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TWO OPTIMISATION PROBLEMS FOR CONVEX BODIES

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Abstract

In this paper, we will show that the spherical symmetric slices are the convex bodies that maximise the volume, the surface area and the integral of mean curvature when the minimum width and the circumradius are prescribed and the symmetric 2-cap-bodies are the ones which minimise the volume, the surface area and the integral of mean curvature given the diameter and the inradius.

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1. Introduction

For a planar convex set K, there are many functionals defining properties of K: the perimeter $P = P(K)$, the area $A = A(K)$, the minimum width $w = w(K)$, the diameter $d = d(K)$, the inradius $r = r(K)$ and the circumradius $R = R(K)$. There are many inequalities comparing the sizes of these functionals (see, for example, [2, 4, 6, 7, 14, 16, 17]).

Following Blaschke's famous work [1], in 1961, Santaló [14] proposed mapping the family of compact planar convex sets into a compact region $[0, 1] \times [0, 1] \subset E^2$, which is called the *Santaló diagram*. The collection of inequalities determined by a Santaló diagram constitutes a *complete system of inequalities*. Since there are six functionals (P, A, w, d, r, R), the Santaló diagram has 20 cases giving bounds for one of the functionals in terms of two others. Santaló [14] provided the solutions for (A, P, w), (A, P, r), (A, P, R), (A, d, w), (P, d, w) and (d, r, R). The case (d, w, r) was solved by Hernández Cifre via an imaginative method in [8]; she also solved (P, d, r) and (P, d, R) in [9]. The cases (d, w, R) and (w, r, R) were concluded by Hernández Cifre and Gomis in [13]. Böröczky *et al.* obtained the cases (A, r, R) and (P, r, R) in [3]. Complete systems of inequalities for 3-rotational symmetric planar convex sets are discussed in [11].

For a three-dimensional convex body K, the volume $V = V(K)$, the surface area $S = S(K)$ and the integral of the mean curvature $M = M(K)$ are very significant quantities besides the minimum width w, the diameter d, the inradius r and the circumradius R of K. It is an interesting problem to find the convex bodies in three-dimensional Euclidean space E^3 which have maximum or minimum volume, surface area and integral of mean curvature when some of the other functionals are fixed. Some higher dimensional discussions have appeared in [10] and [12].

In this paper, inspired by [3], we derive two new groups of inequalities relating the volume, the surface area and the integral of the mean curvature with the minimum width and the circumradius and then with the diameter and the inradius of a convex body K in E^3. We prove the following two theorems.

THEOREM 1.1. *Let K be a compact convex body in the Euclidean space E^3 and R and w its circumradius and minimum width. Then*

$$V(K) \leq \pi\Big(wR^2 - \frac{w^3}{12}\Big), \tag{1.1}$$

$$S(K) \leq \pi\Big(2R^2 - \frac{w^2}{2} + 2wR\Big), \tag{1.2}$$

$$M(K) \leq 2\pi w + 2\pi\sqrt{R^2 - \frac{w^2}{4}}\arccos\frac{w}{2R}, \tag{1.3}$$

and the equality signs in (1.1)–(1.3) hold if and only if K is the spherical symmetric slice, denoted by K^s, that is, the part of the ball $B^3(R)$ bounded by two parallel planes equidistant from the centre O of $B^3(R)$ and a distance w apart (see Figure 1(a)).

THEOREM 1.2. *Let K be a compact convex body in the Euclidean space E^3 and r and d its inradius and diameter. Suppose that there is a diameter of K which intersects the inscribed ball of K. Then*

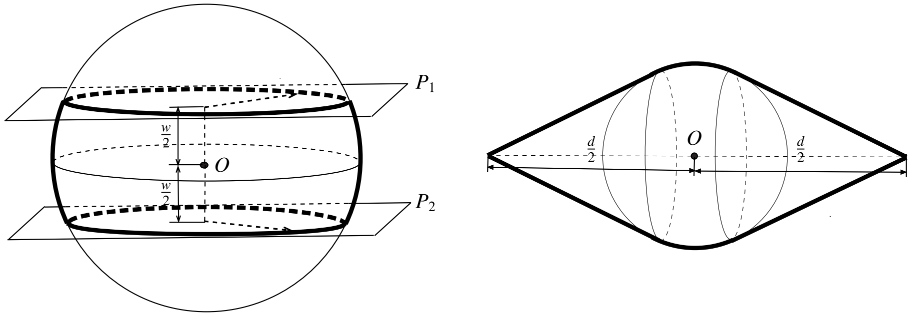
$$V(K) \geq \frac{\pi r^2}{3}\Big(\frac{4r^2}{d} + d\Big), \tag{1.4}$$

$$S(K) \geq \pi r\Big(\frac{4r^2}{d} + d\Big), \tag{1.5}$$

$$M(K) \geq \pi\Big(\frac{4r^2}{d} + d\Big), \tag{1.6}$$

and the equality signs in (1.4)–(1.6) hold if and only if K is the symmetric 2-cap-body, denoted by K_2^c, that is, the convex hull of the ball $B^3(r)$ and two points symmetric with respect to the centre O of $B^3(r)$ and a distance d apart (see Figure 1(b)).

We deal with Theorem 1.1 in Section 2 and Theorem 1.2 in Section 3.



(a) Spherical symmetric slice (b) Symmetric 2-cap-body

FIGURE 1. Spherical symmetric slice and symmetric 2-cap-body.

2. Maximising the volume, the surface area and the integral of the mean curvature

In order to prove Theorem 1.1, we first establish the following two lemmas.

LEMMA 2.1. *Let K be a compact convex body in the Euclidean space E^3 and $B^3(R)$ and w its circumscribed ball and minimum width, respectively. Let P_1 and P_2 be two parallel support planes of K, where each P_i is perpendicular to the direction $\vec{u}$ of minimum width. Then K cannot lie in any hemisphere of $B^3(R)$, unless the intersection of P_1 or P_2 with K is a disc with centre O and radius R.*

PROOF. Since $B^3(R)$ is the circumscribed ball of K, K is contained in $B^3(R)$ and both P_1 and P_2 intersect $B^3(R)$. If the conclusion fails, then P_1 and P_2 must intersect the same hemisphere of $B^3(R)$ and neither of these two planes passes through the centre O of $B^3(R)$, or one of them passes through O but the intersection of K and this plane is not a disc with centre O and radius R. Denote by $\widetilde{K}$ the zone bounded by P_1, P_2 and $B^3(R)$; then $K \subset \widetilde{K}$ (see Figure 2). So, we can move $B^3(R)$ in the direction $\vec{u}$ until the centre O belongs to $\widetilde{K}$, and then narrow the radius of $B^3(R)$ until it intersects K, which contradicts the definition of circumscribed ball. □

LEMMA 2.2. *Let P_1 and P_2 be two parallel planes which intersect different hemispheres of $B^3(R)$ and denote by K the convex body bounded by P_1, P_2 and $B^3(R)$. Denote by $\tilde{x}$ and $\tilde{y}$ the distances from the centre O of $B^3(R)$ to P_1 and P_2 (see Figure 3). If $\tilde{x} + \tilde{y} = w < 2R$, then*

$$V(K) \leq \pi\Big(wR^2 - \frac{w^3}{12}\Big), \tag{2.1}$$

$$S(K) \leq \pi\Big(2R^2 - \frac{w^2}{2} + 2wR\Big), \tag{2.2}$$

$$M(K) \leq 2\pi w + 2\pi\sqrt{R^2 - \frac{w^2}{4}}\arccos\frac{w}{2R}, \tag{2.3}$$

and the equality signs in (2.1)–(2.3) hold if and only if $\tilde{x} = \tilde{y} = w/2$, that is, $K = K^s$.

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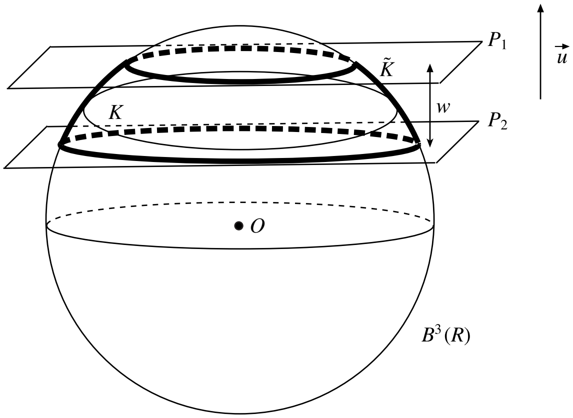

FIGURE 2. P_1 and P_2 intersect the same hemisphere of $B^3(R)$.

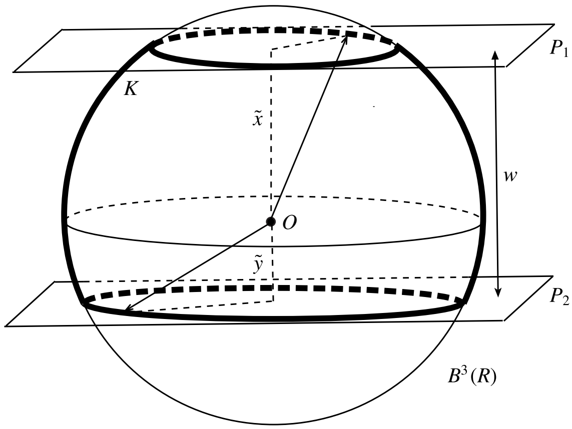

FIGURE 3. K is bounded by P_1, P_2 and $B^3(R)$.

PROOF. Since K can be generated by revolving the curve $x = \sqrt{R^2 - y^2}$ $(-\tilde{y} \le y \le \tilde{x})$ about the y-axis, its volume V and lateral area S_3 can be expressed by

$$V(K) = \int_{-\tilde{y}}^{\tilde{x}} \pi x^2 \, dy = \pi\big(R^2(\tilde{x} + \tilde{y}) - \tfrac{1}{3}(\tilde{x}^3 + \tilde{y}^3)\big),$$

$$S_3(K) = \int_{-\tilde{y}}^{\tilde{x}} 2\pi x \sqrt{1 + x'^2} \, dy \;=\; 2\pi R(\tilde{x} + \tilde{y}).$$

Denote by S_i the area of the domain $P_i \cap K$, $i = 1, 2$. Since $P_1 \cap K$ and $P_2 \cap K$ are discs, it follows that $S_1(K) = \pi(R^2 - \tilde{x}^2)$ and $S_2(K) = \pi(R^2 - \tilde{y}^2)$. So, the surface area is

$$S(K) = S_1(K) + S_2(K) + S_3(K) = \pi(2R^2 - \tilde{x}^2 - \tilde{y}^2 + 2R(\tilde{x} + \tilde{y})).$$

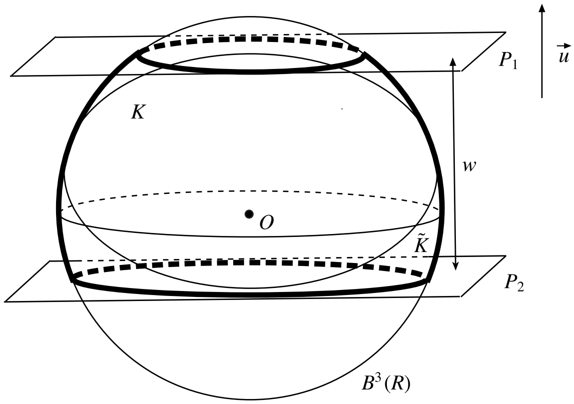

FIGURE 4. $\widetilde{K}$ contains K.

Let H be the mean curvature of K and α_1 and α_2 the exterior dihedral angles along the edges of $P_1 \cap K$ and $P_2 \cap K$. From [5] and [6],

$$M(K) = \int_{S_3} H\,d\sigma + \frac{1}{2}\int_{\partial S_1} \alpha_1\,ds + \frac{1}{2}\int_{\partial S_2} \alpha_2\,ds$$
$$= 2\pi(\tilde{x} + \tilde{y}) + \pi\Big(\sqrt{R^2 - \tilde{x}^2}\arccos\frac{\tilde{x}}{R} + \sqrt{R^2 - \tilde{y}^2}\arccos\frac{\tilde{y}}{R}\Big).$$

Eliminating $\tilde{y}$ by using $\tilde{x} + \tilde{y} = w$ gives

$$V(K) = \pi(wR^2 - \tfrac{1}{3}(\tilde{x}^3 + (w - \tilde{x})^3)),$$
$$S(K) = \pi(2R^2 + 2wR - \tilde{x}^2 - (w - \tilde{x})^2),$$
$$M(K) = 2\pi w + \pi\Big(\sqrt{R^2 - \tilde{x}^2}\arccos\frac{\tilde{x}}{R} + \sqrt{R^2 - (w - \tilde{x})^2}\arccos\frac{w - \tilde{x}}{R}\Big).$$

The functionals $V(K)$, $S(K)$ and $M(K)$ can be regarded as functions of $\tilde{x}$. Some simple computations show that the maxima of these functions are attained only when $\tilde{x} = w/2$. Notice, in the case of $M(K)$, that the function $(u/\sqrt{1 - u^2})\arccos u$ is strictly monotonic increasing on $(0, 1)$. Thus, (2.1)–(2.3) follow and the equality signs hold if and only if $\tilde{x} = \tilde{y} = w/2$, that is, $K = K^s$. □

PROOF OF THEOREM 1.1. If $R = w/2$, then the results are obvious. Let P_1 and P_2 be the support planes in the direction $\vec{u}$ of minimum width and $B^3(R)$ the circumscribed ball of K. If $R > w/2$, for a convex body K in E^3, by Lemma 2.1, P_1 and P_2 intersect different hemispheres of $B^3(R)$ (see Figure 4) or P_1 (or P_2) passes through O and the intersection of K and this plane is a disc with centre O and radius R.

Let $\widetilde{K}$ be the zone bounded by P_1, P_2 and $B^3(R)$, so that $K \subset \widetilde{K}$. From [6], $V(K) \le V(\widetilde{K})$, $S(K) \le S(\widetilde{K})$ and $M(K) \le M(\widetilde{K})$.

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For the first case in which P_1 and P_2 intersect different hemispheres of $B^3(R)$, it follows from Lemma 2.2 that

$$V(K) \le V(\widetilde{K}) \le \pi\Big(wR^2 - \frac{w^3}{12}\Big),$$
$$S(K) \le S(\widetilde{K}) \le \pi\Big(2R^2 - \frac{w^2}{2} + 2wR\Big),$$
$$M(K) \le M(\widetilde{K}) \le 2\pi w + 2\pi\sqrt{R^2 - \frac{w^2}{4}}\arccos\frac{w}{2R},$$

and the equality signs hold if and only if $K = \widetilde{K} = K^s$.

For the second case in which P_1 (say) passes through O and its intersection with K is a disc with centre O and radius R, by a calculation similar to that in Lemma 2.2,

$$V(K) \le V(\widetilde{K}) = \pi\Big(wR^2 - \frac{w^3}{3}\Big) < \pi\Big(wR^2 - \frac{w^3}{12}\Big),$$
$$S(K) \le S(\widetilde{K}) = \pi(2R^2 - w^2 + 2wR) < \pi\Big(2R^2 - \frac{w^2}{2} + 2wR\Big),$$
$$\begin{aligned} M(K) \le M(\widetilde{K}) &= 2\pi w + \frac{\pi^2 R}{2} + \pi\sqrt{R^2 - w^2}\arccos\frac{w}{R} \\ &< 2\pi w + 2\pi\sqrt{R^2 - \frac{w^2}{4}}\arccos\frac{w}{2R}. \end{aligned}$$ □

3. Minimising the volume, the surface area and the integral of the mean curvature

PROOF OF THEOREM 1.2. By assumption, there is a diameter of K, denoted by AB, which intersects the inscribed ball $B^3(r)$. Denote by $\widetilde{K} = \mathrm{conv}\{B^3(r), A, B\}$ the convex hull of $B^3(r)$ and the two points A and B. Then $\widetilde{K} \subset K$; hence, according to [6], $V(K) \ge V(\widetilde{K})$, $S(K) \ge S(\widetilde{K})$ and $M(K) \ge M(\widetilde{K})$.

Let $\tilde{x}$ and $\tilde{y}$ be the distances from the centre O of $B^3(r)$ to the points A and B. Let π_1 be the plane which passes through the three points O, A and B. Denote by D the intersection of π_1 and $\widetilde{K}$. It is clear that D is the convex hull of $B^3(r) \cap \pi_1$ and the two points A, B. Set $\sin\theta_1 = r/\tilde{x}$, $\sin\theta_2 = r/\tilde{y}$ (see Figure 5).

The volume and the surface area of the 'cap' about point A are denoted by V_1, S_1 and those of B are V_2, S_2. The 'cap' body of point A can be generated by revolving the domain D_1 about the x-axis, where D_1 is constructed by the curves $y_1 = \tan\theta_1 x$ $(0 \le x \le \tilde{x} - r\sin\theta_1)$, $y_2 = \sqrt{r^2 - (x - \tilde{x})^2}$ $(\tilde{x} - r \le x \le \tilde{x} - r\sin\theta_1)$ and the x-axis. Hence,

$$V_1(\widetilde{K}) = \int_0^{\tilde{x} - r\sin\theta_1} \pi y_1^2\,dx - \int_{\tilde{x}-r}^{\tilde{x} - r\sin\theta_1} \pi y_2^2\,dx = \frac{\pi r^2}{3}\frac{(\tilde{x} - r)^2}{\tilde{x}}$$

and

$$S_1(\widetilde{K}) = \int_0^{\tilde{x} - r\sin\theta_1} 2\pi y_1\sqrt{1 + y_1'^2}\,dx = \frac{\pi r(\tilde{x}^2 - r^2)}{\tilde{x}}.$$

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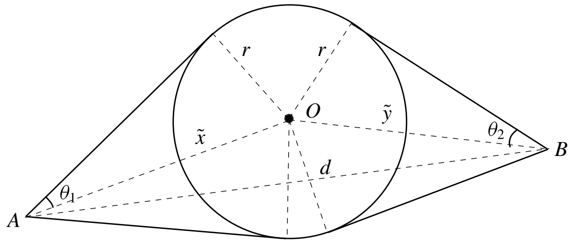

FIGURE 5. The intersection of π_1 and $\widetilde{K}$.

The 'cap' body of point B can be generated by revolving the domain D_2 about the x-axis, where D_2 is constructed by the curves $y_3 = -\tan\theta_2 x$ $(-\tilde{y} + r\sin\theta_2 \le x \le 0)$, $y_4 = \sqrt{r^2 - (x + \tilde{y})^2}$ $(-\tilde{y} + r\sin\theta_2 \le x \le r - \tilde{y})$ and the x-axis. By a similar argument,

$$V_2(\widetilde{K}) = \int_{-\tilde{y}+r\sin\theta_2}^{0} \pi y_3^2\,dx - \int_{-\tilde{y}+r\sin\theta_2}^{r-\tilde{y}} \pi y_4^2\,dx = \frac{\pi r^2}{3}\frac{(\tilde{y}-r)^2}{\tilde{y}},$$

$$S_2(\widetilde{K}) = \int_{-\tilde{y}+r\sin\theta_2}^{0} 2\pi y_3\sqrt{1 + y_3'^2}\,dx = \frac{\pi r(\tilde{y}^2 - r^2)}{\tilde{y}}.$$

Let $\widetilde{S}_1$ and $\widetilde{S}_2$ be the surface areas of $B^3(r)$ covered by the two 'caps' about points A and B. Using the same method as above,

$$\widetilde{S}_1(\widetilde{K}) = \int_{\tilde{x}-r}^{\tilde{x}-r\sin\theta_1} 2\pi y_2\sqrt{1 + y_2'^2}\,dx = 2\pi r^2(1 - \sin\theta_1),$$

$$\widetilde{S}_2(\widetilde{K}) = \int_{-\tilde{y}+r\sin\theta_2}^{r-\tilde{y}} 2\pi y_4\sqrt{1 + y_4'^2}\,dx = 2\pi r^2(1 - \sin\theta_2).$$

Hence,

$$V(\widetilde{K}) = V_1(\widetilde{K}) + V_2(\widetilde{K}) + \frac{4\pi r^3}{3} = \frac{\pi r^2}{3}\Big(\tilde{x} + \tilde{y} + r^2\frac{\tilde{x}+\tilde{y}}{\tilde{x}\tilde{y}}\Big) \triangleq \frac{\pi r^2}{3} g(\tilde{x}, \tilde{y}),$$

$$S(\widetilde{K}) = S_1(\widetilde{K}) + S_2(\widetilde{K}) + (4\pi r^2 - \widetilde{S}_1(\widetilde{K}) - \widetilde{S}_2(\widetilde{K})) = \pi r\Big(\tilde{x} + \tilde{y} + r^2\frac{\tilde{x}+\tilde{y}}{\tilde{x}\tilde{y}}\Big) \triangleq \pi r\, g(\tilde{x}, \tilde{y}).$$

By considering the first derivatives of $g(\tilde{x}, \tilde{y})$ with respect to x and y, it follows that $g(\tilde{x}, \tilde{y})$ is strictly monotonic increasing in each variable. If $\tilde{x} + \tilde{y} > d$, there exists a positive real number $\tilde{x}'$ such that $\tilde{x} - \tilde{x}' + \tilde{y} = d$, and $\tilde{x} - \tilde{x}' > 0$ (since OA, AB and OB form a triangle), so $g(\tilde{x}, \tilde{y}) > g(\tilde{x} - \tilde{x}', \tilde{y})$. Therefore, we need only consider the case $\tilde{x} + \tilde{y} = d$. The function $g(\tilde{x}, \tilde{y})$ has its minimum at the point $(a/2, a/2)$ under the condition $\tilde{x} + \tilde{y} = a$. So,

$$V(K) \ge V(\widetilde{K}) = \frac{\pi r^2}{3} g(\tilde{x}, \tilde{y}) \ge \frac{\pi r^2}{3} g\Big(\frac{d}{2}, \frac{d}{2}\Big) = \frac{\pi r^2}{3}\Big(\frac{4r^2}{d} + d\Big),$$

$$S(K) \ge S(\widetilde{K}) = \pi r\, g(\tilde{x}, \tilde{y}) \ge \pi r\, g\Big(\frac{d}{2}, \frac{d}{2}\Big) = \pi r\Big(\frac{4r^2}{d} + d\Big).$$

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For the integral of the mean curvature of K, it is well known that a general cap-body (not necessarily symmetric) satisfies the relation $S = Mr$ (see [15, pages 367–368]). It is obvious that the relation holds for $\widetilde{K}$; hence,

$$M(K) \geq M(\widetilde{K}) = \frac{1}{r} S(\widetilde{K}) \geq \pi\Big(\frac{4r^2}{d} + d\Big),$$

and the equality is attained if and only if $K = \widetilde{K}$ and $\tilde{x} = \tilde{y} = d/2$, that is, $K = K_2^c$. □

REMARK 3.1. The hypothesis that there exists a diameter which intersects the convex body K in E^3 is necessary. For example, if K is a tetrahedron, all its edges are diameters, but none of them intersects its inscribed ball.

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