Sums of the first n odd integers

CHRIS SANGWIN

There is a long tradition of mathematicians valuing collections of proofs, e.g. [1] is a famous recent collection. In the past, professional mathematicians have also written many different proofs of single important theorems, e.g. [2]. Proofs of the Pythagorean Theorem were collected by [3], and [4] discusses mathematical style through proofs of the irrationality of $\sqrt{2}$. The recent work of [5] is a discussion of mathematical style via a comparison of 99 different 'proofs' the following 'theorem'.

Let $x \in \mathbb{R}$. If $x^3 - 6x^2 + 11x - 6 = 2x - 2$ then $x = 1$ or $x = 4$.

Reference [5] is one of the most interesting and enjoyable mathematics books I have read in many years. Some of the proofs from [5] were used by [6] recently as research materials to investigate the nature of mathematical explanations. I wanted to undertake research on rigour and insight in mathematics education but was dissatisfied with the choice of theorem and, to a lesser extent, the selection of proofs used. So I set about searching for what is, in my view, a more suitable theorem for my research purposes and ultimately I chose the following.

Theorem 1: The sum of the first n odd integers, starting from 1, is n^2.

In this paper, I collect, and discuss, proofs of Theorem 1. Thirteen proofs from this collection are being used for research into whether students understand insight and rigour as separate constructs, and that work will be published elsewhere. This paper however records the full collection together with comments about the proofs. I was also motivated to create this collection partly for personal interest, and with a clear expectation that my collection of proofs will be useful for future discussions with students about proof in a broad range of aspects. The famous collection *Proofs from THE BOOK*, [1], contains proofs which, by their very nature, are atypical mathematical arguments; that is precisely why they have been collected together. Part of education involves learning to read, write and understand agreed forms of proof, which can be somewhat repetitive. For example, proof by induction could be reduced to a mantra to recite but at some important level it is precisely the fact that these 'patterns of reasoning' (adopting Pólya's phrase, [7]) are accepted as standard arguments which releases the student from worrying 'Is this a proof?'. I hope other teachers find this collection a useful starting point for discussions of their own.

Expressed in algebraic notation, Theorem 1 becomes

$$1 + 3 + 5 + 7 + \dots + (2n - 1) = \sum_{k=1}^{n} (2k - 1) = n^2. \qquad (1)$$

https://doi.org/10.1017/mag.2023.3 Published online by Cambridge University Press

Theorem 1 is related to finding formulae for

$$\sum_{k=1}^{n} k^m \text{ for } m = 0,\ 1,\ 2,\ 3,\ 4,\ \ldots . \tag{2}$$

For example, $m = 1$ gives $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n + 1)$ and $m = 2$ gives $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$. The sums of powers of integers have an important history, e.g. [8], [9], [10], [11] and [12].

Theorem 1 is classical, see Example 3, Pictorial I below. According to [13] the first use of the name 'mathematical induction' is found in [14] which calls it *successive induction*, and Theorem 1 is included by De Morgan as an example proof. Knowledge of the historical importance of these two proofs contributed to selecting Theorem 1 specifically from the possible finite integer series.

I set about writing as many proofs as possible, from memory, before searching for different approaches to establishing Theorem 1 itself or summation formulae more generally. I then searched a wide variety of textbooks, including those explicitly teaching proof as a subject, e.g. [15], [16], and a range of historic textbooks. For my research purposes, the cubic solved by [5] is too elementary, so elementary that one might question whether the result deserves the description 'theorem' beyond a very formal sense. My research participants might view this merely as a rather routine calculation and not a theorem at all. Also, some of the proofs in [5] are outlandish and are unlikely to ever arise 'naturally' in a discussion with students. I have given attribution to any proofs I found, adding evidence that these are indeed arguments put forward by others as legitimate proofs. Where proofs below are unattributed then I wrote them myself; however the arguments are certainly not novel*. Some arguments adapt well-known proof techniques used in this particular situation. This search readily uncovered many proofs of the result, together with many statements of the theorem as a recreational problem or an exercise. Indeed, it turns out that I am not the first person to suggest students use Theorem 1 to collect different proof techniques themselves.

> 3.12 Find $1 + 3 + 5 + \ldots + (2n - 1)$, the sum of the first n odd numbers. (List as many different approaches as you can.). [17, I. p. 81]

The theorem gives a formula for the sum of a series and so it is unreasonable to expect to find a fully non-constructive proof. Some proofs merely state and establish the full result. Other proofs essentially solve the problem 'What is the sum of any number of successive odd numbers, beginning from

* Current legal movement restrictions as a response to suppress covid-19 mean most of my books, which are in my office at the University of Edinburgh, are not at the time of writing available to me for more systematic review.

https://doi.org/10.1017/mag.2023.3 Published online by Cambridge University Press

unity?', without anticipating the answer (n^2) at the outset. Within those proofs which solve the problem some work only for this situation whereas others clearly and readily generalise to a range of other situations. [18] refers to this as generality: '*The idea of the proof generalizes to a larger class of theorems*'. Another of their criteria is connectedness: '*The proof idea connects to proof ideas of other theorems*'. For example, two of the proofs make direct use of (2), whereas other proofs connect to more advanced areas of mathematics.

The rest of this paper contains the example proofs.

The problem is stated in [19, p. 95] as a recreational puzzle, without proof.

Example 1: Recreational

If odde numbers *bee continually added from the unitie successively, there will bee made all square numbers, and if* cubike numbers *bee added successuvely from unitie, there will be likewise made* square numbers.

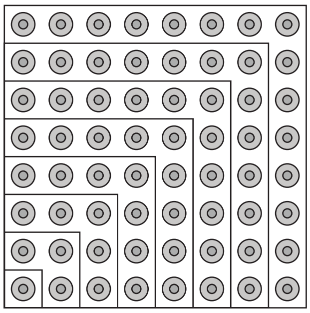
Example 2: Experimental evidence

$$1 = 1 = 1^2$$
$$1 + 3 = 4 = 2^2$$
$$1 + 3 + 5 = 9 = 3^2$$
$$1 + 3 + 5 + 7 = 16 = 4^2$$
$$1 + 3 + 5 + 7 + 9 = 25 = 5^2$$
$$1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2$$
$$\vdots$$
$$1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2.$$

When asked to rank the proofs from most to least rigourous, students clearly and consistently ranked experimental evidence as the least rigorous 'proof'. The following pictorial proofs were ranked the next least rigourous. However students were asked which proof gave more 'understanding of why the theorem is true' and the following pictorial proof was one of the highest ranking proofs for understanding. See [20] for an interesting discussion about the relationship of pictures to rigorous proof. The following 'proof by picture' is attributed to Nicomachus of Gerasa, circa 100CE: [21, p. 243], see also [22, p. 71]. There are many references to this picture, throughout the books I consulted, e.g. [23, I. p. 442].

https://doi.org/10.1017/mag.2023.3 Published online by Cambridge University Press

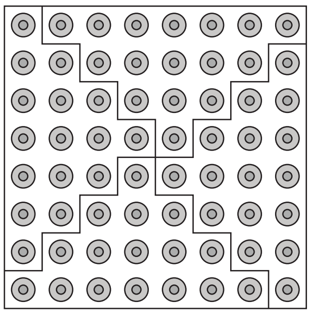
Example 3: Pictorial I



$$1 + 3 + 5 + 7 + \dots + (2n - 1) = n^2.$$

The next 'proof by picture' was not so well received by students. In the form below, it was ranked very poorly on both rigour and in giving students understanding. However, adding two sentences explaining (1) the stepped triangle at the bottom has $1 + 3 + 5 + \dots + (2n - 1)$ dots and (2) four copies of this triangle can be fitted together to give a square, changed significantly how much the proof helped students to understand the theorem.

Example 4: Pictorial II [22, p. 72]



$$1 + 3 + 5 + 7 + \dots + (2n - 1) = \tfrac{1}{4}(2n)^2 = n^2.$$

Example 5: Expert's approach

$$1 + 3 + 5 + 7 + \dots + (2n - 1)$$

is an arithmetic progression with difference 2 and n terms. The first term $a_1 = 1$, and the last term $a_n = 2n - 1$, and the sum of an AP is $\tfrac{1}{2}n\left(a_1 + a_n\right)$, which in this case is $\tfrac{1}{2}n\left(1 + 2n - 1\right) = n^2$.

The arithmetic progression approach is found in [24, §422], [25, p. 219], and many others. Fibonacci, [26, p. 260], uses this result to illustrate summing an arithmetic progression.

https://doi.org/10.1017/mag.2023.3 Published online by Cambridge University Press

Example 6: Reversed list

Write the terms twice, with the second list reversed.

$$\begin{array}{ccccccccccc} 1 & + & 3 & + & 5 + & \dots & + & 2n-3 & + & 2n-1 \\ 2n-1 & + & 2n-3 & + & 2n-5 + & \dots & + & 3 & + & 1 \end{array}$$

Each column has total $2n$ and there are n columns. So the total is $2n^2$, proving that $\sum_{k=1}^{n}(2k - 1) = n^2$.

The reversed argument is often part of an apocryphal story which suggests Gauss summed the integers from 1 to 100 by reversing the lists and adding, [27].

Example 7: Telescope

Notice that $2k - 1 = k^2 - (k - 1)^2$, so that adding up we have

$$\sum_{k=1}^{n}(2k - 1) = \sum_{k=1}^{n} k^2 - (k - 1)^2 .$$

Then in

$$\sum_{k=1}^{n} k^2 - (k-1)^2 = \left(1^2 - 0^2\right) + \left(2^2 - 1^2\right) + \left(3^2 - 2^2\right) + \ \dots \ + \left(n^2 - (n-1)^2\right)$$

all terms cancel except two, one from the first term and one from the last, i.e. $-0^2 + n^2$, leaving n^2.

Example 8: Backwards reasoning [28, p. 210]

The Fundamental Theorem of Finite Differences says that $S_n = \sum_{k=1}^{n} a_k$ if, and only if, (i) $a_{n+1} = S_{n+1} - S_n$, and (ii) $S_1 = a_1$.

Consider $S_n = n^2$ then

$$S_{n+1} - S_n = (n + 1)^2 - n^2 = 2n + 1.$$

Take $a_{n+1} = 2n + 1$ and then $a_n = 2n - 1$ from which $S_1 = 1^2 = 2 \times 1 - 1 = a_1$. Hence $\sum_{k=1}^{n}(2k - 1) = n^2$.

The Telescoping argument is self-contained, but makes use of the ellipsis $(\dots)$. The ellipsis also occurred in Experimental evidence in vertical form. In many situations the ellipsis can be an aid to readability by creating succinct written arguments and here it is a helpful abbreviation. In general the ellipsis is ambiguous since the reader has to spot a pattern, and hence the ellipsis lacks formal rigour. The Backwards argument side-steps the lack of rigour by applying a theorem to avoid this problem. However, the proof of the Fundamental Theorem given by [28, p. 210] does no more than use an ellipsis anyway, and does not use a formal induction: applying a Fundamental Theorem may appear more rigorous, but it relies on the strength of the foundations upon which it is built. It is interesting that fashion plays a role in what we do and what we do not teach. The Theory of Finite Differences itself is now completely neglected as a subject [29]. The sums of the powers of the integers, i.e. (2), do not seem to appear widely in core undergraduate courses despite being an important and classical topic. The sums of the powers of the integers provide a way to prove Theorem 1 and systematically sum integer series of a similar type.

Example 9: Rearranging I

We use the standard results $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ and $\sum_{k=1}^{n} 1 = n$ and rearrange:

$$\sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = 2\frac{n(n+1)}{2} - n = n^2.$$

Example 10: Rearranging II

We use the standard result $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ and rearrange:

$$\sum_{k=1}^{n} \underbrace{2k-1}_{odd} = \underbrace{(1+2+3+\dots+2n)}_{all} - \underbrace{(2+4+6+\dots+2n)}_{even}$$
$$= (1+2+3+\dots+2n) - 2(1+2+3+\dots+n).$$

Hence

$$\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{2n} k - 2\sum_{k=1}^{n} k$$
$$= \frac{2n(2n+1)}{2} - 2\frac{n(n+1)}{2} = n^2.$$

The Telescope and the Backwards reasoning proof generalise but do not solve. Rearranging I solves and generalises, but Rearranging II solves but does not generalise! Some of the following proofs both generalise and solve the problem (e.g. linear system and undetermined coefficients).

Example 11: Historical form

The sum of any number of successive odd numbers, beginning from unity, is a square number, namely the square of half the even number which follows the last odd number. Let this proposition be true in any one single instance; that is, n being some whole number, let $1, 3, 5, \dots$ up to $2n+1$ put together give $(n+1)^2$. Then the next odd number being $2n+3$, the sum of all the odd numbers up to $2n + 3$ will be $(n + 1)^2 + 2n + 3$, or $n^2 + 4n + 4$, or $(n + 2)^2$. But $n + 2$ is half of the even number next following $2n + 3$: consequently, if the proposition be true of any one set of odd numbers, it is true of one more. But it is true of the first odd number 1, for this is the square of half the even number next following. Consequently, being true of 1, it is true of $1 + 3$; being true of $1 + 3$, it is true of $1 + 3 + 5$; being true of $1 + 3 + 5$, it is true of $1 + 3 + 5 + 7$, and so on, *ad infinitum*.

This is De Morgan's proof. One thing to note about it is the rhetorical style in which he states the induction hypothesis without any algebraic symbolism, making for densely packed text. The following is a more modern presentation of induction. In our research the modern proof by induction below was, by far, considered the most rigorous proof by the student research participants.

Example 12: Induction

Let $P(n)$ be the statement $\sum_{k=1}^{n} (2k - 1) = n^2$.

Since $\sum_{k=1}^{1} (2k - 1) = 1 = 1^2$ we see $P(1)$ is true.

Assume $P(n)$ is true then

$$\sum_{k=1}^{n+1} (2k + 1) = \sum_{k=1}^{n} (2k - 1) + \big(2(n + 1) - 1\big) = n^2 + 2n + 1 = (n + 1)^2.$$

Hence $P(n + 1)$ is true.

Since $P(1)$ is true and $P(n + 1)$ follows from $P(n)$ we conclude that $P(n)$ is true for all n by the principle of mathematical induction.

The following proof by contradiction is a proof by infinite descent, also known as Fermat's method of descent. Any induction can be reformulated as such a proof by contradiction, but, to my taste at least, an induction (when available) is more direct and has greater clarity. I wrote some thoughts about proof by contradiction elsewhere, [30].

Example 13: Contradiction

To prove $\forall n \in \mathbb{N} : \sum_{k=1}^{n} (2k - 1) = n^2$, assume, for a contradiction, that $\exists n \in \mathbb{N} : \sum_{k=1}^{n} (2k - 1) \neq n^2$. Let m be the smallest such example. Note, $m > 1$ since $(2 \times 1) - 1 = 1^2$. If $\sum_{k=1}^{m} (2k - 1) > m^2$ then

$$\sum_{k=1}^{m} (2k - 1) = 2m - 1 + \sum_{k=1}^{m-1} (2k - 1) > m^2$$

and so

$$\sum_{k=1}^{m-1} (2k - 1) > m^2 - 2m + 1 = (m - 1)^2 .$$

This proves $\sum_{k=1}^{m-1} (2k - 1) \neq (m - 1)^2$, which contradicts the minimality of m. The case $\sum_{k=1}^{m-1} (2k - 1) < (m - 1)^2$ leads to an identical contradiction.

An anonymous reviewer of this Article asked why this argument is split into cases > and <, rather than using ≠ throughout the argument. The law of trichotomy states that every real number is either positive, negative, or zero. This particular contradiction argument uses the law of trichotomy to create three exhaustive cases. An alternative argument could be created using ≠ but that is not the particular contradiction I choose to write. Which is better style? This is precisely the sort of discussion collections of proofs of the same theorem, such as this, is designed to provoke.

The following two proofs rely on assuming that the sum takes a particular algebraic form.

Example 14: Linear system

Since the sum is always an integer and $S_n = \sum_{k=1}^{n} (2k - 1) \leqslant n(2n - 1)$ the growth of S_n is quadratic in n. We therefore assume

$$\sum_{k=1}^{n} (2k - 1) = an^2 + bn + c \text{ for all } n \in \mathbb{N}.$$

Since this formula holds for all n it must hold for $n = 1, 2, 3$. Hence

$$\begin{aligned} 1 &= a + b + c, & (n = 1), \\ 1 + 3 &= 4a + 2b + c, & (n = 2), \\ 1 + 3 + 5 &= 9a + 3b + c, & (n = 3). \end{aligned}$$

This is a linear system in a, b, c which we set up as

$$\begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}.$$

The matrix clearly has non-zero determinant, so the system has a unique solution. This solution is (exercise to check) $a = 1$, $b = c = 0$. Hence $\sum_{k=1}^{n} (2k - 1) = n^2$.

https://doi.org/10.1017/mag.2023.3 Published online by Cambridge University Press

Example 15: Undetermined coefficients [31, p. 408]

Assume $1 + 3 + 5 + 7 + \dots + (2n - 1) = A + Bn + Cn^2 + Dn^3 + En^4 + \dots$. Then

$$1 + 3 + 5 + 7 + \dots + (2n - 1) + \big(2(n + 1) - 1\big)$$
$$= A + B(n + 1) + C(n + 1)^2 + Dn(n + 1)^3 + E(n + 1)^4 + \dots .$$

Subtracting:

$$2n + 1 = B + C(2n + 1) + D\big(3n^2 + 3n + 1\big) + E\big(4n^3 + 6n^2 + 4n + 1\big) + \dots .$$

Equating powers of $n^2, n^3, \dots$ on both sides we see $D = E = \dots = 0$. Therefore

$$2n + 1 = B + C(2n + 1).$$

Hence $B = 0$ and $C = 1$, from which $A = 0$ and so $\sum_{k=1}^{n} (2k - 1) = n^2$.

The particular form of undetermined coefficients used here makes an assumption of a series solution, rather than a polynomial. Historically, mathematicians were less worried about the rigours of convergence than we are today and I have consciously chosen this historical form here to contrast with the assumption of a particular polynomial in the previous Linear system proof. The use of an ansatz, i.e. an educated guess, is a relatively common proof gambit.

Differentiating a geometric progression $1 + r + r^2 + \dots + r^{n-1}$ repeatedly with respect to r is one way to establish formulae for the sums of the powers of the integers, (2). An identical approach can be used to prove Theorem 1, but the following proof is somewhat forced and the calculus steps have a level of technical difficulty out of proportion with the complexity of Theorem 1.

Example 16: Calculus

The sum of a geometric progression $1 + r + r^2 + \dots + r^{n-1}$ is given by $\sum_{k=0}^{n-1} r^k = \dfrac{r^n - 1}{r - 1}$ for $r \neq 1$. So for $r \neq 1$

$$\sum_{k=1}^{n} \left(r^2\right)^k = \frac{r^2\left(r^{2n} - 1\right)}{r^2 - 1} \text{ and so } \sum_{k=1}^{n} r^{2k-1} = \frac{r^{2n+1} - r}{r^2 - 1}.$$

Differentiating both sides with respect to r (details left as an exercise)

$$\sum_{k=1}^{n} (2k - 1) r^{2k-2} = \frac{r^{2n}\left((2n - 1)r^2 - 2n - 1\right) + \left(r^2 + 1\right)}{(r - 1)^2 (r + 1)^2}.$$

Since we cannot directly substitute $r = 1$ on the right-hand side we take the limit (two applications of l'Hôpital)

$$\sum_{k=1}^{n} (2k - 1) = \lim_{r \to 1} \frac{r^{2n}\left((2n - 1)r^2 - 2n + 1\right) + \left(r^2 + 1\right)}{(r - 1)^2 (r + 1)^2} = n^2.$$

https://doi.org/10.1017/mag.2023.3 Published online by Cambridge University Press

The following proofs illustrate other, more advanced, mathematical ideas. Given the variety of proofs above, these are unlikely to be used as serious proofs in an elementary setting. These proofs could be used to make use of a very familiar result (Theorem 1) to illustrate applying a new technique. I make no further comments about the individual proofs.

Example 17: Linear difference equation

$S_n = \sum_{k=1}^{n} 2k - 1$ gives the linear difference equation with constant coefficients

$$S_{n+1} - S_n = (2n + 1), \tag{3}$$

$$S_{n+2} - S_{n+1} = \big(2(n + 1) + 1\big). \tag{4}$$

Subtracting (3) from (4) gives

$$S_{n+2} - 2S_{n+1} + S_n = \big(2(n + 1) + 1\big) - (2n + 1) = 2. \tag{5}$$

It follows that

$$S_{n+3} - 2S_{n+2} + S_{n+1} = 2. \tag{6}$$

Subtracting (5) from (6) gives the following homogenous linear difference equation with constant coefficients

$$S_{n+3} - 3S_{n+2} + 3S_{n+1} - S_n = 0.$$

Substitute in the standard ansatz $S_n = \lambda^n$ gives $\lambda^{n+3} - 3\lambda^{n+2} + 3\lambda^{n+1} - \lambda^n = 0$, so $\lambda = 1$ is a repeated root of the homogenous equation. Since we have a repeated root, the general solution takes the form

$$S_n = c\lambda^n + bn\lambda^n + an^2\lambda^n = an^2 + bn + c.$$

Substituting this into $S_{n+1} - S_n = (2n + 1)$ gives

$$a(n + 1)^2 + b(n + 1) + c = 2n + 1.$$

Equating coefficients, and using $S_1 = 1$, gives $S_n = \sum_{k=1}^{n} 2k - 1 = n^2$.

Example 18: Matrix power

$S_n = \sum_{k=1}^{n} 2k - 1$ gives rise to a linear difference equation with constant coefficients $S_{n+1} - S_n = a_n$. Define

$$\mathbf{x}_n = \begin{pmatrix} S_n \\ a_n \\ t_n \end{pmatrix}$$

https://doi.org/10.1017/mag.2023.3 Published online by Cambridge University Press

then we have the vector equation $\mathbf{x}_{n+1} = M\mathbf{x}_n$ where

$$\begin{pmatrix} S_{n+1} \\ a_{n+1} \\ t_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_n \\ a_n \\ t_n \end{pmatrix}, \ \mathbf{x}_1 = \begin{pmatrix} S_1 \\ a_1 \\ t_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Note $t_n = 1$ for all n, and $\mathbf{x}_n = M^{n-1}\mathbf{x}_1$. Unfortunately M is not diagnonalisable, (otherwise we could readily calculate the matrix power). If $M = PJP^{-1}$ where J is in Jordan form we can calculate the power of the matrix, and the power of each Jordan block can be written as:

$$J_k(\lambda)^n = \begin{pmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \cdots & \cdots & \binom{n}{k-1}\lambda^{n-k+1} \\ & \lambda^n & \binom{n}{1}\lambda^{n-1} & \cdots & \cdots & \binom{n}{k-2}\lambda^{n-k+2} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & \lambda^n & \binom{n}{1}\lambda^{n-1} \\ & & & & & \lambda^n \end{pmatrix}.$$

Calculating the Jordan form in our case:

$$M = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix},$$

so

$$M^n = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1^n & \binom{n}{1}1^{n-1} & \binom{n}{2}1^{n-2} \\ 0 & 1^n & \binom{n}{1}1^{n-1} \\ 0 & 0 & 1^n \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

https://doi.org/10.1017/mag.2023.3 Published online by Cambridge University Press

$$= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & n & \frac{1}{2}n(n-1) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & n & n(n+1) \\ 0 & 1 & 2n \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, using $\mathbf{x}_n = M^{n-1}\mathbf{x}_1$,

$$\begin{pmatrix} S_n \\ a_n \\ t_n \end{pmatrix} = \begin{pmatrix} 1 & n-1 & (n-1)n \\ 0 & 1 & 2(n-1) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} n^2 \\ 2n-1 \\ 1 \end{pmatrix}.$$

Example 19: z-transform

$x(n) = \sum_{k=1}^{n} 2k - 1$ gives a linear difference equation

$$x(n+1) - x(n) = 2n + 1,\ x(0) = 0.$$

Taking the one-sided z-transform $X(z) = Z[x(n)]$ gives

$$zX(z) - x(0) - X(z) = \frac{2z}{(z-1)^2} + \frac{x}{(z-1)}$$

$$X(z)(z-1) = \frac{z(z+1)}{(z-1)^2}.$$

Giving $X(z) = \dfrac{z(z+1)}{(z-1)^3}$. The inverse z-transform gives $x(n) = n^2$.

Example 20: Generating functions [32, pp. 249-250]

Let $S = 1 + 3x + 5x^2 + 7x^3 + \dots$ and consider $(1 + 3x) + 2x(S - 1) - x^2S$:

$$(1 + 3x) + 2x(S - 1) - x^2S$$

$$= 1 + 3x + (6x^2 + 10x^3 + 14x^4 + \dots) - (x^2 + 3x^3 + 5x^4 + \dots)$$

$$= 1 + 3x + 5x^2 + 7x^3 + \dots = S.$$

So $S = (1 + 3x) + 2x(S - 1) - x^2S$, and solving for S gives the *generating function*

$$\sum_{k=0}^{\infty} (2k-1)x^k = \frac{1+x}{(1-x)^2}. \tag{7}$$

https://doi.org/10.1017/mag.2023.3 Published online by Cambridge University Press

Similarly

$$(2n + 1)x^n + (2n + 3)x^{n+1} + (2n + 5)x^{n+2} + \dots$$
$$= \sum_{k=n}^{\infty} (2k + 1)x^k = \frac{(2n + 1)x^n - (2n - 1)x^{n+1}}{(1 - x)^2}.$$

Hence

$$\sum_{k=n}^{\infty} (2k - 1)x^k = \frac{1 + x - (2n + 1)x^n + (2n - 1)x^{n+1}}{(1 - x)^2}.$$

Taking the limit as $x \to 1$ we get $\sum_{k=1}^{n} (2k - 1) = n^2$.

The generating function (7) is derived in a different way by [33, pp. 181-185]. The power series $\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \dots$ and its derivative $\frac{1}{(1 - x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$ can be substituted into

$$1 + 3x + 5x^2 + 7x^3 + \dots$$
$$= 2\left(1 + 2x + 3x^2 + 4x^3 + \dots\right) - \left(1 + x + x^2 + x^3 + \dots\right)$$

to derive (7). The rest of the argument is identical.

Conclusion

Reference [5] is an inspirational book, destined (I hope) to become a classic. [34, p. 14] opens by saying: ‘The main use of a model is the pleasure derived from making it’, and I have derived considerable personal pleasure in collecting together these proofs of Theorem 1. A selection of these proofs is being used to investigate conceptions of insight and rigour, and I hope publishing the full collection of proofs is helpful for future educational research. Lastly, I suggest this collection of proofs might be a useful resource for teachers when discussing proof with their students, e.g. to consider different styles of proof and the relative merits of different proofs, both in relation to a particular theorem and more generally.

References

1. M. Aigner and G. Ziegler, *Proofs from THE BOOK* (6th edn.), Springer (2018).
2. D. Kalman, Six ways to sum a series *The College Mathematics Journal*, **24**(5): (1993) pp. 402-421.
3. E. Loomis, *The pythagorean proposition*, National Council of Teachers of Mathematics, Washington (1968).
4. L. Duchêne and A. LeBlanc, *Rationnel mon Q*, Hermann (2010).

5. P. Ording, *99 Variations on a proof*, Princeton University Press, (2019).
6. J. P. Mejía-Ramos, T. Evans, C. Rittberg and M. Inglis, Mathematicians' assessments of the explanatory value of proofs, *Axiomathes*, **3** (2021) pp. 1-25.
7. G. Pólya, *Mathematics and plausible reasoning, Vol.1: Induction and analogy in mathematics, Vol.2: Patterns of plausible inference*, Princeton University Press (1954).
8. A. W. F. Edwards, Sums of powers of integers: a little of the history, *Math. Gaz*. **66** (March 1982) pp. 22-28.
9. R. P. Burn, Sums of powers of integers—how Fermat may have found them, *Math. Gaz*. **94** (March 2010) pp. 18-26.
10. R. V. W. Murphy, Summing powers of natural numbers, *Math. Gaz*. **93** (July 2009) pp. 279-285.
11. F. T. Howard, Sums of powers of integers via generating functions, *The Fibonacci Quarterly*, **34**(3) (1996) pp. 244-256.
12. D. Kalman, Sums of powers by matrix methods, *The Fibonacci Quarterly*, **28**(1) (1990) pp. 60-71.
13. F. Cajori, Origin of the name 'mathematical induction', *Amer. Math. Monthly*, **25** (1918) pp. 197-201.
14. A. De Morgan, Mathematical induction, *The Penny Cyclopedia* **XII** (1838) pp. 465-466.
15. J. J. Rotman, *Journey into mathematics: an introduction to proofs*, Prentice Hall, (1998).
16. D. J. Velleman, *How to prove it: a structured approach*, Cambridge University Press (2006).
17. G. Pólya, *Mathematical discovery: on understanding, learning, and teaching problem solving*, Wiley (1962).
18. M. Raman-Sundström and L. Ohman, Mathematical fit: a case study, *Philosophia Mathematica*, **26**(2) (2016) pp. 184-210.
19. W. Oughtred, *Mathematical recreations*, T. Cotes, London (1633).
20. J. R. Brown, *Philosophy of mathematics: a contemporary introduction to the world of proofs and pictures*, Routledge (2008).
21. M. L. D'Ooge, *Nicomachus of Gerasa: introduction to arithmetic*, MacMillan (1926).
22. R. B. Nelson, *Proofs without words: exercises in visual thinking*, The Mathematical Association of America (1993).
23. Franciscus van Schooten, *Exercitationes mathematicae libri quinque*, Lugd. Batavor.: Ex officina elzeviriorum (1657).
24. L. Euler, *Elements of algebra* (3rd edn.), Longman, Hurst, Rees, Orme and Co. (1822). Translated by J. Hewlett from the French, with the notes of M. Bernoulli and the additions of M. de La Grange.
25. J. Day, *An introduction to algebra, being the first part of a course of mathematics* (32nd edn.), Durrie & Peck (1838).

26. L. E. Sigler, *Fibonacci's Liber Abaci: Leonardo Pisaro's book of calculation*, Springer (2002).
27. B. Hayes, Gauss's day of reckoning, *American Scientist* **94**(3) (2006).
28. C. O. Tuckey and W. Armistead, *Algebra* (3rd edn), Longmans, Green & Co. (1962).
29. G. Boole, *A Treatise on the calculus of finite differences*, Cosimo (2007).
30. G. Kinnear and C. J. Sangwin, Contradiction and contrapositive, what is the difference? *Scottish Mathematics Council Journal* **48** (2018) pp. 84-90.
31. I. Todhunter, *Algebra for the use of colleges and schools* (5th edn), MacMillian (1897).
32. T. Lund, *The elements of algebra designed for the use of students in the university* (14th edn), Longman, Brown, Green and Longmans (1852).
33. L. Euler, Introduction to *Analysis of the Infinite*, volume I, Springer, (1988). Translated by J. Blanton from the Latin *Introductio in analysin infinitorum* (1748).
34. H. M. Cundy and A. P. Rollett, *Mathematical Models*, Oxford University Press (1961).

10.1017/mag.2023.3

CHRIS SANGWIN
School of Mathematics,
University of Edinburgh, EH9 3FD
e-mail: *C.J.Sangwin@ed.ac.uk*

Quotations for *Nemo* (continued from page 9)

3. He could take the small clutching child from his nurse's arms with an iteration grimly discountenanced, in respect to their contents, by the glass doors of high cabinets.

4. 'But we can wait a long time,' said poor Catherine in a tone which was meant to express the humblest conciliation, but which had upon her father's nerves the effect of an iteration not characterised by tact.

5 Oh, he was princely indeed: that came out more and more with every word he said and with the particular way he said it, and M could feel his monitress stiffen almost with anguish against the increase of his spell and then hurl herself as a desperate defence of it into the quite confessed poorness of violence, of iteration.

6. His answer came, promptly, with his reawakened wrath: it was of course exactly what they wanted, and what they were "at" him for daily, with the iteration of people who couldn't for their life understand a man's liability to decent feelings