar{\Lambda}$, λ_1, and λ_2 obtained by fitting to the hadron masses to compute the form factors and decay rates, without making any model dependent assumptions. An example of this is given in Problems 2–3.

4.4 $\Lambda_b \to \Lambda_c e \bar{\nu}_e$ decay

The HQET predictions for $\Lambda_b \to \Lambda_c$ form factors were discussed earlier in Sec. 2.11. Recall that the most general form factors are

$$
\begin{aligned}
\langle \Lambda_c(p', s') | \bar{c} \gamma^\nu b | \Lambda_b(p, s) \rangle &= \bar{u}(p', s') [f_1 \gamma^\nu + f_2 v^\nu + f_3 v'^\nu] u(p, s), \\
\langle \Lambda_c(p', s') | \bar{c} \gamma^\nu \gamma_5 b | \Lambda_b(p, s) \rangle &= \bar{u}(p', s') [g_1 \gamma^\nu + g_2 v^\nu + g_3 v'^\nu] \gamma_5 u(p, s),
\end{aligned}
\qquad (4.27)
$$

where $p' = m_{\Lambda_c} v'$ and $p = m_{\Lambda_b} v$. It is convenient for the HQET analysis to consider the form factors f_j and g_j as functions of the dimensionless variable

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$w = v \cdot v'$. Heavy quark symmetry implies that

$$\langle \Lambda_c(v', s')|\bar{c}_{v'}\Gamma b_v|\Lambda_b(v, s)\rangle = \zeta(w)\, \bar{u}(v', s')\Gamma u(v, s), \tag{4.28}$$

with $\zeta(1) = 1$. Consequently the form factors are

$$f_1 = g_1 = \zeta(w), \qquad f_2 = f_3 = g_2 = g_3 = 0. \tag{4.29}$$

In Sec. 3.4 perturbative QCD corrections to the matching of heavy quark currents were computed. For the vector current, new operators of the form $v^\mu \bar{c}_{v'} b_v$ and $v'^\mu \bar{c}_{v'} b_v$ were induced with calculable coefficients. These additional terms do not represent any loss of predictive power because Eq. (4.28) gives the matrix elements of these new operators in terms of the same Isgur-Wise function $\zeta(w)$.

In this section, nonperturbative corrections suppressed by $\Lambda_{\rm QCD}/m_{c,b}$ are considered. These corrections arise from two sources. There are time-ordered products of the $1/m_Q$ terms in the Lagrangian with the heavy quark current. These terms can be thought of as correcting the hadron states in HQET at order $1/m_Q$, or equivalently, as producing a $1/m_Q$ correction to the current, and leaving the states unchanged. For example, the chromomagnetic $1/m_c$ correction to the Lagrangian gives a correction to the current $\bar{c}_{v'}\Gamma b_v$ of

$$-i\frac{a(\mu)}{2}\int d^4x\, T\left(g\bar{c}_{v'}\frac{\sigma^{\mu\nu}G_{\mu\nu}}{2m_c}c_{v'}\Big|_x \; \bar{c}_{v'}\Gamma b_v\Big|_0\right). \tag{4.30}$$

Spin symmetry implies that for $\Lambda_b \to \Lambda_c$ matrix elements in HQET, the above quark–gluon operator is equivalent to the hadronic operator

$$\bar{\Lambda}^{(c)}(v', s')\sigma_{\mu\nu}\frac{(1+\not{v}')}{2}\Gamma\Lambda^{(b)}(v, s)\frac{X^{\mu\nu}}{m_c}, \tag{4.31}$$

where $X_{\mu\nu}$ depends on v and v' and is antisymmetric in its indices μ and ν. The $\sigma_{\mu\nu}$ matrix must be next to $\bar{\Lambda}^{(c)}(v', s')$, and the Γ matrix must be next to $\Lambda^{(b)}(v, s)$ because these matrices were next to $\bar{c}_{v'}$ and b_v in Eq. (4.30). The projector $(1+\not{v}')/2$ arises because $\sigma_{\mu\nu}$ and Γ were multiplied on the right and left by $c_{v'}$ and $\bar{c}_{v'}$, respectively, in Eq. (4.30). The only possibility for X is $X_{\mu\nu} \propto v_\mu v'_\nu - v_\nu v'_\mu$, with the constant of proportionality a function of w. With this form for $X_{\mu\nu}$, Eq. (4.31) is zero since $(1+\not{v}')\sigma^{\mu\nu}(1+\not{v}')v'_\mu = 0$. Thus the chromomagnetic $1/m_c$ correction to the charm quark part of the Lagrangian has no effect on the $\Lambda_b \to \Lambda_c e\bar{\nu}_e$ form factors. Clearly, the same conclusion holds for the $1/m_b$ chromomagnetic correction to the bottom quark part of the Lagrangian.

The kinetic energies of the bottom and charm quarks do not violate heavy quark spin symmetry so they preserve $f_2 = f_3 = g_2 = g_3 = 0$ and can be absorbed into a redefinition of the Isgur-Wise function $\zeta(w)$. It is important to know if this correction to ζ preserves the normalization condition $\zeta(1) = 1$ at zero recoil. One can show that the normalization is preserved by an argument similar to that used in proving the Ademollo-Gatto theorem. The $1/m_Q$ kinetic energy term in the Lagrange density changes the $|\Lambda_Q(v,s)\rangle$ state in HQET to the state $|\Lambda_Q(v,s)\rangle + (\varepsilon/m_Q)|S_Q(v,s)\rangle + \cdots$, where $|S_Q(v,s)\rangle$ is a state orthogonal to $|\Lambda_Q(v,s)\rangle$, ε is of the order of $\Lambda_{\rm QCD}$, and the ellipses denote terms suppressed by more powers of $1/m_Q$. At zero recoil, $\bar{c}_v \Gamma b_v$ is a charge of heavy quark spin flavor symmetry so it takes $|\Lambda_b(v,s)\rangle$ to the state $|\Lambda_c(v,s)\rangle$, which is orthogonal to $|S_c(v,s)\rangle$. Consequently at order $1/m_Q$ the heavy quark kinetic energies preserve Eq. (4.29) and do not change the normalization of ζ at zero recoil. Equivalently, one can use an analysis analogous to that for the chromomagnetic operator. The time-ordered product

$$-i \int d^4x\, T\left(g\bar{c}_{v'} \frac{D_\perp^2}{2m_c} c_{v'}\bigg|_x \bar{c}_{v'} \Gamma b_v \bigg|_0 \right) \tag{4.32}$$

is equivalent to the hadronic operator

$$\bar{\Lambda}^{(c)}(v',s') \frac{(1+\not{v}')}{2} \Gamma \Lambda^{(b)}(v,s) \frac{\chi_1}{m_c}, \tag{4.33}$$

where χ_1 is an arbitrary function of w. Similarly, the b-quark kinetic energy gives a correction term

$$\bar{\Lambda}^{(c)}(v',s') \Gamma \frac{(1+\not{v})}{2} \Lambda^{(b)}(v,s) \frac{\chi_1}{m_b}. \tag{4.34}$$

The two χ_1's are the same (see Problem 4), since one can relate the form of the matrix elements of the two possible time-ordered products by $v \leftrightarrow v'$ and $c \leftrightarrow b$. Equations (4.33) and (4.34) give the following correction terms to the form factors:

$$\begin{aligned} \delta f_1 &= \chi_1 \left(\frac{1}{m_c} + \frac{1}{m_b} \right), \\ \delta g_1 &= \chi_1 \left(\frac{1}{m_c} + \frac{1}{m_b} \right), \\ \delta f_2 &= \delta f_3 = \delta g_2 = \delta g_3 = 0. \end{aligned} \tag{4.35}$$

This corresponds to a redefinition of the Isgur-Wise function:

$$\zeta(w) \to \zeta(w) + \chi_1(w) \left(\frac{1}{m_c} + \frac{1}{m_b} \right). \tag{4.36}$$

At zero recoil, for $m_b = m_c$, the vector current matrix element is normalized, since it is a symmetry generator of the full QCD theory. Since $\zeta(1) = 1$, this implies that $\chi_1(1) = 0$. As a result, the effects of χ_1 can be reabsorbed into ζ by the redefinition in Eq. (4.36), without affecting the normalization at zero recoil.

In addition to the $1/m_Q$ corrections to the Lagrange density, there are order $1/m_Q$ terms that correct the relation between currents in full QCD and HQET.

These terms arise when one includes the $1/m_Q$ corrections to the relation between the quark fields in QCD and HQET. At tree level,

$$Q = e^{-im_Q v\cdot x}\left(1 + i\frac{\not{D}}{2m_Q}\right)Q_v, \tag{4.37}$$

where the relation in Eq. (2.43) and the solution for $\mathfrak{Q}_v$ in Eq. (4.3) have been used. One could equally well have a $\perp$ subscript on the covariant derivative. These two forms for Eq. (4.37) are equivalent, since the difference vanishes by the equation of motion $(v\cdot D)Q_v = 0$. Using Eq. (4.37) the relation between the QCD current and HQET operators to order $1/m_Q$ is

$$\begin{aligned} \bar{c}\gamma^\nu b &= \bar{c}_{v'}\left(\gamma^\nu - \frac{i\overleftarrow{D}_\mu}{2m_c}\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\frac{iD_\mu}{2m_b}\right)b_v, \\ \bar{c}\gamma^\nu\gamma_5 b &= \bar{c}_{v'}\left(\gamma^\nu\gamma_5 - \frac{i\overleftarrow{D}_\mu}{2m_c}\gamma^\mu\gamma^\nu\gamma_5 + \gamma^\nu\gamma_5\gamma^\mu\frac{iD_\mu}{2m_b}\right)b_v. \end{aligned} \tag{4.38}$$

Heavy quark spin symmetry implies for $\Lambda_b \to \Lambda_c$ matrix elements in HQET, one can use

$$\bar{c}_{v'}i\overleftarrow{D}_\mu\Gamma b_v = \bar{\Lambda}^{(c)}(v',s')\Gamma\Lambda^{(b)}(v,s)[Av_\mu + Bv'_\mu], \tag{4.39}$$

where A and B are functions of w. The equation of motion $(iv'\cdot D)c_{v'} = 0$ implies that contracting v'^μ into the above give zero, so

$$B = -Aw. \tag{4.40}$$

The function A can be expressed in terms of $\bar{\Lambda}_\Lambda$ and the Isgur-Wise function ζ. To show this note that

$$\begin{aligned} &\langle\Lambda_c(v',s')|i\partial_\mu(\bar{c}_{v'}\Gamma b_v)|\Lambda_b(v,s)\rangle \\ &\quad = \left[\left(m_{\Lambda_b} - m_b\right)v_\mu - \left(m_{\Lambda_c} - m_c\right)v'_\mu\right]\langle\Lambda_c(v',s')|\bar{c}_{v'}\Gamma b_v|\Lambda_b(v,s)\rangle \\ &\quad = \bar{\Lambda}_\Lambda(v-v')_\mu\zeta\bar{u}(v',s')\Gamma u(v,s). \end{aligned} \tag{4.41}$$

So for $\Lambda_b \to \Lambda_c$ matrix elements in HQET,

$$\begin{aligned} i\partial_\mu(\bar{c}_{v'}\Gamma b_v) &= \bar{c}_{v'}i\overleftarrow{D}_\mu\Gamma b_v + \bar{c}_{v'}\Gamma iD_\mu b_v \\ &= \bar{\Lambda}_\Lambda(v-v')_\mu\zeta\bar{\Lambda}^{(c)}(v',s')\Gamma\Lambda^{(b)}(v,s). \end{aligned} \tag{4.42}$$

Contracting v^μ into this and using the equation of motion $(iv^\mu D_\mu)b_v = 0$ implies that

$$A(1-w^2) = \bar{\Lambda}_\Lambda\zeta(1-w), \tag{4.43}$$

giving

$$A = \frac{\bar{\Lambda}_\Lambda \zeta(w)}{1+w}. \tag{4.44}$$

In summary, putting all the pieces together gives

$$\bar{c}_{v'} i\overleftarrow{D}_\mu \Gamma b_v = \frac{\bar{\Lambda}_\Lambda \zeta}{1+w}\ \bar{\Lambda}^{(c)}(v', s')\Gamma \Lambda^{(b)}(v, s)(v_\mu - w v'_\mu). \tag{4.45}$$

For the operator with the derivative on the bottom quark, one uses

$$\begin{aligned} \bar{c}_{v'} \Gamma i D_\mu b_v &= -(\bar{b}_v i\overleftarrow{D}_\mu \bar{\Gamma} c_{v'})^\dagger \\ &= -\frac{\bar{\Lambda}_\Lambda \zeta}{1+w}\ \bar{\Lambda}^{(c)}(v', s')\Gamma \Lambda^{(b)}(v, s)(v'_\mu - w v_\mu). \end{aligned} \tag{4.46}$$

Using these results with $\Gamma = \gamma^\mu \gamma^\nu$, and so on, together with the fact that the effect of $1/m_Q$ corrections to the Lagrangian can be absorbed into a redefinition of ζ, yields the following expression for the $\Lambda_b \to \Lambda_c e \bar{\nu}_e$ form factors at order $1/m_Q$:

$$\begin{aligned} f_1 &= \left[1 + \left(\frac{\bar{\Lambda}_\Lambda}{2m_c} + \frac{\bar{\Lambda}_\Lambda}{2m_b}\right)\right]\zeta(w), \\ f_2 &= -\frac{\bar{\Lambda}_\Lambda}{m_c}\left(\frac{1}{1+w}\right)\zeta(w), \\ f_3 &= -\frac{\bar{\Lambda}_\Lambda}{m_b}\left(\frac{1}{1+w}\right)\zeta(w), \\ g_1 &= \left[1 - \left(\frac{\bar{\Lambda}_\Lambda}{2m_c} + \frac{\bar{\Lambda}_\Lambda}{2m_b}\right)\left(\frac{1-w}{1+w}\right)\right]\zeta(w), \\ g_2 &= -\frac{\bar{\Lambda}_\Lambda}{m_c}\left(\frac{1}{1+w}\right)\zeta(w), \\ g_3 &= \frac{\bar{\Lambda}_\Lambda}{m_b}\left(\frac{1}{1+w}\right)\zeta(w). \end{aligned} \tag{4.47}$$

The leading order predictions for the form factors in Eq. (4.29) involved a single unknown function $\zeta(w)$. The result including $1/m_Q$ corrections involves a single unknown function, as well as the nonperturbative constant $\bar{\Lambda}_\Lambda$. Many of the leading order relations survive even when the $1/m_Q$ corrections are included in Λ_b decay form factors. In the next section, we will see that fewer relations hold for meson decay including $1/m_Q$ corrections, but some important ones continue to hold even at this order. The Λ_{QCD}/m_Q corrections are expected to be numerically small, of the order of ∼10–20%.

At the zero-recoil point $w = 1$, the matrix elements of the vector and axial vector currents in Λ_b decay become

$$\begin{aligned}\langle \Lambda_c(p', s')|\bar{c}\gamma^\nu b|\Lambda_b(p, s)\rangle &= [f_1 + f_2 + f_3]v^\nu\, u(p', s')u(p, s), \\ \langle \Lambda_c(p', s')|\bar{c}\gamma^\nu\gamma_5 b|\Lambda_b(p, s)\rangle &= g_1\, \bar{u}(p', s')\gamma^\nu\gamma_5 u(p, s).\end{aligned} \tag{4.48}$$

One can see from Eq. (4.47) that at $w = 1$, $f_1 + f_2 + f_3$ and g_1 do not receive any nonperturbative $1/m_Q$ corrections, so that the decay matrix element has no $1/m_Q$ corrections at zero recoil, a result known as Luke's theorem. Note that the individual form factors can have $1/m_Q$ corrections at zero recoil, but the matrix element does not. A similar result will be proven for B decays in the next section.

4.5 $\bar{B} \to D^{(*)} e\bar{\nu}_e$ decay and Luke's theorem

The analysis of $1/m_Q$ corrections for $\Lambda_b \to \Lambda_c$ semileptonic decay can be repeated for $\bar{B} \to D^{(*)}$ semileptonic decay. To determine the $1/m_Q$ corrections using the weak currents in Eq. (4.38), one needs the matrix elements of $\bar{c}_{v'} i\overleftarrow{D}_\mu \Gamma b_v$ and $\bar{c}_{v'}\Gamma i D_\mu b_v$ between $\bar{B}$ and $D^{(*)}$ meson states at leading order in $1/m_Q$. For this, one can use

$$\begin{aligned}\bar{c}_{v'} i\overleftarrow{D}_\mu \Gamma b_v &= \mathrm{Tr}\, \bar{H}^{(c)}_{v'} \Gamma H^{(b)}_v M_\mu(v, v') \\ \bar{c}_{v'}\Gamma i D_\mu b_v &= -(\bar{b}_v i\overleftarrow{D}_\mu \bar{\Gamma} c_{v'})^\dagger = -\mathrm{Tr}\, \bar{H}^{(c)}_{v'} \Gamma H^{(b)}_v \bar{M}_\mu(v', v)\end{aligned} \tag{4.49}$$

where

$$M_\mu(v, v') = \xi_+(v + v')_\mu + \xi_-(v - v')_\mu - \xi_3\gamma_\mu \tag{4.50}$$

is the most general bispinor constructed out of v and v'. There is no term proportional to $\epsilon_{\mu\alpha\beta\nu}v^\alpha v'^\beta\gamma^\nu\gamma_5$ since it can be eliminated by using the three-γ matrix identity in Eq. (1.119) to write

$$-i\epsilon_{\mu\alpha\beta\nu}v^\alpha v'^\beta\gamma^\nu\gamma_5 = \gamma_\mu \not{v}\not{v}' - v_\mu\not{v}' - w\gamma_\mu + v'_\mu\not{v}, \tag{4.51}$$

which can be absorbed into the other terms using $H^{(b)}_v \not{v} = -H^{(b)}_v$, $\not{v}'\bar{H}^{(c)}_{v'} = -\bar{H}^{(c)}_{v'}$. The equation of motion, $(iv' \cdot D)c_{v'} = 0$, implies that

$$\xi_+(w + 1) - \xi_-(w - 1) + \xi_3 = 0. \tag{4.52}$$

By an argument similar to that used to derive Eq. (4.41), one finds that for $\bar{B} \to D^{(*)}$ matrix elements,

$$\begin{aligned}i\partial_\mu(c_{v'}\Gamma b_v) &= \bar{c}_{v'} i\overleftarrow{D}_\mu \Gamma b_v + c_{v'}\Gamma i D_\mu b_v \\ &= -\bar{\Lambda}(v - v')_\mu\, \xi\, \mathrm{Tr}\, \bar{H}^{(c)}_{v'} \Gamma H^{(b)}_v,\end{aligned} \tag{4.53}$$

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which implies using Eqs. (4.49) and (4.50) that

$$\xi_-(w) = \frac{1}{2}\bar{\Lambda}\xi(w). \tag{4.54}$$

When combined with Eq. (4.52), this yields

$$\xi_+(w) = \frac{w-1}{2(w+1)}\bar{\Lambda}\xi(w) - \frac{\xi_3(w)}{w+1}. \tag{4.55}$$

The $1/m_Q$ corrections to the $\bar{B} \to D^{(*)}$ form factors that were defined in Eq. (2.84) from the $1/m_Q$ terms in the currents given in Eq. (4.38) are

$$\begin{aligned}
\delta h_+ &= [(1+w)\xi_+ + \xi_3]\left(\frac{1}{2m_c} + \frac{1}{2m_b}\right) - (w-1)\xi_-\left(\frac{1}{2m_c} + \frac{1}{2m_b}\right),\\
\delta h_- &= [(1+w)\xi_+ + 3\xi_3]\left(\frac{1}{2m_c} - \frac{1}{2m_b}\right) - (w+1)\xi_-\left(\frac{1}{2m_c} - \frac{1}{2m_b}\right),\\
\delta h_V &= \xi_-\left(\frac{1}{m_c} + \frac{1}{m_b}\right) - \xi_3\left(\frac{1}{m_b}\right),\\
\delta h_{A_1} &= \xi_+\left(\frac{1}{m_c} + \frac{1}{m_b}\right) + \frac{\xi_3}{1+w}\left(\frac{1}{m_c} + \frac{2-w}{m_b}\right),\\
\delta h_{A_2} &= (\xi_+ - \xi_-)\left(\frac{1}{m_c}\right),\\
\delta h_{A_3} &= -\xi_3\left(\frac{1}{m_b}\right) + \xi_-\left(\frac{1}{m_b}\right) + \xi_+\left(\frac{1}{m_c}\right),
\end{aligned} \tag{4.56}$$

where ξ_+ and ξ_- are given in Eqs. (4.54) and (4.55).

One also needs to evaluate the $1/m_Q$ corrections from the Lagrangian. The time-ordered product of the c-quark chromomagnetic operator with the weak currents, Eq. (4.30), can be written as

$$\mathrm{Tr}\, \bar{H}_{v'}^{(c)} \sigma_{\mu\nu} \frac{(1+\not{v}')}{2} \Gamma H_v^{(b)} \frac{X^{\mu\nu}}{2m_c}, \tag{4.57}$$

as for the $\Lambda_b \to \Lambda_c$ case. The only difference is that $X_{\mu\nu}$ is now a general bispinor that is antisymmetric in μ and ν. The most general form for $X_{\mu\nu}$ that does not give a vanishing contribution is

$$X_{\mu\nu} = i\chi_2(v_\mu\gamma_\nu - v_\nu\gamma_\mu) - 2\chi_3\sigma_{\mu\nu}. \tag{4.58}$$

A similar result holds for the b-quark chromomagnetic moment. The c-quark kinetic energy term gives a time-ordered product contribution

$$-\mathrm{Tr}\, \bar{H}_{v'}^{(c)} \frac{(1+\not{v}')}{2} \Gamma H_v^{(b)} \frac{\chi_1}{m_c}, \tag{4.59}$$

with a similar expression for the b-quark kinetic energy. These give

$$\begin{aligned}
\delta h_+ &= \chi_1\left(\frac{1}{m_c}+\frac{1}{m_b}\right) - 2(w-1)\chi_2\left(\frac{1}{m_c}+\frac{1}{m_b}\right) + 6\chi_3\left(\frac{1}{m_c}+\frac{1}{m_b}\right),\\
\delta h_- &= 0,\\
\delta h_V &= \chi_1\left(\frac{1}{m_c}+\frac{1}{m_b}\right) - 2(w-1)\chi_2\left(\frac{1}{m_b}\right) - 2\chi_3\left(\frac{1}{m_c}-\frac{3}{m_b}\right),\\
\delta h_{A_1} &= \chi_1\left(\frac{1}{m_c}+\frac{1}{m_b}\right) - 2(w-1)\chi_2\left(\frac{1}{m_b}\right) - 2\chi_3\left(\frac{1}{m_c}-\frac{3}{m_b}\right),\\
\delta h_{A_2} &= 2\chi_2\left(\frac{1}{m_c}\right),\\
\delta h_{A_3} &= \chi_1\left(\frac{1}{m_c}+\frac{1}{m_b}\right) - 2\chi_3\left(\frac{1}{m_c}-\frac{3}{m_b}\right) - 2\chi_2\left(\frac{1}{m_c}+\frac{w-1}{m_b}\right).
\end{aligned} \tag{4.60}$$

The expressions for the form factors are given by adding Eqs. (4.56) and (4.60) to Eq. (2.95). In addition, there are the perturbative corrections discussed in Chapter 3. We will see in the next section that there is a connection between these two seemingly very different kinds of terms.

The $1/m_Q$ corrections to the form factors are parameterized in terms of one unknown constant $\bar{\Lambda}$, and four unknown functions ξ_3, χ_{1-3}, so there are several new functions in the expressions for the meson decay form factors at order $1/m_Q$. At zero recoil, the $\bar{B} \to D$ matrix element of the vector current is normalized when $m_c = m_b$. This gives the constraint $\chi_1(1) + 6\chi_3(1) = 0$. There is also a constraint from the $\bar{B}^* \to D^*$ matrix element being absolutely normalized at $w = 1$ when $m_b = m_c$. We have not computed this matrix element, since it is not relevant for the phenomenology of B decays. However, it is straightforward to compute this matrix element at zero recoil, and show that the constraint is $\chi_1(1) - 2\chi_3(1) = 0$, so that

$$\chi_1(1) = \chi_3(1) = 0. \tag{4.61}$$

Using these relations, one can derive Luke's result for the absence of $1/m_Q$ corrections to the meson matrix elements of the weak currents at zero recoil. The $\bar{B} \to D$ matrix element of the vector current at zero recoil is proportional to $h_+(1)$, and the $\bar{B} \to D^*$ matrix element of the axial current at zero recoil is proportional to $h_{A_1}(1)$. It is easy to see that $\delta h_+(1) = \delta h_{A_1}(1) = 0$ using the results derived above.

The absence of $1/m_Q$ corrections to the matrix elements of the weak currents at zero recoil allows for a precise determination of $|V_{cb}|$ from experimental semileptonic B decay data. Extrapolation of the experimental value for $\mathrm{d}\Gamma(\bar{B} \to D^* e \bar{\nu}_e)/\mathrm{d}w$ toward $w = 1$ gives

$$|V_{cb}||\mathcal{F}_{D^*}(1)| = (35.2 \pm 1.4) \times 10^{-3}, \tag{4.62}$$

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where $\mathcal{F}_{D^*}(w)$ was defined in Eq. (2.87). At zero recoil, the expression for $\mathcal{F}_{D^*}(w)$ simplifies, giving $\mathcal{F}_{D^*}(1) = h_{A_1}(1)$. In the $m_Q \to \infty$ limit $\mathcal{F}_{D^*}(1) = 1$; however there are perturbative and nonperturbative corrections,

$$\mathcal{F}_{D^*}(1) = \eta_A + 0 + \delta_{1/m^2} + \cdots, \tag{4.63}$$

where η_A is the matching coefficient for the axial current, which was determined in Chapter 3 at order α_s. It has been computed to order α_s^2, and is numerically $\eta_A \simeq 0.96$. The zero in Eq. (4.63) indicates the absence of order $1/m_{c,b}$ nonperturbative corrections, and $\delta_{1/m^2} + \cdots$ stands for the nonperturbative corrections of the order of $1/m_Q^2$ and higher. Estimates of these corrections using phenomenological models like the constituent quark model lead to the expectation $\delta_{1/m^2} + \cdots \simeq -0.05$. Putting these results together, and assigning a 100% uncertainty to the model-dependent estimate of the nonperturbative effects yields the theoretical prediction

$$\mathcal{F}_{D^*}(1) = 0.91 \pm 0.05. \tag{4.64}$$

Combining this with the experimental value in Eq. (4.62) yields

$$|V_{cb}| = [38.6 \pm 1.5(\text{exp}) \pm 2.0(\text{th})] \times 10^{-3}, \tag{4.65}$$

for the $b \to c$ element of the CKM matrix.

The theoretical error in Eq. (4.64) is somewhat ad hoc. To have complete confidence that the theoretical uncertainty in the value of V_{cb} is indeed only 5%, and to try and reduce it further, it is necessary to have another high precision determination of $|V_{cb}|$ using a different method. Fortunately, as we shall see in Chapter 6, $|V_{cb}|$ can also be determined using inclusive B decays.

The zero-recoil $\bar{B} \to D$ vector current matrix element also has no order Λ_{QCD}/m_Q corrections, i.e., $h_+(1) = 1 + \mathcal{O}(\Lambda_{\text{QCD}}^2/m_Q^2)$. However, $\bar{B} \to De\bar{\nu}_e$ is not as useful as $\bar{B} \to D^*e\bar{\nu}_e$ for determining V_{cb}. There are two reasons for this. First, the differential decay rate for $\bar{B} \to De\bar{\nu}_e$ vanishes faster as $w \to 1$ than the differential decay rate for $\bar{B} \to D^*e\bar{\nu}_e$. This makes the extrapolation to zero recoil more difficult. Second, $\mathcal{F}_D(1)$ depends on both $h_+(1)$ and $h_-(1)$, and $h_-(1)$ does receive $\mathcal{O}(\Lambda_{\text{QCD}}/m_Q)$ corrections.

4.6 Renormalons

Suppose QCD perturbation theory is used to express some quantity f as a power series in α_s:

$$f(\alpha_s) = f(0) + \sum_{n=0}^{\infty} f_n \alpha_s^{n+1}. \tag{4.66}$$

Typically, this perturbation series for f is an asymptotic series and has zero radius of convergence. The convergence can be improved by defining the Borel transform of f,

$$B[f](t) = f(0)\delta(t) + \sum_{n=0}^{\infty} \frac{f_n}{n!}\, t^n, \tag{4.67}$$

which is more convergent than the original expansion in Eq. (4.66). The original series for $f(\alpha_s)$ can be recovered from the Borel transform $B[f](t)$ by the inverse Borel transform

$$f(\alpha_s) = \int_0^\infty dt\, e^{-t/\alpha_s} B[f](t). \tag{4.68}$$

If the integral in Eq. (4.68) exists, the perturbation series in Eq. (4.66) for $f(\alpha_s)$ is Borel summable, and Eq. (4.68) gives a definition for the sum of the series. While this provides a definition for the sum of the series in Eq. (4.66), it does not mean that it gives the complete, nonperturbative value for f. For example, $\exp(-1/\alpha_s)$ has the power series expansion

$$\exp(-1/\alpha_s) = 0 + 0\alpha_s + 0\alpha_s^2 + \cdots \tag{4.69}$$

whose sum is zero. If there are singularities in $B[f](t)$ along the path of integration, the Borel sum of f is ambiguous. The inverse Borel transform must be defined by deforming the contour of integration away from the singularity, and the inverse Borel transform in general depends on the deformation used.

Singularities in the Borel transform $B[f](t)$ arise from factorial growth in the coefficients f_n at high orders in perturbation theory. For example, suppose that for large n, f_n is of the order of

$$f_n \sim a w^n (n+k)! \tag{4.70}$$

The Borel transform then has a pole of order $k+1$ at $t = 1/w$:

$$B[f](t) \sim \frac{a k!}{(1-wt)^{k+1}} + \text{less singular}. \tag{4.71}$$

One source of singularities in $B[f]$ in QCD is infrared renormalons. Infrared renormalons are ambiguities in perturbation theory arising from the fact that the gluon coupling gets strong for soft gluons. The infrared renormalons produce a factorial growth in the coefficients f_n, which gives rise to poles in the Borel transform $B[f]$. The renormalon ambiguities have a power law dependence on the momentum transfer Q^2. For example, a simple pole at $t = t_0$ in $B[f]$ introduces an ambiguity in f, depending on whether the integration contour is deformed to pass above or below the renormalon pole. The difference between

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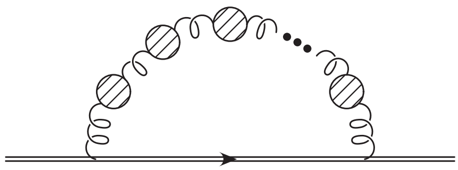

Fig. 4.1. The bubble chain sum. The blob is the gluon vacuum polarization at one loop.

the two choices is proportional to

$$\delta f \sim \oint_C dt\, e^{-t/\alpha_s(Q)} B[f](t) \sim \left(\frac{\Lambda_{\rm QCD}}{Q}\right)^{2\beta_0 t_0}, \tag{4.72}$$

where β_0 defined in Eq. (1.90) is proportional to the leading term in the QCD β function that governs the high-energy behavior of the QCD coupling constant, and the contour C encloses t_0. It is useful to write the Borel transform $B[f](t)$ in terms of the variable $u = \beta_0 t$. The form of the renormalon singularity in Eq. (4.72) then implies that a renormalon at u_0 produces an ambiguity in f that is of the order of $(\Lambda_{\rm QCD}/Q)^{2u_0}$. This ambiguity is canceled by a corresponding ambiguity in nonperturbative effects such as in the matrix elements of higher dimension operators.

Clearly, one is not able to sum the entire QCD perturbation series to determine the renormalon singularities. Typically, one sums bubble chains of the form given in Fig. 4.1. One can consider a formal limit in which the bubble chain sum is the leading term. Take QCD with N_f flavors in the limit $N_f \to \infty$, with $a = N_f \alpha_s$ held fixed. Feynman diagrams are computed to leading order in α_s, but to all orders in a. Terms in the bubble sum of Fig. 4.1 with any number of bubbles are equally important in this limit, since each additional fermion loop contributes a factor $\alpha_s N_f$, which is not small. QCD is not an asymptotically free theory in the $N_f \to \infty$ limit, so the procedure used is to write the Borel transform as a function of u but still study renormalons for positive u. The singularities in u are taken to be the renormalons for asymptotically free QCD. This procedure is a formal way of doing the bubble chain sum while neglecting other diagrams.

The Borel transform of the sum of Feynman graphs containing a single bubble chain can be readily obtained by performing the Borel transform before doing the final loop integral. In the Landau gauge, the bubble chain sum is

$$G(\alpha_s, k) = \sum_{n=0}^{\infty} \frac{i}{k^2}\left(\frac{k_\mu k_\nu}{k^2} - g_{\mu\nu}\right)(-\beta_0 \alpha_s N_f)^n [\ln(-k^2/\mu^2) + C]^n, \tag{4.73}$$

where k is the momentum flowing through the gauge boson propagator, C is a constant that depends on the particular subtraction scheme, and $\beta_0 = -1/6\pi$ is the contribution of a single fermion to the β function. In the $\overline{\rm MS}$ scheme,

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$C = -5/3$. The Borel transform of Eq. (4.73) with respect to $\alpha_s N_f$ is

$$B[G](u,k) = \frac{1}{\alpha_s N_f} \sum_{n=0}^{\infty} \frac{i}{k^2}\left(\frac{k_\mu k_\nu}{k^2} - g_{\mu\nu}\right)\frac{(-u)^n}{n!}[\ln(-k^2/\mu^2) + C]^n$$
$$= \frac{1}{\alpha_s N_f} \frac{i}{k^2}\left(\frac{k_\mu k_\nu}{k^2} - g_{\mu\nu}\right)\exp[-u\ln(-k^2 e^C/\mu^2)]$$
$$= \frac{1}{\alpha_s N_f}\left(\frac{\mu^2}{e^C}\right)^u \frac{i}{(-k^2)^{2+u}}(k_\mu k_\nu - k^2 g_{\mu\nu}). \tag{4.74}$$

The $1/\alpha_s$ has been factored out before Borel transforming, because it will be canceled by the factor of g^2 from the gluon couplings to the external fermion line. The Borel transformed loop graphs can be computed by using the propagator in Eq. (4.74) instead of the usual gauge boson propagator in the Landau gauge:

$$(k_\mu k_\nu - k^2 g_{\mu\nu})\frac{i}{(k^2)^2}. \tag{4.75}$$

By construction, HQET has the same infrared physics as the full QCD theory. However, because the ultraviolet physics differs in the two theories (above the scale m_Q at which the theories are matched), the coefficients of operators in the effective theory must be modified at each order in $\alpha_s(m_Q)$ to ensure that physical predictions are the same in the two theories. Such matching corrections were considered in Chapter 3.

Since the two theories coincide in the infrared, these matching conditions depend in general only on ultraviolet physics and should be independent of any infrared physics, including infrared renormalons. However, in a mass-independent renormalization scheme such as dimensional regularization with $\overline{\text{MS}}$, such a sharp separation of scales cannot be achieved. It is easy to understand why infrared renormalons appear in matching conditions. Consider the familiar case of integrating out a W boson and matching onto a four-Fermi interaction. The matching conditions at one loop involve subtracting one-loop scattering amplitudes calculated in the full and effective theories, as indicated in Fig. 4.2, where C_0 is the lowest order coefficient of the four-Fermi operator, and C_1 is the α_s correction. For simplicity, neglect all external momenta and particle masses, and consider the region of loop integration where the gluon is soft. When $k = 0$, the two theories are identical and the graphs in the two theories are identical. This

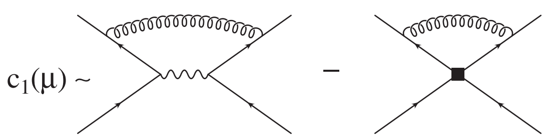

Fig. 4.2. Matching condition for the four-Fermi operator.

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is the well-known statement that infrared divergences cancel in matching conditions. However, for finite (but small) k, the two theories differ at $\mathcal{O}(k^2/M_W^2)$ when one retains only the lowest dimension operators in the effective theory. Therefore, the matching conditions are sensitive to soft gluons at this order, and it is not surprising that the resulting perturbation series is not Borel summable and has renormalon ambiguities starting at $\mathcal{O}(\Lambda_{\text{QCD}}^2/M_W^2)$.

However, this ambiguity is completely spurious and does not mean that the effective field theory is not well defined. Since the theory has only been defined to a fixed order, an ambiguity at higher order in $1/M_W$ is irrelevant. The renormalon ambiguity corresponded to the fact that the two theories differed in the infrared at $\mathcal{O}(k^2/M_W^2)$. When operators suppressed by an additional power of $1/M_W^2$ in the effective theory are consistently taken into account, the two theories will coincide in the infrared up to $\mathcal{O}(k^4/M_W^4)$, and any ambiguity is then pushed up to $\mathcal{O}(\Lambda_{\text{QCD}}^4/M_W^4)$. Consistently including $1/M_W^4$ suppressed operators pushes the renormalon to $\mathcal{O}(\Lambda_{\text{QCD}}^6/M_W^6)$, and so on. In general, a renormalon at $u = u_0$ in the coefficient function of a dimension D operator is canceled exactly by a corresponding ambiguity in matrix elements of operators of dimension $D + 2u_0$, so that physical quantities are unambiguous. This cancellation is a generic feature of all effective field theories, and it also occurs in HQET.

The HQET Lagrangian has an expansion in inverse powers of the heavy quark mass, which can be formally written as

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \cdots + \mathcal{L}_{\text{light}}, \\ \mathcal{L}_0 &= \bar{Q}_v (i D \cdot v) Q_v - \delta m \bar{Q}_v Q_v, \end{aligned} \tag{4.76}$$

on scaling out the phase factor $\exp(-i m_0 v \cdot x)$ from the heavy quark field. Here m_0 is a mass that can differ from m_Q by an amount of order Λ_{QCD}, $\mathcal{L}_{\text{light}}$ is the QCD Lagrangian for the light quarks and gluons, Q_v is the heavy quark field, and $\mathcal{L}_k$ are terms in the effective Lagrangian for the heavy quark that are of order $1/m_0^k$. There are two mass parameters for the heavy quark in Eq. (4.76), the expansion parameter of HQET m_0, and the residual mass term δm. The two parameters are not independent; one can make the redefinition $m_0 \to m_0 + \Delta m$, $\delta m \to \delta m - \Delta m$. A particularly convenient choice is to adjust m_0 so that the residual mass term δm vanishes. Most HQET calculations are done with this choice of m_0, and this is the choice we have used so far in this book, but it is easy to show that the same results are obtained with a different choice of m_0. The HQET mass m_0 when $\delta m = 0$ is often referred to in the literature as the pole mass m_Q, and we will follow this practice here.

Like all effective Lagrangians, the HQET Lagrangian is nonrenormalizable, so a specific regularization prescription must be included as part of the definition of the effective theory. An effective field theory is used to compute physical quantities in a systematic expansion in a small parameter, and the effective

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Lagrangian is expanded in this small parameter. The expansion parameter of the HQET is $\Lambda_{\rm QCD}/m_0$. One can then use "power counting" to determine what terms in the effective theory are relevant to a given order in the $1/m_0$ expansion. For example, to second order in $1/m_0$, one needs to study processes to first order in $\mathcal{L}_2$, and to second order in $\mathcal{L}_1$. It is useful to have a renormalization procedure that preserves the power counting. We choose to use dimensional regularization with $\overline{\rm MS}$, and nonperturbative matrix elements must be interpreted in this scheme. A nonperturbative calculation of a matrix elements, e.g., using lattice Monte Carlo methods, can be converted to $\overline{\rm MS}$ by means of a perturbative matching procedure.

There is a renormalon in the relation between the renormalized mass at short distances (such as the $\overline{\rm MS}$ mass $\bar{m}_Q$) and the pole mass of the heavy quark at $u = 1/2$, which produces an ambiguity of the order of $\Lambda_{\rm QCD}$ in the relation between the pole mass and the $\overline{\rm MS}$ mass. The heavy quark mass in HQET and the $\overline{\rm MS}$ mass at short distances are parameters in the Lagrangian that must be determined from experiment. Any scheme can be used to compute physical processes, though one scheme might be more advantageous for a particular computation. The $\overline{\rm MS}$ mass at short distances is useful in computing high-energy processes. However, there is no advantage to using the "short distance" mass (such as the running $\overline{\rm MS}$ mass) in HQET. In fact, from the point of view of HQET, this is inconvenient. The effective Lagrangian in Eq. (4.76) is an expansion in inverse powers of m_0. Power counting in $1/m_0$ in the effective theory is only valid if δm is of the order of one (or smaller) in m_0, i.e., only if δm remains finite in the infinite mass limit $m_0 \to \infty$. When m_0 is chosen to be the $\overline{\rm MS}$ mass the residual mass term δm is of the order of m_0 (up to logarithms). This spoils the $1/m_0$ power counting of HQET, mixes the α_s and $1/m_0$ expansions, and breaks the heavy flavor symmetry. For example, using m_0 to be the $\overline{\rm MS}$ mass at $\mu = m_0$, one finds at one loop that

$$\delta m = \frac{4}{3\pi}\alpha_s m_0. \tag{4.77}$$

In $b \to c$ decays, including this residual mass term in the heavy c-quark Lagrangian causes $1/m_c$ operators such as $\bar{c}_{v'}\overleftarrow{\not{D}}\Gamma b_v/m_c$ to produce effects that are suppressed by α_s rather than $\Lambda_{\rm QCD}/m_c$. While physical quantities calculated in this way must be the same as those calculated by using the pole mass, it unnecessarily complicates the power counting to use a definition for m_0 that leaves a residual mass term that is not finite in the $m_0 \to \infty$ limit. Better choices for the expansion parameter of HQET are the heavy meson mass (with δm of the order of $\Lambda_{\rm QCD}$), and the pole mass (with $\delta m = 0$).

The $\overline{\rm MS}$ mass at short distances can be determined (in principle) from experiment without any renormalon ambiguities proportional to $\Lambda_{\rm QCD}$. The $\overline{\rm MS}$ quark mass can be related to other definitions of the quark mass by using QCD

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perturbation theory. The connection between the Borel-transformed pole mass and the $\overline{\text{MS}}$ mass is

$$B[m_Q](u) = \bar{m}_Q\delta(u) + \frac{\bar{m}_Q}{3\pi N_f}\left[\left(\frac{\mu^2}{\bar{m}_Q^2}\right)^u e^{-uC}6(1-u)\frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)} - \frac{3}{u} + R_{\Sigma_1}(u)\right], \tag{4.78}$$

where $\bar{m}_Q$ is the renormalized $\overline{\text{MS}}$ mass at the subtraction point μ, and the constant $C = -5/3$ and the function $R_{\Sigma_1}(u)$ have no singularities at $u = 1/2$. Equation (4.78) has a renormalon singularity at $u = 1/2$, which is the leading infrared renormalon in the pole mass. Writing $u = 1/2 + \Delta u$, we have

$$B[m_Q](u = 1/2 + \Delta u) = -\frac{2\mu e^{-C/2}}{3\pi N_f\,\Delta u} + \cdots, \tag{4.79}$$

where the ellipses denote terms regular at $\Delta u = 0$. We will only work to leading order in $1/m_0$, so poles to the right of $u = 1/2$, which are related to ambiguities at higher order in $1/m_0$, are irrelevant. Although m_Q is formally ambiguous at Λ_{QCD}, we have argued that physical quantities that depend on m_Q are unambiguously predicted in HQET. We now demonstrate this explicitly for a ratio of form factors in Λ_b semileptonic decay.

The matrix element of the vector current for the semileptonic decay $\Lambda_b \to \Lambda_c e\bar{\nu}_e$ decay is parameterized by the three decay form factors $f_{1-3}(w)$ defined in Eq. (4.27). In the limit $m_b,\ m_c \to \infty$, and at lowest order in α_s, the form factors f_2 and f_3 vanish. We will consider α_s and $1/m_c$ corrections, but work in the $m_b \to \infty$ limit. Consider the ratio $r_f = f_2/f_1$, which vanishes at lowest order in α_s and $1/m_c$. The corrections to r_f can be written in the form

$$r_f(\alpha_s, w) \equiv \frac{f_2(w)}{f_1(w)} = -\frac{\bar{\Lambda}_\Lambda}{m_c}\frac{1}{(1+w)} + f_r(\alpha_s, w), \tag{4.80}$$

where the function $f_r(\alpha_s, w)$ is a perturbatively calculable matching condition from the theory above $\mu = m_c$ to the effective theory below $\mu = m_c$, and the $\bar{\Lambda}_\Lambda$ term arises from $1/m_c$ suppressed operators in HQET. At one loop (see Problem 5 of Chapter 3),

$$f_r(\alpha_s, w) = -\frac{2\alpha_s}{3\pi}\frac{1}{\sqrt{w^2-1}}\ln\left(w + \sqrt{w^2-1}\right). \tag{4.81}$$

The ratio $r_f = f_2/f_1$ is an experimentally measurable quantity and does not have a renormalon ambiguity. The standard form for r_f in Eq. (4.80) is obtained by using HQET with the pole mass as the expansion parameter. The HQET parameter $\bar{\Lambda}_\Lambda$ is the baryon mass in the effective theory, i.e., it is the baryon mass m_{Λ_c} minus the pole mass of the c quark. The pole mass has the leading renormalon

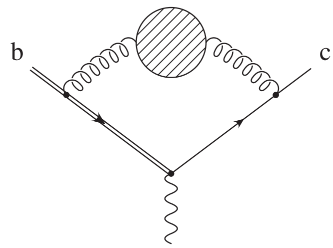

Fig. 4.3. The bubble chain sum for the radiative correction to the vector current form factors.

ambiguity at $u = 1/2$ given in Eq. (4.79), which produces an ambiguity in the $1/m_c$ contribution to f_2/f_1 given by the first term in Eq. (4.80). There must therefore also be a renormalon at $u = 1/2$ in the radiative correction to f_2/f_1 given by the second term in Eq. (4.80). It is straightforward to show that this is indeed the case.

The Borel-transformed series $B[f_r](u, w)$ in the $1/N_f$ expansion is easily calculated from the graph in Fig. 4.3, using the Borel-transformed propagator in Eq. (4.74). The Borel transform of the Feynman diagram is

$$B[\text{graph}] = \frac{i}{\alpha_s N_f}\frac{4}{3}g^2\left(\frac{\mu^2}{e^C}\right)^u \times \int \frac{d^4k}{(2\pi)^4}\frac{\gamma^\nu(m_c\not{v}' + \not{k} + m_c)\gamma^\alpha v^\mu(k_\mu k_\nu - k^2 g_{\mu\nu})}{(k^2 + 2m_c k\cdot v')(-k^2)^{2+u}k\cdot v}. \tag{4.82}$$

The radiative correction to f_2 (which determines f_r) is obtained from the terms in Eq. (4.82) that are proportional to v^α. Combining denominators using Eq. (1.45) and Eq. (3.6), extracting the terms proportional to v^α and performing the momentum integral, we obtain

$$B[f_r](u, w) = \frac{4(u-2)}{3\pi N_f(1+u)}\left(\frac{\mu^2}{e^C}\right)^u m_c \times \int_0^\infty d\lambda \int_0^1 dx \frac{(1-x)^{1+u}\,x}{\left[\lambda^2 + 2\lambda m_c x w + m_c^2 x^2\right]^{1+u}}. \tag{4.83}$$

Rescaling $\lambda \to x m_c \lambda$ and performing the x integral gives

$$B[f_r](u, w) = \frac{4}{3\pi N_f}\left(\frac{\mu^2}{m_c^2 e^C}\right)^u \frac{(u-2)\Gamma(1-2u)\Gamma(1+u)}{\Gamma(3-u)} \times \int_0^\infty d\lambda \frac{1}{[\lambda^2 + 2\lambda w + 1]^{1+u}}. \tag{4.84}$$

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This expression has a pole at $u = 1/2$. Expanding in $\Delta u = u - 1/2$ gives

$$\begin{aligned} B[f_r](u = 1/2 + \Delta u, w) \\ &= \frac{2\mu}{3\pi N_f m_c e^{C/2}} \frac{1}{\Delta u} \int_0^\infty d\lambda \frac{1}{[\lambda^2 + 2\lambda w + 1]^{3/2}} + \cdots \\ &= \frac{2\mu}{3\pi N_f m_c e^{C/2}} \frac{1}{\Delta u} \frac{1}{1+w}, \end{aligned} \tag{4.85}$$

where the ellipsis denotes terms that are regular at $u = 1/2$.

The Borel singularity in Eq. (4.85) cancels the singularity in the first term of Eq. (4.80) at all values of w, so that the ratio of form factors $r_f(\alpha_s, w) = f_2(w)/f_1(w)$ has no renormalon ambiguities. Therefore the standard HQET computation of the $1/m_c$ correction to f_2/f_1 using the pole mass and the standard definition of $\bar{\Lambda}_\Lambda$ gives an unambiguous physical prediction for the ratio of form factors.

The cancellation of renormalon ambiguities has been demonstrated by explicit computation in this example, but the result holds in general.

4.7 $v \cdot A = 0$ gauge

Calculations in HQET can be performed in almost any gauge. However, in the $v \cdot A = 0$ gauge, HQET perturbation theory is singular. Consider tree-level Qq elastic scattering in the rest frame $v = v_r$. In HQET, an on-shell heavy quark has a four velocity v and a residual momentum k that satisfies $v \cdot k = 0$. Suppose the initial heavy quark has zero residual momentum and the final quark has residual momentum $k = (0, \mathbf{k})$. The tree-level Feynman diagram in Fig. 4.4 gives the Qq scattering amplitude

$$\mathcal{M} = -g^2 \bar{u}_Q T^A u_Q \frac{i}{\mathbf{k}^2} \bar{u}_q T^A \not{v} u_q, \tag{4.86}$$

in the Feynman or Landau gauge, where u_Q and u_q are the heavy and light quark spinors, respectively. The current conservation equation $\bar{u}_q \not{k} u_q = 0$ was used to simplify the result.

In the $v \cdot A = 0$ gauge, the gluon propagator is

$$\frac{-i}{k^2 + i\varepsilon} \left[g_{\mu\nu} - \frac{1}{v \cdot k}(k_\mu v_\nu + v_\mu k_\nu) + \frac{1}{(v \cdot k)^2} k_\mu k_\nu \right]. \tag{4.87}$$

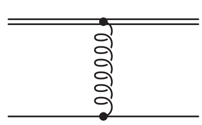

Fig. 4.4. Heavy quark + light quark scattering amplitude at tree level.

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The heavy quark kinetic energy cannot be treated as a perturbation in this gauge, because then $v \cdot k = 0$ and the gluon propagator is ill defined. Including the heavy quark kinetic energy in the Lagrangian, the residual momentum of the outgoing heavy quark becomes $k^\mu = (\mathbf{k}^2/2m_Q, \mathbf{k})$ and $v \cdot k = \mathbf{k}^2/2m_Q$ is not zero. Note that the factors of $1/(v \cdot k)$ in Eq. (4.87) lead to $2m_Q/\mathbf{k}^2$ terms in the gluon propagator, so that the $v \cdot A = 0$ gauge can mix different orders in the $1/m_Q$ expansion.

It is instructive to see how the scattering amplitude in Eq. (4.86) arises in the $v \cdot A = 0$ gauge. The amplitude comes from the QQA vertex that is due to the heavy quark kinetic energy term $-\bar{Q}_v D_\perp^2/(2m_Q) Q_v$. Although this is a $1/m_Q$ term in the Lagrangian, it can contribute to a leading-order amplitude in the $v \cdot A = 0$ gauge. The Feynman rule for $Q_v(k') \to Q_v(k) + A_\mu$ vertex arising from an insertion of the kinetic energy operator is $i(g/2m_Q)(k_\perp + k'_\perp)_\mu = i(g/2m_Q)(k+k')_\mu - i(g/2m_Q) v \cdot (k+k') v_\mu$. In the case we are considering, v is chosen so that $k' = 0$. The part proportional to v_μ doesn't contribute, since $v \cdot A = 0$. Since $\bar{u}_q \not{k} u_q = 0$ only the $v_\mu k_\nu + v_\nu k_\mu$ term in the gluon propagator contributes, and one can show that it reproduces Eq. (4.86) for large values of m_Q.

In the $v \cdot A = 0$ gauge the heavy quark kinetic energy must be considered as a leading operator for on-shell scattering processes, because we have just seen that it is the QQA vertex from this $1/m_Q$ operator that gives rise to the leading Qq on-shell scattering amplitude.

4.8 NRQCD

HQET is not the appropriate effective field theory for systems with more than one heavy quark. In HQET the heavy quark kinetic energy is neglected. It occurs as a small $1/m_Q$ correction. At short distances the static potential between heavy quarks is determined by one gluon exchange and is a Coulomb potential. For a $Q\bar{Q}$ pair in a color singlet, it is an attractive potential, and the heavy quark kinetic energy is needed to stabilize a $Q\bar{Q}$ meson. For $Q\bar{Q}$ hadrons (i.e., quarkonia) the kinetic energy plays a very important role, and it cannot be treated as a perturbation.

In fact the problem is more general than this. Consider, for example, trying to calculate low-energy QQ scattering in the center of a mass frame using HQET. Setting $v = v_r$ for each heavy quark, and using initial and final residual momenta $k_\pm = (0, \pm\mathbf{k})$ and $k'_\pm = (0, \pm\mathbf{k}')$ respectively, we find the one-loop Feynman diagram, Fig. 4.5, gives rise to a loop integral,

$$\int \frac{d^n q}{(2\pi)^n} \frac{i}{(q^0 + i\varepsilon)} \frac{i}{(-q^0 + i\varepsilon)} \frac{i}{(q+k_+)^2 + i\varepsilon} \frac{i}{(q+k'_+)^2 + i\varepsilon}. \tag{4.88}$$

The q^0 integration is ill defined because it has poles above and below the real axis

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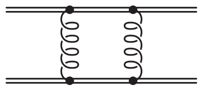

Fig. 4.5. One-loop contribution to QQ scattering.

at $q^0 = \pm i\varepsilon$. This problem is cured by not treating the heavy quark kinetic energy as a perturbation but including it in the leading-order terms. Then the denominators of the two heavy quark propagators become $E + q^0 - \mathbf{q}^2/2m_Q + i\varepsilon$ and $E - q^0 - \mathbf{q}^2/2m_Q + i\varepsilon$, where $E = \mathbf{k}^2/2m_Q = \mathbf{k}'^2/2m_Q$. Closing the q^0 contour in the upper half-plane, we find Eq. (4.88) is dominated (for large m_Q) by the residue of the pole at $q^0 = E - \mathbf{q}^2/2m_Q + i\varepsilon$ and is proportional to m_Q. That is why we obtained an infinite answer for Eq. (4.88) by using the $m_Q \to \infty$ limit of the fermion propagators.

Properties of quarkonia are usually predicted as a power series in v/c, where v is the magnitude of the relative $Q\bar{Q}$ velocity and c is the speed of light. For these systems the appropriate limit of QCD to examine is the $c \to \infty$ limit. In this limit the QCD Lagrangian becomes an effective field theory called NRQCD. For finite c there are corrections suppressed by powers of $1/c$. In particle physics we usually set $\hbar = c = 1$. Making the factors of c explicit, we find the QCD Lagrangian density is

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G^B_{\mu\nu} G^{B\mu\nu} - c\bar{Q}(i\not{D} - m_Q c)Q. \tag{4.89}$$

In the above the zero component of a partial derivative is

$$\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}, \tag{4.90}$$

and D is the covariant derivative

$$D_\mu = \partial_\mu + \frac{ig}{c} A^B_\mu T^B. \tag{4.91}$$

The gluon field strength tensor $G^B_{\mu\nu}$ is defined in the usual way except that $g \to g/c$.

Although c is explicit, $\hbar$ has been set to unity. All dimensionful quantities can be expressed in units of length $[x]$ and time $[t]$, i.e., $[E] \sim 1/[t]$ and $[p] \sim 1/[x]$. The Lagrangian $L = \int d^3x \mathcal{L}$ has units of $1/[t]$ since the action $S = \int \mathcal{L} dt$ is dimensionless. It is straightforward to deduce that the gluon field has units $[A] \sim 1/\sqrt{[x][t]}$ and the strong coupling $g \sim \sqrt{[x]/[t]}$. The fermion field has units $[\psi] \sim 1/[x]^{3/2}$ while its mass has units $[m_Q] \sim [t]/[x]^2$. With these units $m_Q c^2$ has dimensions of energy and the strong fine structure constant $\alpha_s = g^2/4\pi c$ is dimensionless.

For the fermion field Q the transition from QCD to NRQCD is analogous to the derivation of HQET. The heavy quark field is rewritten as

$$Q = e^{-im_Q c^2 t}\left[1 + \frac{i\not{D}_\perp}{m_Q c} + \cdots\right]\begin{pmatrix}\psi \\ 0\end{pmatrix}, \tag{4.92}$$

where ψ is a two-component Pauli spinor. Using this field redefinition, we find the part of the QCD Lagrange density involving Q becomes

$$\mathcal{L}_\psi = \psi^\dagger\left[i\left(\frac{\partial}{\partial t} + igA_0^B T^B\right) + \frac{\nabla^2}{2m_Q}\right]\psi + \cdots, \tag{4.93}$$

where the ellipses denote terms suppressed by powers of $1/c$. Note that the heavy quark kinetic energy is now leading order in $1/c$. The replacement $g \to g/c$ was necessary to have a sensible $c \to \infty$ limit.

Among the terms suppressed by a single power of $1/c$ is the gauge completion of the kinetic energy:

$$\mathcal{L}_{\text{int}} = \frac{ig}{2m_Q c}\mathbf{A}^C[\psi^\dagger T^C \nabla\psi - (\nabla\psi)^\dagger T^C \psi]. \tag{4.94}$$

There is also a $1/c$ term involving the color magnetic field $\mathbf{B}^C = \nabla \times \mathbf{A}^C$.

It is convenient to work in Coulomb gauge, $\nabla \cdot \mathbf{A}^C = 0$. Then the part of the action that involves the gluon field strength tensor and is quadratic in the gauge fields simplifies to

$$\begin{aligned}-\frac{1}{4}\int d^3x G^C_{\mu\nu}G^{C\mu\nu} &\to \frac{1}{2}\int d^3x G^C_{0i}G^C_{0i} - \frac{1}{4}\int d^3x G^C_{ij}G^C_{ij} \\ &= \frac{1}{2}\int d^3x \left(\partial_i A_0^C\right)^2 + \left(\partial_0 A_i^C\right)^2 - \left(\partial_i A_j^C\right)^2 + \text{ non-Abelian terms.}\end{aligned} \tag{4.95}$$

The non-Abelian terms are suppressed by factors of $1/c$. [The above derivation implicitly assumes that $m_Q v^2 \gg \Lambda_{\text{QCD}}$.]

In Eq. (4.95), the zero component of the gauge field has no time derivatives. Therefore, it does not represent a propagating degree of freedom. Neglecting terms suppressed by factors of $1/c$, the Lagrangian only contains terms quadratic and linear in the field A_0^C. Hence the functional integral over A_0^C can be performed exactly by completing the square. The effects of A_0^C exchange are then reproduced by an instantaneous potential $V(\mathbf{x}, \mathbf{y})$ that is proportional to the Fourier transform of the momentum–space propagator,

$$V(\mathbf{x}, \mathbf{y}) = g^2\int \frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\frac{1}{\mathbf{k}^2} = \frac{g^2}{4\pi|\mathbf{x}-\mathbf{y}|}. \tag{4.96}$$

The transverse gluons $\mathbf{A}^\mathbf{C}$ do not couple to the quarks at leading order in the $1/c$ expansion. Neglecting terms suppressed by $1/c$, we find the effective Lagrangian

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for the interaction of nonrelativistic quarks is

$$L_{\mathrm{NRQCD}} = \int d^3x\ \psi^\dagger\left(i\frac{\partial}{\partial t} + \frac{\boldsymbol{\nabla}^2}{2m_Q}\right)\psi \\ - \int d^3x_1 \int d^3x_2 \psi^\dagger(\mathbf{x}_1, t) T^A \psi(\mathbf{x}_1, t) V(\mathbf{x}_1, \mathbf{x}_2) \psi^\dagger(\mathbf{x}_2, t) T^A \psi(\mathbf{x}_2, t). \tag{4.97}$$

The Hamiltonian

$$H = \int d^3x \psi^\dagger i \frac{\partial}{\partial t}\psi - L \tag{4.98}$$

has the familiar form used in nonrelativistic many-body theory. When restricting one's attention to the two heavy quark sector, the effective theory reduces to ordinary nonrelativistic quantum mechanics.

4.9 Problems

1. For any doublet of heavy hadrons $H_\pm^{(Q)}$ with spins $j_\pm = s_\ell \pm 1/2$, show that

$$m_{H_\pm^{(Q)}} = m_Q + \bar{\Lambda}_H - \frac{\lambda_{H,1}}{2m_Q} \pm n_\mp \frac{\lambda_{H,2}}{2m_Q},$$

   where $n_\pm = 2j_\pm + 1$ and $\lambda_{H,1}$ and $\lambda_{H,2}$ are defined in Eqs. (4.23). We have inserted an extra subscript H because the values of the matrix elements depend on the particular doublet.

2. For the ground-state doublet of mesons, let $\{\bar{\Lambda}_H, \lambda_{H,1}, \lambda_{H,2}\} = \{\bar{\Lambda}, \lambda_1, \lambda_2\}$ and for the excited $s_\ell = 3/2$ mesons let $\{\bar{\Lambda}_H, \lambda_{H,1}, \lambda_{H,2}\} = \{\bar{\Lambda}^*, \lambda_1^*, \lambda_2^*\}$. Show that

$$\bar{\Lambda}^* - \bar{\Lambda} = \frac{m_b(\bar{m}_B^* - \bar{m}_B) - m_c(\bar{m}_D^* - \bar{m}_D)}{m_b - m_c},$$
$$\lambda_1^* - \lambda_1 = 2m_c m_b \frac{(\bar{m}_B^* - \bar{m}_B) - (\bar{m}_D^* - \bar{m}_D)}{m_b - m_c},$$

   where

$$\bar{m}_H = \frac{n_- m_{H_-} + n_+ m_{H_+}}{n_+ + n_-}.$$

3. In Problems 6–9 of Chapter 2, the leading $m_Q \to \infty$ predictions for the $\bar{B} \to D_1 e\bar{\nu}_e$ and $\bar{B} \to D_2^* e\bar{\nu}_e$ form factors were derived. In this problem, the $1/m_Q$ corrections are included.

   (a) For $\bar{B} \to D_1$ and $\bar{B} \to D_2^*$ matrix elements, argue that

$$\bar{c}_{v'}\, i\overleftarrow{D}_\lambda \Gamma\, b_v = \mathrm{Tr}\left\{S_{\sigma\lambda}^{(c)} \bar{F}_{v'}^\sigma \Gamma H_v^{(b)}\right\},$$
$$\bar{c}_{v'}\, \Gamma i D_\lambda\, b_v = \mathrm{Tr}\left\{S_{\sigma\lambda}^{(b)} \bar{F}_{v'}^\sigma \Gamma H_v^{(b)}\right\},$$

   where

$$S_{\sigma\lambda}^{(Q)} = v_\sigma\left[\tau_1^{(Q)} v_\lambda + \tau_2^{(Q)} v'_\lambda + \tau_3^{(Q)} \gamma_\lambda\right] + \tau_4^{(Q)} g_{\sigma\lambda},$$

   and the functions $\tau_i^{(Q)}$ depend on w. (They are not all independent.)

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(b) Show that the heavy quark equation of motion implies

$$w\tau_1^{(c)} + \tau_2^{(c)} - \tau_3^{(c)} = 0,$$
$$\tau_1^{(b)} + w\tau_2^{(b)} - \tau_3^{(b)} + \tau_4^{(b)} = 0.$$

(c) Further relations between the τ's follow from

$$i\partial_\nu(\bar{c}_{v'}\,\Gamma\, b_v) = (\bar{\Lambda} v_\nu - \bar{\Lambda}^* v'_\nu)\bar{c}_{v'}\,\Gamma\, b_v.$$

Show that this equation implies the relations

$$\begin{aligned}
\tau_1^{(c)} + \tau_1^{(b)} &= \bar{\Lambda}\tau,\\
\tau_2^{(c)} + \tau_2^{(b)} &= -\bar{\Lambda}^*\tau,\\
\tau_3^{(c)} + \tau_3^{(b)} &= 0,\\
\tau_4^{(c)} + \tau_4^{(b)} &= 0,
\end{aligned}$$

where τ was defined in Problem 9 of Chapter 2. The relations in parts (b) and (c) imply that all the $\tau_j^{(Q)}$'s can be expressed in terms of $\tau_1^{(c)}$ and $\tau_2^{(c)}$.

(d) Using the results from parts (a)–(c), show that the corrections to the currents give the following corrections to the form factors:

$$\begin{aligned}
\sqrt{6}\,\delta f_A &= -\epsilon_b(w-1)[(\bar{\Lambda}^* + \bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2]\\
&\quad - \epsilon_c[4(w\bar{\Lambda}^* - \bar{\Lambda})\tau - 3(w-1)(\tau_1 - \tau_2)],\\
\sqrt{6}\,\delta f_{V_1} &= -\epsilon_b(w^2-1)[(\bar{\Lambda}^* + \bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2]\\
&\quad - \epsilon_c[4(w+1)(w\bar{\Lambda}^* - \bar{\Lambda})\tau - 3(w^2-1)(\tau_1 - \tau_2)],\\
\sqrt{6}\,\delta f_{V_2} &= -3\epsilon_b[(\bar{\Lambda}^* + \bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] - \epsilon_c[(4w-1)\tau_1 + 5\tau_2)],\\
\sqrt{6}\,\delta f_{V_3} &= \epsilon_b(w+2)[(\bar{\Lambda}^* + \bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2]\\
&\quad + \epsilon_c[4(w\bar{\Lambda}^* - \bar{\Lambda})\tau + (2+w)\tau_1 + (2+3w)\tau_2],
\end{aligned}$$

for $\bar{B} \to D_1 e\bar{\nu}_e$. For $\bar{B} \to D_2^* e\bar{\nu}_e$ show that the corrections to the form factors are

$$\begin{aligned}
\delta k_V &= -\epsilon_b[(\bar{\Lambda}^* + \bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] - \epsilon_c[\tau_1 - \tau_2],\\
\delta k_{A_1} &= -\epsilon_b(w-1)[(\bar{\Lambda}^* + \bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] - \epsilon_c(w-1)[\tau_1 - \tau_2],\\
\delta k_{A_2} &= -2\epsilon_c\tau_1,\\
\delta k_{A_3} &= \epsilon_b[(\bar{\Lambda}^* + \bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] - \epsilon_c[\tau_1 + \tau_2].
\end{aligned}$$

Here $\epsilon_c = 1/(2m_c)$, $\epsilon_b = 1/(2m_b)$ and $\tau_1 = \tau_1^{(c)}$, $\tau_2 = \tau_2^{(c)}$.

(e) The zero-recoil matrix elements of the weak current are determined by $f_{V_1}(1)$. The $1/m_Q$ corrections to the current imply that

$$\sqrt{6} f_{V_1}(1) = -8\epsilon_c(\bar{\Lambda}^* - \bar{\Lambda})\tau(1).$$

Show that the $1/m_Q$ corrections to the states do not alter this relation.

4. Explain why the χ_1's from the charm and bottom quark kinetic energies are the same.
5. Show that the $\bar{B}^* \to D^*$ matrix element implies that $\chi_1(1) - 2\chi_3(1) = 0$ for the $1/m_Q$ corrections to the $\bar{B} \to D^{(*)}$ form factors that arise from the chromomagnetic term in the Lagrangian.
6. Verify Eq. (4.77) for the relation between the $\overline{\text{MS}}$ mass and the pole mass.
7. Calculate the order $\Lambda_{\text{QCD}}/m_{c,b}$ corrections to the form factor ratios R_1 and R_2 defined in Chapter 2. Express the result in terms of $\bar{\Lambda}$, ξ_3 and χ_{1-3}.

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