THE WAKE OF A FINITE ROTATING DISC

L. M. LESLIE

(Received 10 November 1969; revised 11 February 1970)

Communicated by A. F. Pillow

1. Introduction

When a disc rotates in a fluid at rest, fluid near the disc acquires azimuthal momentum because of the viscous torque of the disc and outwards radial momentum under the action of centrifugal forces. The resultant flow is essentially a swirling jet. Away from the disc continuity requires the existence of an axial flow towards the disc to compensate for the fluid which has been thrown outwards. If the disc is finite there is a discontinuity in the boundary conditions at the edge of the disc where the no-slip condition is suddenly replaced by a condition of zero stress in the plane of the disc. The flow discharged from the edge of the disc is essentially a wake embedded in a swirling radial jet. It appears that no investigation of this wake has yet been made.

Before proceeding, the results of research into a closely related flow problem, namely, the wake that forms downstream from the trailing edge of a flat plate placed parallel to a uniform stream, should be examined. The boundary layer equations for the flow near a semi-infinite flat plate were first solved by Blasius [1]. Goldstein [3] went one step further by solving the boundary layer equations for the wake that develops downstream from the trailing edge of a finite flat plate. Goldstein assumed that the boundary layer approximation is valid downstream from the trailing edge itself and obtained a solution for the near wake, the near wake being defined by $0 < x \ll l$, where x denotes distance downstream from the trailing edge and l is the length of the plate. However, as has been shown by Stewartson [6], [7], the boundary layer approximation fails in the neighbourhood of the trailing edge and the full Navier-Stokes must be used there. Goldstein's solution is therefore not appropriate to all of the near wake but to a slightly smaller domain which Stewartson refers to as the Goldstein wake. Stewartson [7] has demonstrated that for high Reynolds number flows the neighbourhood of the trailing edge, which connects the Blasius flow over the plate with the Goldstein wake, may be split into two main regions. Suppose that the Reynolds number of the flow is defined by $\mathrm{Re} = U_0 l/\nu$, where U_0 is the free stream velocity and ν is the kinematic viscosity; that ε is defined by $\varepsilon^8 = \mathrm{Re}^{-1}$; and that y measures perpendicular distance from the x axis. Then Stewartson has shown that the neighbourhood of the trailing edge consists of 'a central region in which $x \sim \varepsilon^6 l$, $y \sim \varepsilon^6 l$ . . . and an intermediate region in which $x \sim \varepsilon^3 l$, $y \sim \varepsilon^3 l$'.

The primary aim of this paper is to show how the method developed by Goldstein for the flat plate may be extended to obtain a solution for the near wake of the discharge from a finite rotating disc. As in the case of the flat plate, this solution is not valid in the neighbourhood of the edge of the disc. Any examination of the flow in this neighbourhood would necessarily involve the full Navier-Stokes equations and no such analysis will be attempted here. Following Stewartson, the domain for which Goldstein's solution is appropriate will be referred to as the Goldstein wake of a finite rotating disc. Smith [5] has extended Goldstein's method to advance the solution for the free convection boundary layer along a vertical plate past the level at which a discontinuity in plate temperature occurs. Smith has included a detailed discussion of the method and only a brief outline is necessary here. The method, as originally used by Goldstein, consists in forming two expansions for the stream function, one valid near the wake axis (the inner expansion) and the other valid away from the wake axis (the outer expansion). The inner expansion satisfies the boundary conditions at

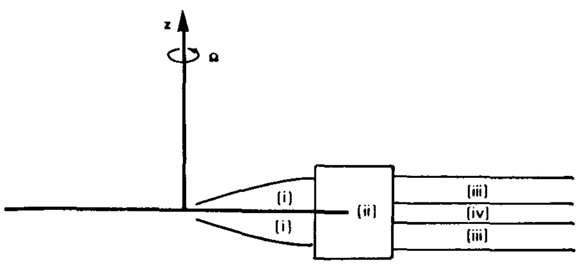

Figure (1a)

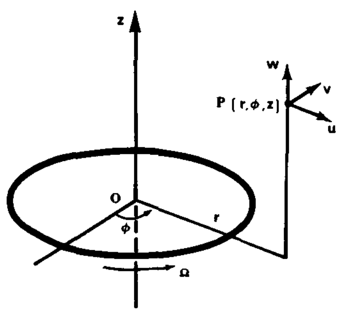

Figure (1b)

Fig. 1(a): Illustrates the various flow regions near the edge of the disc: in (i) Cochran's (1934) solution is valid; in (ii) the boundary layer equations are not valid; while (iii) and (iv) are the Goldstein outer and inner wake regions respectively. (b) Shows the co-ordinate system used.

https://doi.org/10.1017/S1446788700013707 Published online by Cambridge University Press

the wake axis and its form is suggested by the Blasius solution. The outer expansion satisfies the constraints imposed on the solution far from the wake axis and the two expansions are matched by re-arranging the asymptotic expression for the inner expansion into the same form as the outer expansion. For the rotating disc Goldstein's method is extended by forming expansions not only of the stream function but also of the azimuthal velocity, and the form of the inner expansions is derived from Cochran's [2] improvement on the original solution by von Karman [8] for the flow near an infinite rotating disc. The main flow regions near the edge of the disc are illustrated in Fig. 1(a).

2. The equations of motion

Let (r, φ, z) be cylindrical polar co-ordinates with origin at the centre of the disc and z axis of rotation. The velocity vector is (u, v, w) and the finite disc is of radius a.

The boundary layer equations for flow in the Goldstein wake near a rotating disc are, as given by Schlichting [4]:

$$u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} = \nu \frac{\partial^2 u}{\partial z^2}, \tag{2.1}$$

$$u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{uv}{r} = \nu \frac{\partial^2 v}{\partial z^2}, \tag{2.2}$$

$$w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \frac{\partial^2 w}{\partial z^2}, \tag{2.3}$$

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0. \tag{2.4}$$

Since the boundary layer equations involve the assumption that p and w are independent of r (2.3) may be written as

$$\frac{p}{\rho} + \tfrac{1}{2}w^2 - \nu \frac{\partial w}{\partial z} = \text{const.} \tag{2.5}$$

It remains only to solve (2.1), (2.2), (2.4) subject to the boundary conditions of the problem. Then p may be calculated from (2.5).

For an infinite plate rotating in a still environment the boundary conditions are

$$\begin{aligned} z &= 0: & u &= w = 0, & v &= r\Omega; \\ z &= \infty: & u &= v = 0, & w &\text{ finite}; \end{aligned} \tag{2.6}$$

where Ω is the angular velocity of the disc. Equations (2.1), (2.2), (2.4), (2.6) have been solved by von Kármán [8] and Cochran [2]. They defined new variables F, G, H, Z, by

$$\begin{aligned} u &= r\Omega F(Z), \\ v &= r\Omega G(Z), \\ w &= \sqrt{\nu\Omega} H(Z), \\ z &= \sqrt{\frac{\nu}{\Omega}} Z. \end{aligned}$$

The functions F, G, H satisfy the following simultaneous system of ordinary differential equations

$$\begin{aligned} F^2 + F'H - G^2 - F'' &= 0 \\ 2FG + HG' - G'' &= 0, \\ 2F + H' &= 0, \end{aligned} \tag{2.7}$$

together with the transformed boundary conditions

$$\begin{aligned} Z = 0: &\quad F = H = 0, \quad G = 1; \\ Z = \infty: &\quad F = G = 0, \quad H \text{ finite}. \end{aligned}$$

The functions F, G, H, F', G' have been tabulated by Cochran.

3. The continuation problem

The continuation problem is to advance the solution to equations (2.1), (2.2), (2.4) radially outwards, given the Cochran velocity profiles at the edge of the disc. A transformation to non-dimensional variables of form appropriate to the outflow from the disc edge is

$$\begin{aligned} u &= a\Omega U, \\ v &= a\Omega V, \\ w &= \sqrt{\nu\Omega} W, \\ r &= a(1+R), \\ z &= \sqrt{\frac{\nu}{\Omega}} Z. \end{aligned}$$

The transformed equations are

$$\begin{aligned} U\frac{\partial U}{\partial R} + W\frac{\partial U}{\partial Z} - \frac{V^2}{1+R} &= \frac{\partial^2 U}{\partial Z^2}, \\ U\frac{\partial V}{\partial R} + W\frac{\partial V}{\partial Z} + \frac{UV}{1+R} &= \frac{\partial^2 V}{\partial Z^2}, \\ \frac{\partial}{\partial R}\{(1+R)U\} + \frac{\partial}{\partial Z}\{(1+R)W\} &= 0. \end{aligned} \tag{3.1}$$

https://doi.org/10.1017/S1446788700013707 Published online by Cambridge University Press

The profiles of U, V, W at $R = 0$, which are the Cochran profiles F, G, H, may be expanded in Maclaurin series

$$
\begin{aligned}
U &= a_1 Z + a_2 Z^2 + a_3 Z^3 + \cdots, \\
V &= b_0 + b_1 Z + b_2 Z^2 + \cdots,
\end{aligned}
\tag{3.2}
$$

where, by successive differentiation of equations (2.7) it is easily shown that

$$
\begin{aligned}
a_1 &= 0.510, & b_1 &= -0.616; \\
a_2 &= -1/2!, & b_2 &= 0; \\
a_3 &= -2b_1/3!, & b_3 &= 2a_1/3!.
\end{aligned}
$$

4. The inner solution

From the third of equations (3.1) there is a stream function ψ such that

$$
\begin{aligned}
U &= \frac{1}{1+R}\frac{\partial \psi}{\partial Z}, \\
W &= -\frac{1}{1+R}\frac{\partial \psi}{\partial R}.
\end{aligned}
\tag{4.1}
$$

Following Goldstein, new independent variables ξ and η are defined by

$$
\begin{aligned}
\xi &= R^{\frac{1}{3}}, \\
\eta &= ZR^{-\frac{1}{3}}/3,
\end{aligned}
\tag{4.2}
$$

and the stream function and azimuthal velocity are taken to be

$$
\begin{aligned}
\psi &= \xi^2 f(\xi, \eta), \\
V &= g(\xi, \eta).
\end{aligned}
\tag{4.3}
$$

Then, from (4.1) and (4.3)

$$
\begin{aligned}
U &= \frac{\xi}{3(1+\xi^3)} f_\eta, \\
W &= -\frac{\xi^{-1}}{3(1+\xi^3)}(2f + \xi f_\xi - \eta f_\eta),
\end{aligned}
\tag{4.4}
$$

and the initial conditions (3.2) become

$$
\begin{aligned}
U &= a_1(3\eta\xi) + a_2(3\eta\xi)^2 + \cdots, \\
V &= b_0 + b_1(3\eta\xi) + b_2(3\eta\xi)^2 + \cdots.
\end{aligned}
\tag{4.5}
$$

A comparison of (4.4) and (4.5) suggests the following power series expansions in ξ for f and g:

$$f = \sum_{m=0}^{\infty} h_m(\eta)\xi^m, \tag{4.6}$$
$$g = \sum_{m=0}^{\infty} k_m(\eta)\xi^m.$$

If these series are substituted in equations (4.4) and (3.1), two sets of ordinary differential equations are obtained by equating powers of ξ to zero. The first four equations of each set are:

$$\begin{aligned}
h_0''' + 2h_0 h_0'' - h_0'^2 &= 0,\\
h_1''' + 2h_0 h_1'' - 3h_0' h_1' + 3h_0'' h_1 &= -27k_0^2,\\
h_2''' + 2h_0 h_2'' - 4h_0' h_2' + 4h_0'' h_2 &= 2h_1'^2 - 3h_1 h_1'' - 54k_0 k_1,\\
h_3''' + 2h_0 h_3'' - 5h_0' h_3' + 5h_0'' h_3 &= 5h_1' h_2' - 3h_1 h_2'' - 4h_1'' h_2\\
&\quad -27k_1^2 - 54k_0 k_2 - 4h_0'^2 + 2h_0 h_0'';
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
k_0'' + 2h_0 h_0' &= 0,\\
k_1'' + 2h_0 k_1' - h_0' k_1 &= -3h_1 k_0',\\
k_2'' + 2h_0 k_2' - 2h_0' k_2 &= h_1' k_1 - 3h_1 k_1' - 4h_2 k_0',\\
k_3'' + 2h_0 k_3' - 3h_0' k_3 &= 3h_0' k_0 + 2h_1' k_2 + h_2' k_1 - 3h_1 k_2' - 4h_2 k_1' - 5h_3 k_0'.
\end{aligned} \tag{4.8}$$

The boundary conditions to be satisfied by U, V and W in the region $R > 0$ are

$$Z = 0 : W = \frac{\partial U}{\partial Z} = \frac{\partial V}{\partial Z} = 0;$$
$$Z = \infty : U = V = 0, \ W \text{ finite}.$$

The boundary conditions at $Z = 0$ transform to

$$\eta = 0 : h_m = h_m'' = k_m' = 0, \tag{4.9}$$

while the boundary conditions as $\eta \to \infty$ are obtained by equating the series expansions of U and V with the initial conditions in the form (4.5). Thus

$$\begin{aligned}
&\lim_{\eta\to\infty} \frac{h_0'}{\eta} = 3^2 a_1,\\
&\lim_{\eta\to\infty} \frac{h_1'}{\eta^2} = 3^3 a_2,\\
&\lim_{\eta\to\infty} \frac{h_2'}{\eta^3} = 3^4 a_3,\\
&\lim_{\eta\to\infty} \frac{h_3' - h_0'}{\eta^4} = 3^5 a_4;
\end{aligned} \tag{4.10a}$$

https://doi.org/10.1017/S1446788700013707 Published online by Cambridge University Press

(4.10b)
$$\lim_{\eta\to\infty} \frac{k_m}{\eta^m} = 3^m b_m \qquad (m = 0, 1, 2, 3).$$

The solution for h_0

The equation for h_0 was investigated by Goldstein op. cit. He obtained the following solution

(4.11)
$$h_0 = \beta_0 \eta + \beta_0^2 \frac{\eta^3}{3!} - 2\beta_0^3 \frac{\eta^5}{5!} + 10\beta_0^4 \frac{\eta^7}{7!} \cdots,$$

which is a MacLaurin series whose coefficients are computed by successive differentiations of equation (4.7) for f_0. Let $H_0(\eta)$ be the solution to equation (4.7) for h_0 with $\beta_0 = 1$. This solution is related to h_0 by

(4.12)
$$h_0(\eta) = \beta_0^{\frac{1}{2}} H_0(\beta_0^{\frac{1}{2}}\eta).$$

Therefore

$$\lim_{\eta\to\infty} \frac{h_0'(\eta)}{\eta} = \beta_0^{\frac{3}{2}} \lim_{\eta\to\infty} \frac{H_0'(\eta)}{\eta},$$
$$= \beta_0^{\frac{3}{2}} \lim_{\eta\to\infty} H_0''(\eta).$$

From the first of equations (4.10a)

$$\lim_{\eta\to\infty} \frac{h_0'(\eta)}{\eta} = 3^2 a_1,$$
$$= \alpha_0,$$

which implies

$$\beta_0 = \{\alpha_0 / \lim_{\eta\to\infty} H_0''(\eta)\}^{\frac{2}{3}}.$$

The asymptotic behaviour of h_0 as $n \to \infty$ is, according to Goldstein

$$h_0 \sim \tfrac{1}{2}\alpha_0 \eta'^2$$

where $n' = \eta + \delta_0$, δ_0 being an arbitrary constant. This may be written as

(4.13)
$$h_0 \sim A_0 \eta^2 + B_0 \eta + C_0.$$

$h_0(\eta)$ is the solution to equation (4.7) for h_0 with $H_0'(0) = 1$, $H_0(0) = H_0''(0) = 0$. Thus $H_0(\eta)$, $H_0'(\eta)$ and $H_0''(\eta)$ may be obtained numerically using a Runge-Kutta technique and it was found that for large values of η

(4.14)
$$H_0(\eta) \sim 0.42357\,(\eta + 0.65364)^2.$$

From (4.12) and (4.14) the values of β_0 and δ_0 may be found. They are

$$\beta_0 = 3.0849,$$
$$\delta_0 = 0.37215.$$

Equation (4.7) for h_0 may now be solved, using β_0 as the initial value of h_0'.

https://doi.org/10.1017/S1446788700013707 Published online by Cambridge University Press

The solution for k_0

The solution to equation (4.8) for k_0, satisfying boundary conditions (4.9) and (4.10b) is a constant,

$$k_0 = b_0. \tag{4.15}$$

The solution for h_1

Let $H_{1p}(\eta)$ be a particular integral of the equation (4.7) for h_1 and $H_{1c}(\eta)$ the complementary function for which $H_{1c}(0) = H''_{1c}(0) = 0$ and $H'_{1c}(0) = 1$. Then

$$H_{1c}(\eta) = \eta + 3\beta_0 \frac{\eta^3}{3!} - 6\beta_0^2 \frac{\eta^5}{5!} + \cdots,$$

and, choosing a particular integral such that $H_{1p}(0) = H'_{1p}(0) = H''_{1p}(0) = 0$,

$$H_{1p}(\eta) = -27b_0^2 \left(\frac{y^3}{3!} - \beta_0 \frac{y^5}{5!} + \cdots \right).$$

The solution for h_1 is therefore

$$h_1(\eta) = H_{1p}(\eta) + \lambda_1 H_{1c}(\eta), \tag{4.16}$$

where λ_1 is to be determined from (4.10a). Asymptotically, the equation for f_1 is satisfied by

$$A_1 \eta'^3 + \bar{C}_1 \eta' - (9b_0^2 + A_1)\alpha_0^{-1}, \tag{4.17}$$

where exponentially small terms have been neglected. From (4.14a)

$$A_1 = 3^2 a_2. \tag{4.18}$$

Consequently the following asymptotic equality defines λ_1:

$$\lambda_1 \cong (3A_1 \eta' - H''_{1p}(\eta))/H''_{1c}(\eta). \tag{4.19}$$

The functions $H_{1p}(\eta)$, $H_{1c}(\eta)$ and their first two derivatives were tabulated by numerical solution of the homogeneous and non-homogeneous equations respectively. The value of λ_1 was then found by substituting $H''_{1p}(\eta)$ and H''_{1c} into (4.19), for "sufficiently" large η ($\eta = 7$ was found to be large enough to make the exponential terms negligible). The value of $\bar{C}_1$ was then found by comparing the derivative of (4.17) with the right-hand side of (4.1b) for "sufficiently" large η.

If (4.17) is expressed as a polynomial in η, it is noticed that the asymptotic behaviour of h_1 is

$$h_1 \sim A_1 \eta^3 + B_1 \eta^2 + C_1 \eta + D_1, \tag{4.20}$$

where A_1 is given by (4.18) and B_1, C_1, and D_1 are related to A_1 and $\bar{C}_1$ by

$$\begin{aligned} B_1 &= 3A_1\delta_0, \\ \bar{C}_1 &= C_1+3A_1\delta_0^2, \\ D_1 &= A_1\delta_0^3+C_1\delta_0-(9b_0^2+A_1)\alpha_0^{-1}. \end{aligned}$$

The solution for k_1

The equation (4.8) for k_1 is homogeneous since k_0 is constant. If K_{1c} is the complementary function satisfying $K_{1c}(0) = 1$, $k'_{1c}(0) = 0$, then

$$k_1 = \mu_1 k_{1c}, \tag{4.21}$$

where μ_1 is to be determined from (4.10b). The asymptotic behaviour of k_1 is easily shown to be

$$\begin{aligned} k_1 &\sim G_1\eta' \\ &\sim G_1\eta+H_1, \end{aligned} \tag{4.22}$$

where $H_1 = \delta_0 G_1$. From (4.10b)

$$G_1 = 3b_1, \tag{4.23}$$

hence, using (4.21), (4.22), (4.23)

$$\mu_1 \cong 3b_1/k'_{1c}. \tag{4.24}$$

The functions $k_{1c}(\eta)$ and $k'_{1c}(\eta)$ were tabulated by numerical solution of the differential equation and the limiting value of the right-hand side of (4.23) was obtained.

The solutions for h_n and k_n ($n = 2, 3$)

The solutions for f_n and g_n are:

$$\begin{aligned} h_2(\eta) &= H_{2p}(\eta)+\lambda_2 H_{2c}(\eta), \\ h_3(\eta) &= H_{3p}(\eta)+\lambda_3 H_{3c}(\eta), \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} k_2(\eta) &= K_{2p}(\eta)+\mu_2 K_{2c}(\eta), \\ k_3(\eta) &= K_{3p}(\eta)+\mu_3 K_{3c}(\eta), \end{aligned} \tag{4.26}$$

where H_{np}, $K_{np}(n = 2, 3)$ are particular integrals and H_{nc}, K_{nc} are complementary functions for the appropriate differential equations of (4.7) and (4.8). The functions H_{np}, H_{nc}, K_{nc} satisfy the same conditions at $\eta = 0$ as H_{1p}, H_{1c}, k_{1c} respectively, while $K_{np}(0) = K'_{np}(0) = 0$. The λ_n and μ_n are to be determined from (4.10a) and (4.10b) respectively, in the same manner that λ_1 and μ_1 were determined.

The asymptotic forms of h_n and k_n may be inferred directly from the results of Smith op. cit. for his free-convection boundary layer problem. Thus h_n and k_n have the following asymptotic behaviour:

$$(4.27)\qquad \begin{aligned} h_2 &\sim A_2\eta^4+B_2\eta^3+C_2\eta^2+D_2\eta+E_2,\\ h_3 &\sim A_3\eta^5+B_3\eta^4+C_3\eta^3+D_3\eta^2+E_3\eta+F_3, \end{aligned}$$

and

$$\begin{aligned} k_2 &\sim G_2\eta^2+H_2\eta+I_2,\\ k_3 &\sim G_3\eta^3+H_3\eta^2+I_3\eta+J_3, \end{aligned}$$

where the coefficients A_2, A_3, etc., are obtained numerically.

5. The outer solution

The outer expansion is obtained as follows. The inner expansion for large η is written in terms of the outer variable Z and, with the assumption that re-arrangement is possible, equations (4.13), (4.15), (4.20), (4.22) and (4.27) combine to give

$$(5.1)\qquad \begin{aligned} \psi \sim\ & A_0(Z/3)^2+A_1(Z/3)^2+\cdots\\ &+\xi(B_0(Z/3)+B_1(Z/3)^2+\cdots)\\ &+\xi^2(C_0+C_1(Z/3)+C_2(Z/3)^2+\cdots)\\ &+\xi^3(D_1+D_2(Z/3)+\cdots),\\ &+\cdots \end{aligned}$$

and

$$(5.2)\qquad \begin{aligned} V \sim\ & k_0+G(Z/3)+G_2(Z/3)^2+\cdots\\ &+\xi(H_1+H_2(Z/3)+\cdots)\\ &+\xi^2(I_2+I_3(Z/3)+\cdots)\\ &+\cdots \end{aligned}$$

The expressions (5.1) and (5.2) suggest the following expansions for the outer solution, provided $ZR^{-\frac{1}{3}}$ is sufficiently large:

$$(5.3)\qquad \begin{aligned} \psi &= \psi_0(Z)+\frac{\xi}{1!}\psi_1(Z)+\frac{\xi^2}{2!}\psi_2(Z)+\cdots\\ V &= V_0(Z)+\frac{\xi}{1!}V_1(Z)+\frac{\xi^2}{2!}V_2(Z)+\cdots, \end{aligned}$$

where

$$(5.4)\qquad \begin{aligned} \psi_0' &= a_1Z+a_2Z^2+a_3Z^3+\cdots,\\ \psi_1 &= \tfrac{1}{3}B_0Z+\tfrac{1}{9}B_1Z^2+\tfrac{1}{27}B_2Z^3+\cdots,\\ \psi_2 &= 2C_0+\tfrac{2}{3}C_1Z+\tfrac{2}{9}C_2Z^2+\cdots,\\ \psi_3 &= 6D_1+2D_2Z+\tfrac{2}{3}D_3Z^2+\cdots, \end{aligned}$$

and

https://doi.org/10.1017/S1446788700013707 Published online by Cambridge University Press

$$
\begin{aligned}
V_0 &= b_0+b_1Z+b_2Z^2+b_3Z^3+\cdots,\\
V_1 &= H_1+\tfrac{1}{3}H_2Z+\tfrac{1}{9}H_3Z^2+\cdots,\\
V_2 &= I_2+\tfrac{1}{3}I_3Z+\cdots,\\
V_3 &= J_3+\cdots
\end{aligned}
\tag{5.5}
$$

When the series (5.3) are substituted into the first two equations of (3.1), the following equations for ψ_n and V_n $(n = 1, 2, 3)$ are obtained:

$$
\begin{aligned}
\psi_1'\psi_0'-\psi_1\psi_0'' &= 0,\\
\psi_2'\psi_0'-\psi_2\psi_0'' &= \psi_1\psi_1''-\psi_1'^2,\\
\psi_3'\psi_0'-\psi_3\psi_0'' &= 6(\psi_0'''+V_0^2)+\psi_1\psi_2''-3\psi_1'\psi_2'+2\psi_1''\psi_2,
\end{aligned}
\tag{5.6}
$$

and

$$
\begin{aligned}
V_1\psi_0'-V_0'\psi_1 &= 0,\\
V_2\psi_0'-V_0'\psi_2 &= \psi_1V_1'-\psi_1'V_1,\\
V_3\psi_0'-V_0'\psi_3 &= 6(V_0'''+V_0\psi_0')+2\psi_2V_1'+\psi_1V_2'\\
&\quad -2\psi_1'V_2-\psi_2'V_1.
\end{aligned}
\tag{5.7}
$$

The first equation of (5.6) may be integrated immediately to yield

$$\psi_1 = k\psi_0', \tag{5.8a}$$

the constant k being found from a comparison of the first two expansions of (5.4). Hence

$$k = B_0/3a_1.$$

The expansion for ψ_2 may now be integrated to give

$$\psi_2 = k^2\psi_0''+l\psi_0', \tag{5.8b}$$

with the first and third expansions of (5.4) implying that

$$l = 2(C_1-3a_2k^2)/3a_1.$$

The solution for ψ_3 is

$$\psi_3 = 6\psi_0'\left(\int_0^z \frac{V_0+\psi_0'''}{\psi_0'^2}\,dZ+m\right)+k^3\psi_0'''+3kl\psi_0''. \tag{5.8c}$$

The coefficient of ψ_0' in the expression for ψ_3 involves an arbitrary constant of integration, m. The expansion for ψ_3 in powers of y is

$$\Psi_3 = k(3a_2k^2-3a_1l)+6(a_1m+a_3k^3+a_2kl)y+0(y^2),$$

and by comparison with the expansion (5.4) for Ψ_3

$$m = (\tfrac{1}{3}D_2-a_3k^2-a_2kl)/a_1.$$

The functions V_n ($n = 1, 2, 3$) are now defined in terms of k, l, m and the initial solutions Ψ_0 and V_0. They are:

$$
\begin{aligned}
V_1 &= kV_0', \\
V_2 &= k^2V_0 + lV_0', \\
&= \frac{6(V_0'' + V_0\Psi_0')}{\Psi_0'} + 6\Psi_0'\left(\int^Z \frac{V_0 + \Psi_0'''}{\Psi_0'^2}\,dZ\right) \\
&\quad + k^3V_0''' + 3kl\Psi_0'\left(\frac{V_0'}{\Psi_0'}\right).
\end{aligned}
\tag{5.9}
$$

6. Discussion of the results

Profiles of the non-dimensional radial and azimuthal velocity components U and V are shown in Figures 2(a) and 2(b) respectively for several values of the non-dimensional radial variable ξ, in the range $0 \leqq \xi \leqq 0.3$. The profiles were plotted by fitting together the inner solution, which is obtained by substituting the solutions for h_m and k_m ($m = 0, 1, 2, 3$) into (4.6), and the outer solution, which is obtained by substituting Ψ_m and V_m, as calculated from (5.8) and (5.9) into (5.3). For values of ξ greater than 0.3 the overlap between the inner and outer solutions is lost, as the $m \geqq 4$ terms neglected in the expansions (4.6) and (5.3) become significant. Goldstein was able to proceed further downstream than this because he was able, with the Blasius solution as the initial profile, to expand his stream function in powers of ξ^{3m+1} compared with the expansion in powers of ξ^{m+1} allowed by the Cochran solution.

It is observed that the velocity profiles have adjusted rapidly to the new boundary condition in the plane $Z = 0$, and are approaching smoothly towards a swirling radial jet flow before the eventual decay at large distances from the disc. As ξ increases the radial velocity profile is steadily filling in and the azimuthal velocity is gradually flattening.

7. Conclusions

It has been shown that the matched asymptotic method used by Goldstein to solve the boundary layer equations for the near wake downstream from the trailing edge of a finite flat plate may be extended to solve the boundary equations for the near wake region of the discharge from the edge of a finite rotating disc. This is, of course, only the first step towards obtaining the complete picture of the discharge and it remains to investigate the flow in the neighbourhood of the edge of the disc and also to continue the boundary layer solution past the near wake region. Any attempt to examine the flow in the neighbourhood of the edge of the disc must necessarily involve the full Navier-Stokes equations, but the flow

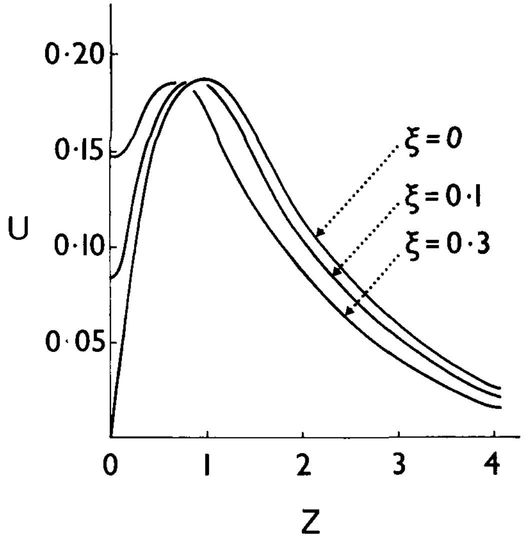

Figure 2(a)

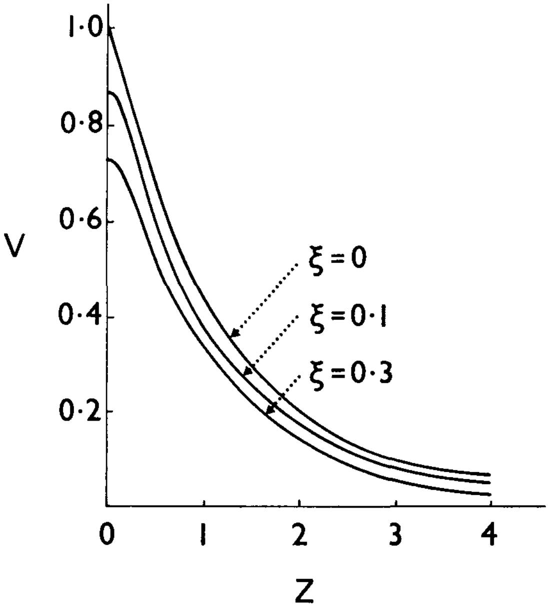

Figure (2b)

Fig. 2. Profiles of the dimensionless radial velocity, U, and azimuthal velocity, V, are plotted in (a) and (b) respectively for several values of ξ in the range $0 \leqq \xi \leqq 0.3$.

https://doi.org/10.1017/S1446788700013707 Published online by Cambridge University Press

past the near wake region should not be difficult to obtain by using one of the many methods available for the continuation of boundary layer solutions. Goldstein's method itself may be used in a step-by-step manner to continue the boundary layer solution indefinitely but this is true in principle only and proves far too tedious in practice. A direct numerical method should prove to be most suitable.

References

[1] H. Blasius, 'Grenzschichten in Flüssigkeiten mit kleiner Reibung', *Z. Math. Phys.* 56 (1908), 1—37.
[2] W. G. Cochran, 'The flow due to a rotating disc', *Proc. Camb. Phil. Soc.* 30 (1934), 365—375.
[3] S. Goldstein, 'Concerning some solutions of the boundary layer equations in hydrodynamics', *Proc. Camb. Phil. Soc.* 26 (1930), 1—30.
[4] H. Schlichting, *Boundary layer theory*, (Pergamon Press (1968), 93—97).
[5] R. K. Smith, 'The laminar free-convection boundary layer on a vertical heated plate in the neighbourhood of a discontinuity in the plate temperature', *J. Aust. Math. Soc.* (1970), 149—168.
[6] K. Stewartson, 'On the flow near the trailing edge of a flat plate', *Proc. Roy. Soc. A.* 306 (1968), 275—290.
[7] K. Stewartson, 'On the flow near the trailing edge of a flat plate II', *Mathematika* 16 (1969), 106—121.
[8] Th. von Karman, 'Über laminare und turbulente Reibung', *Z. angew. Math. Mech.* 1 (1921), 233—252.

Department of Mathematics
Monash University
Clayton, Victoria