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RESEARCH ARTICLE

Homoclinic orbits, multiplier spectrum and rigidity theorems in complex dynamics

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Abstract

The aims of this paper are to answer several conjectures and questions about the multiplier spectrum of rational maps and giving new proofs of several rigidity theorems in complex dynamics by combining tools from complex and non-Archimedean dynamics.

A remarkable theorem due to McMullen asserts that, aside from the flexible Lattès family, the multiplier spectrum of periodic points determines the conjugacy class of rational maps up to finitely many choices. The proof relies on Thurston's rigidity theorem for post-critically finite maps, in which Teichmüller theory is an essential tool. We will give a new proof of McMullen's theorem (and therefore a new proof of Thurston's theorem) without using quasiconformal maps or Teichmüller theory.

We show that, aside from the flexible Lattès family, the length spectrum of periodic points determines the conjugacy class of rational maps up to finitely many choices. This generalizes the aforementioned McMullen's theorem. We will also prove a rigidity theorem for marked length spectrum. Similar ideas also yield a simple proof of a rigidity theorem due to Zdunik.

We show that a rational map is exceptional if and only if one of the following holds: (i) the multipliers of periodic points are contained in the integer ring of an imaginary quadratic field, and (ii) all but finitely many periodic points have the same Lyapunov exponent. This solves two conjectures of Milnor.

Contents

1. Introduction

1.1. *Exceptional endomorphisms*

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism over $\mathbb{C}$ of degree at least 2. It is called *Lattès* if it is semi-conjugate to an endomorphism on an elliptic curve. Further, it is called *flexible Lattès* if it is semi-conjugate to the multiplication by an integer n on an elliptic curve for some $|n| \geq 2$. We say that f is of *monomial type* if it is semi-conjugate to the map $z \mapsto z^n$ on $\mathbb{P}^1$ for some $|n| \geq 2$. We call f *exceptional* if it is Lattès or of monomial type. An endomorphism f is exceptional if and only if some iterate f^n is exceptional. Exceptional endomorphisms are considered as the exceptional examples in complex dynamics.

In this paper, we will prove a criterion for an endomorphism being exceptional via the information of a *homoclinic orbit* of f. See Theorem 2.11 for the precise statement, and see Section 2 for the definition and basic properties of homoclinic orbits. Since every f has plenty of homoclinic orbits, the above criterion turns out to be very useful. A direct consequence is the following characterization of exceptional endomorphisms by the linearity of a *conformal expending repeller* Strategy of the proof of Theorem (CER).

Theorem 1.1. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism over $\mathbb{C}$. Assume that f has a linear CER that is not a finite set. Then, f is exceptional.*

CER is an impotent concept in complex dynamics introduced by Sullivan [Sul86]. See Section 7.1 for its definition and basic properties.

1.2. Rigidity of stable algebraic families

For $d \geq 1$, let $\mathrm{Rat}_d(\mathbb{C})$ be the space of degree d endomorphisms on $\mathbb{P}^1(\mathbb{C})$. It is a smooth quasi-projective variety of dimension $2d+1$ [Sil12]. Let $FL_d(\mathbb{C}) \subseteq \mathrm{Rat}_d(\mathbb{C})$ be the locus of flexible Lattès maps, which is Zariski closed in $\mathrm{Rat}_d(\mathbb{C})$. The group $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ acts on $\mathrm{Rat}_d(\mathbb{C})$ by conjugacy. The geometric quotient

$$\mathcal{M}_d(\mathbb{C}) := \mathrm{Rat}_d(\mathbb{C})/\mathrm{PGL}_2(\mathbb{C})$$

is the (coarse) *moduli space* of endomorphisms of degree d [Sil12]. The moduli space $\mathcal{M}_d(\mathbb{C}) = \mathrm{Spec}\,(\mathcal{O}(\mathrm{Rat}_d(\mathbb{C})))^{\mathrm{PGL}_2(\mathbb{C})}$ is an affine variety of dimension $2d-2$ [Sil07, Theorem 4.36(c)]. Let $\Psi : \mathrm{Rat}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ be the quotient morphism.

An *irreducible algebraic family* f_Λ (of degree d endomorphisms) is an algebraic endomorphism $f_\Lambda : \mathbb{P}^1_\Lambda \to \mathbb{P}^1_\Lambda$ over an irreducible variety Λ, such that for every $t \in \Lambda(\mathbb{C})$, the restriction f_t of f_Λ above t has degree d. Giving an algebraic family over Λ is the same as giving an algebraic morphism $\Lambda \to \mathrm{Rat}_d$. A family f_Λ is called *isotrivial* if $\Psi(\Lambda)$ is a single point.

For every $f \in \mathrm{Rat}_d(\mathbb{C})$ and $n \geq 1$, f^n has exactly $N_n := d^n + 1$ fixed points counted with multiplicity. Their multipliers define a point $s_n(f) \in \mathbb{C}^{N_n}/S_{N_n}$,[1] where S_{N_n} is the symmetric group which acts on $\mathbb{C}^{N_n}$ by permuting the coordinates. The *multiplier spectrum* of f is the sequence $s_n(f), n \geq 1$. An irreducible algebraic family is called *stable* if the multiplier spectrum of f_t does not depend on $t \in \Lambda(\mathbb{C})$.[2]

In 1987, McMullen [McM87] established the following remarkable characterization of stable irreducible algebraic families.

Theorem 1.2 (McMullen). *Let f_Λ be a non-isotrivial stable irreducible algebraic family of degree $d \geq 2$. Then, $f_t \in FL(\mathbb{C})$ for every $t \in \Lambda(\mathbb{C})$.*

McMullen's proof relies on the following Thurston's rigidity theorem for *post-critically finite* (PCF) maps [DH93], in which Teichmüller theory is essentially used. An endomorphism f of degree $d \geq 2$ is called PCF if the critical orbits of f are a finite set.

Theorem 1.3 (Thurston). *Let f_Λ be a non-isotrivial irreducible algebraic family of PCF maps. Then, $f_t \in FL(\mathbb{C})$ for every $t \in \Lambda(\mathbb{C})$.*

[1] Via the symmetric polynomials, we have $\mathbb{C}^{N_n}/S_{N_n} \simeq \mathbb{C}^{N_n}$.
[2] Stability has several equivalent definitions and can be defined for more general families [McM16, Chapter 4].

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In this paper, we will give a new proof of McMullen's theorem without using quasiconformal maps or Teichmüller theory. Since an irreducible algebraic family of PCF maps is automatically stable, this leads to a new proof of Theorem 1.3 without using quasiconformal maps or Teichmüller theory. Except Theorem 2.11, whose proof relies on some basic complex analysis, our proof of Theorem 1.2 only requires some basic knowledge in Berkovich dynamics and hyperbolic dynamics. We explain our strategy of the proof as follows.

Cutting by hypersurfaces, one may reduce to the case that Λ is a smooth affine curve. Let W be the smooth projective compactification of Λ, and let $B := W \setminus \Lambda$. For every $o \in B$, our family induces a non-Archimedean dynamical system on the Berkovich projective line (see Section 4 for details), which encodes the asymptotic behavior of f_t when $t \to o$. Since f_Λ is non-isotrivial and stable, via the study of non-Archimedian dynamics, we show that there is one point $o \in B$ such that when $t \to o$, f_t 'degenerates' to a map taking form $z \mapsto z^m$ in a suitable coordinate, where $2 \le m \le d-1$. The above degeneration $z \mapsto z^m$ is called a *rescaling limit* of f_Λ at o, in the sense of Kiwi [Kiw15] (see Definition 5.4). On the central fiber, it is easy to find a homoclinic orbit satisfying the condition in our exceptional criterion Theorem 2.11. Using an argument in hyperbolic dynamics [Jon98] (see Lemma 6.1), we can deform such homoclinic orbit to nearby fibers preserving the required condition. By Theorem 2.11, f_t is exceptional for all t sufficiently close to o. We conclude the proof by the fact that exceptional endomorphisms that are not flexible Lattès are isolated in the moduli space $\mathcal{M}_d(\mathbb{C})$.

1.3. Length spectrum as moduli

According to the Noetheriality of the Zariski topology on Rat_d, McMullen's rigidity theorem can be reformulated as follows.

Theorem 1.4 (Multiplier spectrum as moduli=Theorem 1.2). *Aside from the flexible Lattès family, the multiplier spectrum determines the conjugacy class of endomorphisms in $\mathrm{Rat}_d(\mathbb{C})$, $d \ge 2$, up to finitely many choices.*

Replace the multipliers by its norm in the definition of multiplier spectrum, and one get the definition of the *length spectrum*. More precisely, for every $f \in \mathrm{Rat}_d(\mathbb{C})$ and $n \ge 1$, we denote by $L_n(f) \in \mathbb{R}^{N_n}/S_{N_n}$ the element corresponding to the multiset $\{|\lambda_1|, \dots, |\lambda_{N_n}|\}$, where $\lambda_i, i = 1, \dots, N_n$ are the multipliers of all f^n-fixed points. The length spectrum of f is defined to be the sequence $L_n(f), n \ge 1$. A priori, the length spectrum contains less information than the multiplier spectrum. But in this paper, we will show that it determines the conjugacy class up to finitely many choices, therefore generalizing Theorem 1.4.

Theorem 1.5 (Length spectrum as moduli). *Aside from the flexible Lattès family, the length spectrum determines the conjugacy class of endomorphisms in $\mathrm{Rat}_d(\mathbb{C})$, $d \ge 2$, up to finitely many choices.*

1.3.1. Strategy of the proof of Theorem 1.5

Let $g \in \mathrm{Rat}_d(\mathbb{C}) \setminus FL_d(\mathbb{C})$. We need to show that the image of

$$Z := \{f \in \mathrm{Rat}_d(\mathbb{C}) \setminus FL_d(\mathbb{C}) |\ L(f) = L(g)\}$$

in $\mathcal{M}_d(\mathbb{C})$ is finite. For $n \ge 0$, set

$$Z_n := \{f \in \mathrm{Rat}_d(\mathbb{C}) \setminus FL_d(\mathbb{C}) |\ L_i(f) = L_i(g), i = 1, \dots, n\}.$$

It is clear that $Z_i, i \ge 1$ is a decreasing sequence of closed subsets of $\mathrm{Rat}_d(\mathbb{C}) \setminus FL_d(\mathbb{C})$ and $Z = \cap_{n \ge 1} Z_n$. For simplicity, we assume that all periodic points of g are repelling. Otherwise, instead of the length spectrum $L(g)$ of all periodic points, we consider the length spectrum $RL(g)$ of all repelling periodic

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points. Such a change only adds some technical difficulties. To get a contradiction, we assume that $\Psi(Z) \in \mathcal{M}_d(\mathbb{C})$ is infinite. Our proof contains two steps.

In Step 1, we show that $Z = Z_N$ for some $N \geq 0$. We first look at the analogue of this step for the multiplier spectrum. The analogue of Z_n is

$$Z'_n := \{f \in \mathrm{Rat}_d(\mathbb{C}) \setminus FL_d(\mathbb{C}) |\ s_i(f) = s_i(g), i = 1, \ldots, n\},$$

which is Zariski closed in $\mathrm{Rat}_d(\mathbb{C}) \setminus FL_d(\mathbb{C})$. Hence, Z'_n is stable when n is large by the Noetheriality. This is how Theorem 1.2 implies Theorem 1.4. In the length spectrum case, since the n-th length map $L_n : \mathrm{Rat}_d(\mathbb{C}) \to \mathbb{R}^{N_n}/S_{N_n}$ takes only real values, it is more natural to view $\mathrm{Rat}_d(\mathbb{C})$ as a real algebraic variety by splitting the complex variable into two real variables via $z = x + iy$. If all $Z_n, n \geq 1$ are real algebraic, we can still conclude this step by the Noetheriality. Unfortunately, this is not true in general (c.f. Theorem 8.10). Since the map L_n^2 sending f to $\{|\lambda_1|^2, \ldots, |\lambda_{N_n}|^2\} \in \mathbb{R}^{N_n}/S_{N_n}$ is semialgebraic, all $Z_n, n \geq 1$ are semialgebraic. The problem is that, in general, closed semialgebraic sets do not satisfy the descending chain condition. We solve this problem by introducing a class of closed semialgebraic sets called *admissible* subsets. Roughly speaking, admissible subsets are the closed subsets that are images of algebraic subsets under étale morphisms. See Section 8.2 for its precise definition and basic properties. We show that admissible subsets satisfy the descending chain condition. Under the assumption that all periodic points of g are repelling, we can show that all Z_n are admissible. The admissibility is only used to prove Theorem 1.5.

Step 1 implies that $Z = Z_N$ is semialgebraic. Since $\Psi(Z)$ is infinite, there is an analytic curve $\gamma \simeq [0, 1]$ contained in Z such that $\Psi(\gamma)$ is not a point. Every $t \in \gamma \subseteq \mathrm{Rat}_d$ defines an endomorphism f_t. After shrinking γ, we may assume that f_0 is not exceptional.

In Step 2, we show that the multiplier spectrum of f_t does not depend on $t \in \gamma$. Once Step 2 is finished, we get a contradiction by Theorem 1.4. Since for every $t \in \gamma$, $L(f_t) = L(g)$, all periodic points of f_t are repelling. For every repelling periodic point x of f_0, there is a real analytic map $\phi_x : \gamma \to \mathbb{P}^1(\mathbb{C})$ such that for every $t \in \gamma$, $\phi_x(t)$ and x have the same minimal period and the norms of their multipliers are same. Using homoclinic orbits, we may construct a CER E_0 of f_0 containing x. It is nonlinear by Theorem 1.1. By Lemma 6.1, for t sufficiently small, E_0 can be deformed to a CER E_t of f_t containing $\phi_x(t)$. Using Sullivan's rigidity theorem [Sul86] (Theorem 7.6), we show that E_0 and E_t are conformally conjugate. In particular, the multipliers of $\phi_x(t)$ are a constant for t sufficiently small. Since γ is real analytic, the multipliers of $\phi_x(t)$ are a constant on γ. Since x is arbitrary, all $f_t, t \in \gamma$ have the same multiplier spectrum. This finishes Step 2.

1.3.2. Further discussion

It is interesting to know whether the uniform version of Theorem 1.5 holds.

Question 1.6. Is there $N \geq 1$ depending only on $d \geq 2$, such that for every $f \in \mathrm{Rat}_d(\mathbb{C}) \setminus FL_d(\mathbb{C})$,

$$\#\Psi(\{g \in \mathrm{Rat}_d(\mathbb{C}) \setminus FL_d(\mathbb{C}) |\ L_i(g) = L_i(f), i = 1, \ldots, N\}) \leq N\,?$$

For every $n \geq 0$, we set

$$R_n := \{(f, g) \in (\mathrm{Rat}_d(\mathbb{C}) \setminus FL_d(\mathbb{C}))^2 |\ L_i(f) = L_i(g), i = 1, \ldots, n\}$$

and

$$R'_n := \{(f, g) \in (\mathrm{Rat}_d(\mathbb{C}) \setminus FL_d(\mathbb{C}))^2 |\ s_i(f) = s_i(g), i = 1, \ldots, n\}.$$

Both of them are decreasing closed subsets of $(\mathrm{Rat}_d(\mathbb{C}) \setminus FL_d(\mathbb{C}))^2$. Since all R'_n are algebraic subsets of $(\mathrm{Rat}_d(\mathbb{C}) \setminus FL_d(\mathbb{C}))^2$, the sequence R'_n is stable for n large. This implies that Theorem 1.4 for the multiplier spectrum is equivalent to its uniform version.

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If one can show that the sequence $R_n, n \geq 0$ is stable (for example, if one can show that R_n are admissible), then Question 1.6 has a positive answer. But at the moment, we only know that R_n are semialgebraic.

1.4. Marked multiplier and length spectrum

By Theorem 1.5 and 1.4, aside from the flexible Lattès family, the length spectrum (and therefore the multiplier spectrum) determines the conjugacy class of endomorphisms of degree $d \geq 2$ *up to finitely many choices*. However, by [Sil07, Theorem 6.62], the multiplier spectrum $f \mapsto s(f)$ (and therefore the length spectrum $f \mapsto L(f)$) is far from being injective when d large. For this reason, we consider the *marked multiplier and length spectrum*. We show that both of them are rigid.

Theorem 1.7 (Marked multiplier spectrum rigidity). *Let f and g be two endomorphisms of $\mathbb{P}^1$ over $\mathbb{C}$ of degree at least 2 such that f is not exceptional. Assume there is a homeomorphism $h : \mathcal{J}(f) \to \mathcal{J}(g)$ such that $h \circ f = g \circ h$ on $\mathcal{J}(f)$. Then, the following two conditions are equivalent.*

(i) *There is a nonempty open set $\Omega \subset \mathcal{J}(f)$ such that, for every periodic point $x \in \Omega$, we have $df^n(x) = dg^n(h(x))$, where n is the period of x;*
(ii) *h extends to an automorphism $h : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ such that $h \circ f = g \circ h$ on $\mathbb{P}^1(\mathbb{C})$.*

Let $U, V \subset \mathbb{P}^1(\mathbb{C})$ be two open sets. A homeomorphism $h : U \to V$ is called *conformal* if h is holomorphic or antiholomorphic in every connected component of U. Note that a conformal map h is holomorphic if and only if h preserves the orientation of $\mathbb{P}^1(\mathbb{C})$.

Theorem 1.8 (Marked length spectrum rigidity). *Let f and g be two endomorphisms of $\mathbb{P}^1$ over $\mathbb{C}$ of degree at least 2 such that f is not exceptional. Assume there is a homeomorphism $h : \mathcal{J}(f) \to \mathcal{J}(g)$ such that $h \circ f = g \circ h$ on $\mathcal{J}(f)$. Then, the following two conditions are equivalent.*

(i) *There is a nonempty open set $\Omega \subset \mathcal{J}(f)$ such that, for every periodic point $x \in \Omega$, we have $|df^n(x)| = |dg^n(h(x))|$, where n is the period of x;*
(ii) *h extends to a conformal map $h : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ such that $h \circ f = g \circ h$ on $\mathbb{P}^1(\mathbb{C})$.*

Note that if $h : \Omega \to h(\Omega)$ is bi-Lipschitz, then it is not hard to show that for n-periodic point $x \in \Omega$, we have $|df^n(x)| = |dg^n(h(x))|$. So the above theorem implies that a locally bi-Lipschitz conjugacy can be improved to a conformal conjugacy on $\mathbb{P}^1(\mathbb{C})$.

Combining Theorem 1.7 and λ-Lemma [McM16, Theorem 4.1], we get a second proof of Theorem 1.2. This proof does not use Teichmüller theory, but we need to use quasiconformal maps and the highly nontrivial Sullivan's rigidity theorem, which is a great achievement in thermodynamic formalism.

Remark 1.9. In Theorem 1.8, in general, h can not be extended to an automorphism on $\mathbb{P}^1(\mathbb{C})$. The complex conjugacy $\sigma : z \mapsto \bar{z}$ induces a mark $h := \sigma|_{\mathcal{J}(f)} : \mathcal{J}(f) \to \overline{\mathcal{J}(f)} = \mathcal{J}(\overline{f})$, preserving the length spectrum. In general, some periodic point of f may have non-real multipliers. Hence, in this case, h cannot be extended to an automorphism on $\mathbb{P}^1(\mathbb{C})$.

Remark 1.10. Theorem 1.8 was proved by Przytycki and Urbanski in [PU99, Theorem 1.9] under the assumptions that both f and g are tame and $\Omega = \mathcal{J}(f)$. See [PU99, Definition 1.1] for the precise definition of tameness. In [Ree84, Theorem 2], Rees showed that there are endomorphisms having dense critical orbits and therefore, are not tame.

The study of marked length spectrum rigidity has been investigated in various settings in dynamics and geometry.

In one-dimensional real dynamics, marked multiplier spectrum rigidity was proved for expanding circle maps (see Shub-Sullivan [SS85]) and for some unimodal maps (see Martens-de Melo [MdM99] and Li-Shen [LS06]).

In the contexts of geodesic flows on Riemannian manifolds with negative curvature, a long-standing conjecture stated by Burns-Katok [BK85] (and probably considered even before) asserted the rigidity

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of marked length spectrum (for closed geodesics). The surface case was proved by Otal [Ota90] and by Croke [Cro90] independently. A local version of the Burns-Katok conjecture in any dimension was proved by Guillarmou-Lefeuvre [GL19].

It was also studied in dynamical billiards. We refer the readers to Huang-Kaloshin-Sorrentino [HKS18], Bálint-De Simoi-Kaloshin-Leguil [BDSKL20], De Simoi-Kaloshin-Leguil [DSKL19] and the references therein.

We prove Theorem 1.8 by combining Theorem 1.1 and Sullivan's rigidity theorem [Sul86] (see Theorem 7.6). More precisely, let o be a repelling fixed point of f. We construct a family C of CERs of f using homoclinic orbits which covers all backward orbits of o. By Theorem 1.1, all of them are nonlinear. We show that their images under h are CERs of g. Applying Sullivan's rigidity theorem, we get that conformal conjugacies link the CERs in C to their images. Two CERs in C have 'large' intersections. Hence, those conformal conjugacies can be patched together. Using this, we get a conformal extension of h to some disk intersecting the Julia set of f. We can further extend it to a global conformal map on $\mathbb{P}^1(\mathbb{C})$.

Theorem 1.7 is a simple consequence of Theorem 1.8 and a result of Eremenko-van Strien [EVS11, Theorem 1] about endomorphisms with real multipliers.

1.5. Zdunik's rigidity theorem

The following rigidity theorem was proved by Zdunik [Zdu90].

Theorem 1.11 (Zdunik). *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism over $\mathbb{C}$ of degree at least 2. Let μ be the maximal entropy measure, and let α be the Hausdorff dimension of μ. Then, μ is absolutely continous with respect to the α-dimensinal Hausdorff measure Λ_α on the Julia set if and only if f is exceptional.*

Zdunik's proof is divided into two parts. The first part was proved in her previous work [PUZ89, Theorem 6] with Przytycki and Urbanski. Later, she proved the second part (hence Theorem 1.11) in [Zdu90]. In this paper we will give a simple proof of the second part using Theorem 1.1.

1.6. Milnor's conjectures on multiplier spectrum

As applications of Theorem 2.11 and Theorem 1.1, we prove two conjectures of Milnor proposed in [Mil06].

Theorem 1.12. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism over $\mathbb{C}$ of degree at least 2. Let K be an imaginary quadratic field. Let O_K be the ring of integers of K. If for every $n \geq 1$ and every n-periodic point x of f, $df^n(x) \in O_K$. Then, f is exceptional.*

The inverse of Theorem 1.12 is also true by Milnor [Mil06, Corollary 3.9 and Lemma 5.6]. In fact, the original conjecture of Milnor concerns the case $K = \mathbb{Q}$. Since imaginary quadratic fields exist (e.g., $\mathbb{Q}(i)$) and they contain $\mathbb{Q}$, Theorem 1.12 implies Milnor's original conjecture.

Some special cases of Milnor's conjecture for integer multipliers are known before by Huguin:

(i) In [Hug22a], the conjecture was proved for quadratic endomorphisms.
(ii) In [Hug21], the conjecture was proved for unicritical polynomials. In fact, Huguin proved a stronger statement, which only assumes the multipliers are in $\mathbb{Q}$ (instead of $\mathbb{Z}$).

Remark 1.13. In the recent preprint [Hug22b], Huguin reproved and strengthened our Theorem 1.12. In his result, the multipliers are only assumed to be contained in an arbitrary number field. Huguin's result relies on an arithmetic equidistribution result for small points proved by Autissier [Aut01] and on a characterization of exceptional maps proved by Zdunik [Zdu14].

The following result confirms another conjecture of Milnor in [Mil06].

Theorem 1.14. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism over $\mathbb{C}$ of degree at least 2. Assume there exists $a > 0$ such that for every but finitely many periodic point x, $f^n(x) = x$, we have $|df^n(x)| = a^n$. Then, f is exceptional.*

Remark 1.15. Theorem 1.14 can also be deduced by a minor modification of an argument of Zdunik [Zdu14].

Let x be a n-periodic point of f. The *Lyapunov exponent* of x is a real number defined by $\frac{1}{n}\log|df^n(x)|$. We let $\Delta(f)$ be the closure of the Lyapunov exponents of periodic points contained in the Julia set. Combining Theorem 1.14 and results due to Gelfert-Przytycki-Rams-Rivera Letelier [GPR10], [GPRRL13], we get the following description of $\Delta(f)$ when f is nonexceptional. A closed interval in $\mathbb{R}$ is called nontrivial if it is not a singleton.

Corollary 1.16. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a nonexceptional endomorphism over $\mathbb{C}$ of degree at least 2. Then, $\Delta(f)$ is a disjoint union of a nontrivial closed interval I and a finite set E (possibly empty). Moreover, there are at most 4 periodic points whose Lyapunov exponents are contained in E, in particular $|E| \leq 4$.*

1.7. Organization of the paper

In Section 2, we prove some basic properties of homoclinic orbits and we prove the fundamental exceptional criterion Theorem 2.11 by using only the information of a homoclinic orbit. In Section 3, we prove Theorem 1.12. In Section 4, we recall some results about dynamics on the Berkovich projective line. In Section 5, we study the rescaling limit via the dynamics on the Berkovich projective line. In Section 6, we give a new proof of McMullen's theorem (Theorem 1.2) by studying rescaling limits. In Section 7, we recall some results about CER, and we prove Theorem 1.1, Theorem 1.7, Theorem 1.8, Theorem 1.14 and Corollary 1.16. Moreover, we give a new proof of Theorem 1.11 and we give another proof of Theorem 1.2. In Section 8, we prove Theorem 1.5.

2. Homoclinic orbits and applications

For an endomorphism f of $\mathbb{P}^1$ of degree at least 2, we denote by $C(f)$ the set of critical points of f and $PC(f) := \cup_{n\geq 1} f^n(C(f))$ the postcritical set. In this section, $\mathbb{P}^1(\mathbb{C})$ is endowed with the complex topology.

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ an endomorphism over $\mathbb{C}$ of degree at least 2. Let o be a repelling fixed point of f. A *homoclinic orbit* [3] of f at o is a sequence of points $o_i, i \geq 0$ satisfying the following properties:

(i) $o_0 = o$, $o_1 \neq o$ and $f(o_i) = o_{i-1}$ for $i \geq 1$;
(ii) $\lim_{i\to\infty} o_i = o$.

Be aware that $o_i, i \geq 0$ is actually a backward orbit.

The main result of this section is Theorem 2.11, which is a criterion for an endomorphism f being exceptional via the information of a homoclinic orbit. We will state and prove this theorem at the end of this section.

2.1. Linearization domain and good return times

Define a *linearization domain* of o to be an open neighborhood U of o such that there is an isomorphism $\phi : U \to \mathbb{D}$ sending o to 0, which conjugates $f|_{U_0} : U_0 \to U$ to the morphism $z \mapsto \lambda z$ via ϕ, where $U_0 = f^{-1}(U) \cap U$ and $\lambda = df(o)$. We call such ϕ a *linearization on U*.

Define g to be the morphism $U \to U$ sending z to $\phi^{-1}(\lambda^{-1}\phi(z))$. It is the unique endomorphism of U satisfiying $f \circ g = \mathrm{id}$.

[3] This terminology was introduced by Milnor [Mil11] in his presentation of Julia's proof that repelling periodic points are dense in the Julia set. The word 'homoclinic orbit' dates back to Poincaré.

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Remark 2.1. By Koenigs' theorem [Mil11, Theorem 8.2], for every repelling point o, there is always a linearization domain U. For every $r \in (0,1]$, $\phi^{-1}(\mathbb{D}(0,r))$ is also a linearization domain of o. In particular, the linearization domains of o form a neighborhood system of o.

Remark 2.2. Since g is injective, for every $x \in U$, $f^{-1}(x) \cap U = g(x)$. In particular, if $o_i \in U$ for $i \geq l$, then $o_i = g^{i-l}(o_l)$ for all $i \geq l$.

The following lemma shows that for every repelling fixed point o, there are many homoclinic orbits.

Lemma 2.3. *For every integer $m \geq 0$ and for every $a \in f^{-m}(o)$, there is a homoclinic orbit $o_i, i \geq 0$ of o such that $o_m = a$.*

Proof. Let U be a linearization domain of o. Since preimages of a are dense in the Julia set, there is $l \geq m$ such that $f^{m-l}(a) \cap U \neq \emptyset$. Pick $o_l \in f^{m-l}(a) \cap U$ and for $i = 0, \ldots, l$, set $o_i := f^{l-i}(o_l)$. Then $o_0 = o$ and $o_m = a$. For $i \geq l+1$, set $o_i := g^{i-l}(o_l)$. Then $o_i, i \geq 0$ is a homoclinic orbit of o. □

Definition 2.4. Let U be a connected open neighborhood of o such that U is contained in a linearization domain. For $i \geq 0$, let U_i be the connected component of $f^{-i}(U)$ containing o_i. An integer $m \geq 1$ is called a *good return time* for the homoclinic orbit and U if

(i) $o_i \in U$ for every $i \geq m$;
(ii) $U_m \subset\subset U$, and $f^m : U_m \to U$ is an isomorphism between U_m and U.

Remark 2.5. If U itself is a linearization domain and m is a good return time, then i is a good return time for all $i \geq m$. Indeed, one has $U_i = g^{i-m}(U_m) \subset\subset U$, and $f^i : U_i \to U$ can be writen as $f^m \circ g^{m-i}$, which is an isomorphism.

Proposition 2.6. *The following statements are equivalent:*

(i) *$o_i \notin C(f)$ for every $i \geq 1$;*
(ii) *there is a linearization domain U and an integer $m \geq 1$ which is a good return time of U;*
(iii) *there is a linearization domain U such that, for every connected open neighborhood V of o, $V \subset U$, there is an integer $m \geq 1$ which is a good return time of V.*

In particular, when $o \notin PC(f)$, (i) (and therefore (ii) and (iii)) are satisfied.

Proof. We first show (i) is equivalent to (ii). To see that (ii) implies (i), let m be a good return time of U. Then, by the definition of good return time, $o_i \notin C(f)$ for $i = 1, \ldots, m$. By Remark 2.5, we conclude that $o_i \notin C(f)$ for every $i \geq 1$. To see that (i) implies (ii), first choose a linearization domain U_0. Let $g : U_0 \to U_0$ be the morphism such that $f \circ g = \mathrm{id}$. Since $\lim_{i\to\infty} o_i = o$, there is $l \geq 1$ such that $o_i \in U_0$ for $i \geq l$. Since $o_i \notin C(f)$ for every $i \geq 1$, we have $d(f^l)(o_l) \neq 0$. So there is an open neighborhood W of o_l in U_0 such that $f^l(W) \subseteq U_0$ and $f^l|_W$ is injective. Pick a linearization domain of U of o contained in $f^l(W)$. Set $U_l := f^{-l}(U) \cap W$. Since g is attracting, there is $m \geq l$ such that $g^{m-l}(U_l) \subset\subset U$. We note that $U_m := f^{-m}(U) \cap U = g^{m-l}(U_l)$. Hence, $U_m \subset\subset U$, and $f^m : U_m \to U$ is an isomorphism. This implies (ii).

It is clear that (iii) implies (ii). It remains to show that (ii) implies (iii). Let $l \geq 1$ be a good return time of U. Let U_i (resp. V_i) be the connected component of $f^{-i}(U)$ (resp. $f^{-i}(V)$) for $i \geq 0$. We have $U_l \subset\subset U$. Since g is attracting, there is $m \geq l$ such that $g^{m-l}(U_l) \subset\subset V$. This implies that m is a good return time of V. □

2.2. *Adjoint sequence of periodic points*

Let U be a linearization domain, and let m be a good return time of U. We construct a sequence of periodic points $q_i, i \geq m$ as follows. By Remark 2.5, for every $i \geq m$, $f^i|_{U_i} : U_i \to U$ is an isomorphism. Since $U_i \subset\subset U$, the morphism $(f^i|_{U_i})^{-1} : U \to U_i$ is strictly attracting with respect to the hyperbolic metric on U. Hence, it has a unique fixed point $q_i \in U_i$. Such q_i is the unique i-periodic point of f which is contained in U_i. Indeed, i is the smallest period of q_i, and q_i is repelling. We call such a sequence an *adjoint sequence* for the homoclinic orbit $o_i, i \geq 0$ with respect to the linearization domain U and the good return time m (we write (U, m) for short). One can say that a sequence of points $q_i, i \geq 0$ is an *adjoint sequence* of the homoclinic orbit $o_i, i \geq 0$ if $q_i, i \geq m$ is an adjoint sequence for $o_i, i \geq 0$ with respect to some (U, m). It is clear that for every adjoint sequence $q_i, i \geq 0$ of $o_i, i \geq 0$, $\lim_{i \to \infty} q_i = o$. The following lemma shows that the adjoint sequences are unique modulo finite terms.

Lemma 2.7. *Let $q_i, i \geq 0$ and $q'_i, i \geq 0$ be two adjoint sequence for $o_i, i \geq 0$. Then, there is $l \geq 0$ such that $q_i = q'_i$ for all $i \geq l$.*

Proof. We only need to prove the case where $q_i, i \geq l$ is an adjoint sequence with respect to (U, l) and $q'_i, i \geq l'$ is an adjoint sequence with respect to (U', l'). Since there is a linearization domain U'' such that $U'' \subseteq U \cap U'$, we may assume that $U' \subseteq U$. After replacing l, l' by $\max\{l, l'\}$, we may assume that $l = l'$. Then, for every $i \geq l$, $U'_i \subseteq U_i$. Then, both q_i and q'_i are the unique i-periodic point of f in U_i. So $q_i = q'_i$ for $i \geq l$. □

2.3. Poincaré's linearization map

Set $\lambda := df(o) \in \mathbb{C}$. Since o is repelling, $|\lambda| > 1$. Let (U, m) be the pair of linearization domain and good return time for $o_i, i \geq 0$, and let $q_i, i \geq 0$ be an adjoint sequence.

A theorem of Poincaré [Mil11, Corollary 8.12] says that there is a morphism $\psi : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ such that $\psi|_{\mathbb{D}}$ gives an isomorphism between $\mathbb{D}$ and U and

$$f(\psi(z)) = \psi(\lambda z) \tag{2.1}$$

for every $z \in \mathbb{C}$. In particular, $\psi|_{\mathbb{D}}^{-1} : U \to \mathbb{D}$ is a linearization of f on U. Such a ψ is called a *Poincaré map*.

The following criterion for exceptional endomorphisms using the Poincaré map ψ is due to Ritt.

Lemma 2.8 [Rit22]. *If the Poincaré map ψ is periodic (i.e., there is a $a \in \mathbb{C}^*$ such that $\psi(z + a) = \psi(z)$ for every $z \in \mathbb{C}$), then f is exceptional.*

Ritt's theorem can be easily generalized as following.

Lemma 2.9. *If there is an affine automorphism $h : \mathbb{C} \to \mathbb{C}$ such that $h(0) \neq 0$ and $\psi \circ h = \psi$, then f is exceptional.*

Proof. Let G be the group of affine automorphisms g of $\mathbb{C}$ satisfying $\psi \circ g = \psi$. We have $h \in G$. It takes form $h : z \mapsto az + b$, $a \in \mathbb{C}^*$ and $b = h(0) \in \mathbb{C}^*$. For every $z \in \mathbb{C}$, we have

$$\psi(\lambda h(\lambda^{-1} z)) = f\psi(h(\lambda^{-1} z)) = f\psi(\lambda^{-1} z) = \psi(z).$$

Hence, the automorphism $g : z \mapsto \lambda h(\lambda^{-1} z) = az + \lambda b$ is contained in G. Then, $T := h^{-1} \circ g : z \mapsto z + a^{-1}(\lambda - 1)b$ is contained in G. Since $b \neq 0$ and $|\lambda| > 1$, T is a nontrivial translation. We conclude the proof by Lemma 2.8. □

Set $P := \lambda^m \psi|_{\mathbb{D}}^{-1}(o_m)$ and $V := \lambda^m (\psi|_{\mathbb{D}}^{-1}(U_m))$. For $i \geq m$, set $Q_i := \psi|_{\mathbb{D}}^{-1}(q_i)$. One has $\psi(V) = U$, $\psi(P) = o$, and $\psi|_V : V \to U$ is an isomorphism. We set $T := (\psi|_V)^{-1} \circ \psi|_{\mathbb{D}} : \mathbb{D} \to V$. Then T is an isomorphism. Similar constructions of T appeared already in the works of Ritt [Rit22] and Eremenko-van Strien [EVS11]. We summarize our construction in the following figure.

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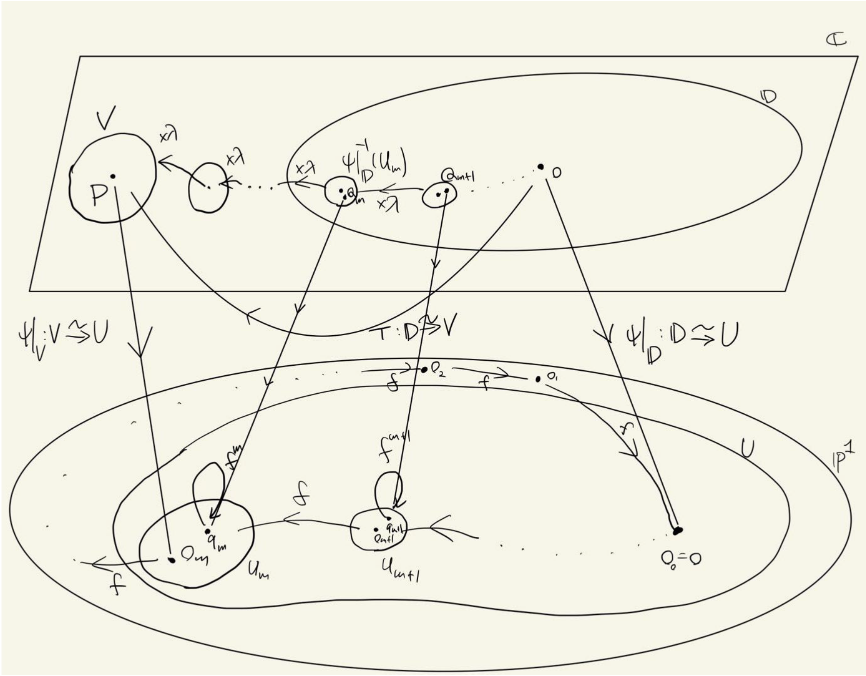

We have $\psi \circ T = \psi$ and $T(0) = P$. Moreover, by our construction, we have for every $i \geq m$, $V = \lambda^i (\psi|_{\mathbb{D}})^{-1}(U_i)$. In particular, $\lambda^i Q_i \in V$. By (2.1) we have

$$\psi(\lambda^i Q_i) = f^i(\psi(Q_i)) = f^i(q_i) = q_i.$$

This implies

$$T(Q_i) = \lambda^i Q_i. \tag{2.2}$$

Since $\lim_{i \to \infty} q_i = o$, we have $\lim_{i \to \infty} Q_i = 0$ and

$$\lim_{i \to \infty} \lambda^i Q_i = P. \tag{2.3}$$

By (2.1), we have for every $i \geq 1$,

$$df^i(\psi(z))d\psi(z) = \lambda^i d\psi(\lambda^i z), \tag{2.4}$$

and by $\psi \circ T = \psi$, we have

$$d\psi(T(z))T'(z) = d\psi(z). \tag{2.5}$$

Set $z = Q_i$. Combine (2.2), (2.4) and (2.5), and we get

$$df^i(q_i)d\psi(\lambda^i Q_i)T'(Q_i) = \lambda^i d\psi(\lambda^i Q_i).$$

Since zeros of a holomorphic function are isolated, as $\lambda^i Q_i \to P$, for i large enough, we have $d\psi(\lambda^i Q_i) \neq 0$. Hence, for i large enough,

$$\lambda^i T'(Q_i)^{-1} = df^i(q_i). \tag{2.6}$$

The following observation will be used in the proof of Theorem 1.12.

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Lemma 2.10. *Set $\theta := 1/T' : \mathbb{D} \to \mathbb{C}$. We have*

$$\lim_{i\to\infty}(df^i(q_i) - \lambda^i\theta(0)) = P\theta'(0).$$

Proof. By (2.3) and (2.6),we have

$$\lim_{i\to\infty}(df^i(q_i) - \lambda^i\theta(0))/P = \lim_{i\to\infty}(df^i(q_i) - \lambda^i\theta(0))/\lambda^i Q_i$$

$$= \lim_{i\to\infty}(df^i(q_i)/\lambda^i - \theta(0))/Q_i = \lim_{i\to\infty}(\theta(Q_i) - \theta(0))/Q_i = \theta'(0),$$

which concludes the proof. □

The following is the main result of this section, which characterizes exceptional endomorphisms by using the multipliers of adjoint sequence of a homoclinic orbit.

Theorem 2.11. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism over $\mathbb{C}$ of degree at least 2. Let o be a repelling fixed point of f such that $df(o) = \lambda$. Let $o_i, i \geq 0$ be a homoclinic orbit of o such that $o_i \notin C(f)$ for every $i \geq 0$. Assume that there is $C \in \mathbb{C}^*$, such that for one (and therefore, every) adjoint sequence $q_i, i \geq 0$ of $o_i, i \geq 0$, $df^i(q_i) = C\lambda^i$ for i large. Then f is exceptional.*

Proof. We may assume that $q_i, i \geq m$ is adjoint with respect to the linearization domain and good return time (U, m) for $o_i, i \geq 0$, and $d(f^i)(q_i) = C\lambda^i$ for all $i \geq m$. By (2.6), we get $T'(Q_i) = C^{-1}$ for $i \geq m$. Since $Q_i \neq 0$ for $i \geq m$ and $\lim_{i\to\infty} Q_i = 0, T' = C^{-1}$ on $\mathbb{D}$. It follows that $T(z) = C^{-1}z + P$ for every $z \in \mathbb{D}$. Then, T extends to the affine endomorphism on $\mathbb{C}$ sending z to $C^{-1}z + P$. One has $\psi = \psi \circ T$ on $\mathbb{C}$. We conclude the proof by Lemma 2.9. □

3. Proof of Milnor's conjecture

In this section, we prove one of Milnor's conjectures (Theorem 1.12). We postpone the proof of another conjecture of Milnor (Theorem 1.14) to Section 7.

Proof of Theorem 1.12. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism over $\mathbb{C}$ of degree at least 2. Let K be an imaginary quadratic field. Assume that for every $n \geq 1$ and every n-periodic point x of f, $df^n(x) \in O_K$.

After replacing f by a suitable positive iterate, we may assume that f has a repelling fixed point $o \notin PC(f)$. Let $o_i, i \geq 0$ be a homoclinic orbit of o. By Proposition 2.6, there is a linearization domain and a good return time (U, m) for $o_i, i \geq 0$. Let $q_i, i \geq m$ be the adjoint sequence for it. Set $\mu_i := df^i(q_i) \in O_K$ for $i \geq m$. Set $\lambda := df(o)$.

Lemma 3.1. *There are $a \in K^*, b \in K$ such that $\mu_i = a\lambda^i + b$ for i large.*

Proof of Lemma 3.1. We view K as a subfield of $\mathbb{C}$. Then, O_K is a discrete subgroup of $(\mathbb{C}, +)$. Set $\mathbb{T} := \mathbb{C}/O_K$ and $\pi : \mathbb{C} \to \mathbb{T}$ the quotient map. Since $\lambda \in O_K$, the multiplication by λ on L descends to an endomorphism $[\lambda]$ on $\mathbb{T}$. By Lemma 2.10, we have

$$\lim_{i\to\infty}(\mu_i - a\lambda^i) = b, \tag{3.1}$$

where $a = \theta(0) = 1/T'(0) \in \mathbb{C}^*$ and $b = P\theta'(0) \in \mathbb{C}$ (See Section 2 for the definitions of T and θ). Since $\mu_i \in O_K, i \geq m$, we get

$$\lim_{i\to\infty}[\lambda]^i\pi(a) = \pi(b).$$

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In particular, $\pi(b)$ is fixed by $[\lambda]$. Since $d[\lambda](b) = \lambda$, $[\lambda]$ is repelling at $\pi(b)$. Hence, for i large, we must have

$$[\lambda]^i \pi(a) = \pi(b). \tag{3.2}$$

Since O_K is discrete in $\mathbb{C}$, by (3.1) and (3.2), we have

$$\mu_i = a\lambda^i + b \text{ for } i \text{ large}. \tag{3.3}$$

There are $n > l \geq m$ such that $\mu_n = a\lambda^n + b$ and $\mu_l = a\lambda^l + b$. This implies that $a, b \in K$. $\square$

After enlarging m, we may assume that $\mu_i = a\lambda^i + b$ for all $i \geq m$. Assume by contradiction that f is not exceptional. By Theorem 2.11, we must have $b \neq 0$. For $\mathbf{p} \in \operatorname{Spec} O_K$, let $K_{\mathbf{p}}$ be the completion of K with respect to $\mathbf{p}$. Denote by $|\cdot|_{\mathbf{p}}$ the $\mathbf{p}$-adic norm on $K_{\mathbf{p}}$ normalized by $|p|_{\mathbf{p}} = p^{-1}$ where $p := \operatorname{char} O_K/\mathbf{p}$. Let $K_{\mathbf{p}}^{\circ}$ be the valuation ring of $K_{\mathbf{p}}$. For $\mu \in O_K$, $\mu \in \mathbf{p}$ if and only if $|\mu|_{\mathbf{p}} < 1$.

Lemma 3.2. *For $\mathbf{p} \in \operatorname{Spec} O_K$ and $\epsilon > 0$, if $\lambda \notin \mathbf{p}$, then there is $N \in \mathbb{Z}_{>0}$ such that $|\lambda^{Ni} - 1|_{\mathbf{p}} < \epsilon$ for all $i \geq 0$.*

Proof of Lemma 3.2. Since $O_K/\mathbf{p}$ is a finite field and $\lambda \notin \mathbf{p}$, there is $l \geq 1$ such that $\lambda^l - 1 \in \mathbf{p}$. Since

$$\lim_{n\to\infty} \lambda^{lp^n} = \lim_{n\to\infty} (1 + (\lambda^l - 1))^{p^n} = 1$$

in the $\mathbf{p}$−adic topology, there is $N \in \mathbb{Z}_{>0}$, such that $|\lambda^N - 1|_{\mathbf{p}} < \epsilon$. Then, for every $i \geq 0$, $|\lambda^{Ni} - 1|_{\mathbf{p}} = |\lambda^N - 1|_{\mathbf{p}} |1 + \lambda^N \cdots + \lambda^{N(i-1)}|_{\mathbf{p}} < \epsilon$. $\square$

Let S be the finite set of prime ideals $\mathbf{p} \in \operatorname{Spec} O_K \setminus \{0\}$ dividing $\lambda(\deg f)! \in O_K$. For every $\mathbf{p} \in \operatorname{Spec} O_K \setminus (S \cup \{0\})$, there is an embedding of field $\tau_K : K \hookrightarrow \mathbb{C}_p$ such that $|\cdot|_{\mathbf{p}}$ is the restriction of the norm on $\mathbb{C}_p$ via this embedding. Recall that $\mathbb{C}_p$ is the completion of the algebraic closure of $\mathbb{Q}_p$. Then, τ_K extends to an isomorphism $\tau : \mathbb{C} \to \mathbb{C}_p$. Via τ, the norm $|\cdot|_{\mathbf{p}}$ extends to a non-Archimedean complete norm on $\mathbb{C}$. By [RL03a, Corollaire 4.7 and Corollaire 4.9] of Rivera-Letelier (or [BIJL14, Corollary 1.6] of Benedetto-Ingram-Jones-Levy), for every $\mathbf{p} \in \operatorname{Spec} O_K \setminus (S \cup \{0\})$, there are at most finitely many integers $i \geq m$ satisfying $|\mu_i|_{\mathbf{p}} < 1$. We claim that for every $i \geq m$, we have $\mu_i = a\lambda^i + b \notin \mathbf{p}$ for every $\mathbf{p} \in \operatorname{Spec} O_K \setminus (S \cup \{0\})$. In fact if there is $\mathbf{p} \in \operatorname{Spec} O_K \setminus (S \cup \{0\})$ such that $a\lambda^i + b \in \mathbf{p}$ for some $i \geq m$, by Lemma 3.2, there is $N \in \mathbb{Z}_{>0}$, such that for all $j \geq 0$, $|\lambda^{Nj} - 1|_{\mathbf{p}} < |a^{-1}|/2$. Then, for every $j \geq m$, we get

$$|\mu_{i+Nj}|_{\mathbf{p}} = |a\lambda^{i+Nj} + b|_{\mathbf{p}} \leq \max\{|a\lambda^i + b|_{\mathbf{p}} + |a\lambda^i(\lambda^{Nj} - 1)|_{\mathbf{p}}\} < 1.$$

Thus, we obtain infinitely many integers $i \geq m$ satisfying $|\mu_i|_{\mathbf{p}} < 1$, which is a contradiction.

Set $S' := \{\mathbf{p} \in S |\ \lambda \in \mathbf{p}\}$ and $S'' = S \setminus S'$. Since $a \neq 0$, there is $l \geq 0$ such that $a\lambda^l + b \neq 0$. Set

$$A := \min(\{|a\lambda^l + b|_{\mathbf{p}}|\ \mathbf{p} \in S''\} \cup \{|b|_{\mathbf{p}}|\ \mathbf{p} \in S'\} \cup \{1\}) > 0.$$

For every $\mathbf{p} \in S'$, there is an integer $M_{\mathbf{p}} \geq m$ such that $|a\lambda^{M_{\mathbf{p}}}|_{\mathbf{p}} < |b|_{\mathbf{p}}$. Then, for every $i \geq M_{\mathbf{p}}$, $\mathbf{p} \in S'$, we have

$$|\mu_i|_{\mathbf{p}} = |b|_{\mathbf{p}} \geq A.$$

For every $\mathbf{p} \in S''$, by Lemma 3.2, there is $N_{\mathbf{p}} \in \mathbb{Z}_{>0}$ such that for every $j \geq 0$, $|\lambda^{N_{\mathbf{p}}j} - 1|_{\mathbf{p}} < |a^{-1}|_{\mathbf{p}} |a\lambda^l + b|_{\mathbf{p}}$. Then, for all $j \geq m$, we have

$$|\mu_{l+N_{\mathbf{p}}j}|_{\mathbf{p}} = |a\lambda^{l+N_{\mathbf{p}}j} + b|_{\mathbf{p}} = |(a\lambda^l + b) + a\lambda^i(\lambda^{N_{\mathbf{p}}j} - 1)|_{\mathbf{p}} = |a\lambda^l + b|_{\mathbf{p}} \geq A.$$

Set $M := \max\{M_{\mathbf{p}} | \mathbf{p} \in S'\}$ and $N := \prod_{\mathbf{p}\in S''} N_{\mathbf{p}}$. For every $i \geq M$, by the above discussion, we get $|\mu_{l+Ni}|_{\mathbf{p}} \geq A$ for all $\mathbf{p} \in S$. Fix an embedding of K in $\mathbb{C}$. For every $\mathbf{p} \in \operatorname{Spec} O_K \setminus \{0\}$, set $n_{\mathbf{p}} := [K_{\mathbf{p}} : \mathbb{Q}_p]$ with $p = \operatorname{char} O_K/\mathbf{p}$. We have $n_{\mathbf{p}} \leq 2$. By product formula, we get, since $|\mu_{l+Ni}|_{\mathbf{p}} = 1$ for all $\mathbf{p} \in \operatorname{Spec} O_K \setminus (S \cup \{0\})$,

$$|\mu_{l+Ni}|^{[K:\mathbb{Q}]} = \prod_{\mathbf{p}\in \operatorname{Spec} O_K\setminus\{0\}} |\mu_{l+Ni}|_{\mathbf{p}}^{-n_{\mathbf{p}}} = \prod_{\mathbf{p}\in S} |\mu_{l+Ni}|_{\mathbf{p}}^{-n_{\mathbf{p}}} \leq A^{-2|S|},$$

where $i \geq m$.

Hence, $\mu_{l+Ni}, i \geq m$ is bounded in $\mathbb{C}$. Since $a \neq 0$ and $|\lambda| > 1$, we get a contradiction. The proof is finished. □

4. The Berkovich projective line

Let $\mathbf{k}$ be a complete valued field with a nontrivial non-Archimedean norm $|\cdot|$. We denote by $\mathbf{k}^\circ$ the valuation ring of $\mathbf{k}$, $\mathbf{k}^{\circ\circ}$ the maximal ideal of $\mathbf{k}^\circ$ and $\tilde{k} = \mathbf{k}^\circ/\mathbf{k}^{\circ\circ}$ the residue field.

In this section, we collect some basic facts about Berkovich's analytification of $\mathbb{P}^1_{\mathbf{k}}$. We refer the readers to [Ber90] for a general discussion on Berkovich space, and to [BR10] for a detailed description of the Berkovich projective line and the dynamics on it.

4.1. Analytification of the projective line

Let $\mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$ be the analytification of $\mathbb{P}^1_{\mathbf{k}}$ in the sense of Berkovich, which is a compact topological space endowed with a structural sheaf of analytic functions. Only its topological structure will be used in this paper. We describe it briefly below.

The analytification $\mathbb{A}^{1,\mathrm{an}}_{\mathbf{k}}$ of the affine line $\mathbb{A}^1_{\mathbf{k}}$ is the space of all multiplicative semi-norms on $\mathbf{k}[z]$ whose restriction to $\mathbf{k}$ coincide with $|\cdot|$, endowed with the topology of pointwise convergence. For any $x \in \mathbb{A}^{1,\mathrm{an}}_{\mathbf{k}}$ and $P \in \mathbf{k}[z]$, it is customary to denote $|P(x)| := |P|_x$, where $|\cdot|_x$ is the semi-norm associated to x.

As a topological space, $\mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$ is the one-point compactification of $\mathbb{A}^{1,\mathrm{an}}_{\mathbf{k}}$. We write $\mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}} = \mathbb{A}^{1,\mathrm{an}}_{\mathbf{k}} \cup \{\infty\}$. More formally, it is obtained by gluing two copies of $\mathbb{A}^{1,\mathrm{an}}_{\mathbf{k}}$ in the usual way via the transition map $z \mapsto z^{-1}$ on the punctured affine line $(\mathbb{A}^1_{\mathbf{k}} \setminus \{0\})^{\mathrm{an}}$.

The Berkovich projective line $\mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$ is an $\mathbb{R}$-tree in the sense that it is uniquely path-connected (see [Jon15, Section 2] for the precise definitions). In particular, for $x, y \in \mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$, there is a well-defined segment $[x, y]$.

For $a \in \mathbf{k}$ and $r \in [0, +\infty)$, we denote $\mathbb{D}(a, r)$ by the closed disk $\mathbb{D}(a, r) := \{x \in \mathbb{A}^{1,\mathrm{an}}_{\mathbf{k}} : |(z-a)(x)| \leq r\}$. One may check that the norm $\sum_{i\geq 0} a_i(z-a)^i \mapsto \max\{|a_i|r^i, i \geq 0\}$ defines a point $\xi_{a,r} \in \mathbb{D}(a, r)$. One may set $x_G := \xi_{0,1}$ and call it the Gauss point.

Remark 4.1. When $r = 0$, $\xi_{a,0}$ is exactly the image of a via the identification $\mathbf{k} = \mathbb{A}^1(\mathbf{k}) \hookrightarrow \mathbb{A}^{1,\mathrm{an}}_{\mathbf{k}}$.

The group $\mathrm{PGL}_2(\mathbf{k})$ acts on $\mathbb{P}^1_{\mathbf{k}}$, and therefore on $\mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$.

Lemma 4.2. *[DF19, Proposition 1.4] For a point $x \in \mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$, $x \in \mathrm{PGL}_2(\mathbf{k}) \cdot x_g$ if and only if it takes form $x = \xi_{a,r}$ for some $a \in \mathbf{k}$ and $r \in |\mathbf{k}^*|$.*

Remark 4.3. The stablizer of $\mathrm{PGL}_2(\mathbf{k})$ at x_g is $\mathrm{PGL}_2(\mathbf{k}^\circ)$, which is open in $\mathrm{PGL}_2(\mathbf{k})$. So for any dense subfield L of $\mathbf{k}$, we have $\mathrm{PGL}_2(L) \cdot x_g = \mathrm{PGL}_2(\mathbf{k}) \cdot x_g$.

4.2. Points in $\mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$

Let $\widehat{\overline{\mathbf{k}}}$ be the completion of the algebraic closure of $\mathbf{k}$. It is still algebraically closed. By [Ber90, Corollary 1.3.6], $\mathrm{Aut}(\widehat{\overline{\mathbf{k}}}/\mathbf{k})$ acts on $\mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{\mathbf{k}}}}$ and we have $\mathbb{P}^1_{\widehat{\overline{\mathbf{k}}}}/\mathrm{Aut}(\widehat{\overline{\mathbf{k}}}/\mathbf{k}) = \mathbb{P}^1_{\mathbf{k}}$. We denote by $\pi : \mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{\mathbf{k}}}} \to \mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$ the quotient map. The points of $\mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$ can be classified into 4 types:

(i) a type 1 point takes form $\pi(a)$ where $a \in \widehat{\overline{\mathbf{k}}} \cup \{\infty\} = \mathbb{P}^{1,\mathrm{an}}_{\widehat{\overline{\mathbf{k}}}}$;
(ii) a type 2 point takes form $\pi(\xi_{x,r})$ where $x \in \widehat{\overline{\mathbf{k}}}$ and $r \in |\widehat{\overline{\mathbf{k}}}^*|$;
(iii) a type 3 point takes form $\pi(\xi_{x,r})$ where $x \in \widehat{\overline{\mathbf{k}}}$ and $r \in \mathbb{R}_{>0} \setminus |\widehat{\overline{\mathbf{k}}}^*|$;
(iv) a type 4 point takes form $\pi(x)$ where x is the pointwise limit of ξ_{x_i,r_i} such that the corresponding discs $\mathbb{D}(x_i, r_i)$ form a decreasing sequence with empty intersection.

See [Ber90, Section 1.4.4] for further details when $\mathbf{k}$ is algebraically closed. See also [Ked11, Proposition 2.2.7] and [Ste19, Section 2.1]. The set of type 1 (resp. type 2) points is dense in $\mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$. Points of type 4 exist only when $\mathbf{k}$ is not spherically complete. If we view $\mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$ as a metric tree, then the end points have type 1 or 4.

For every $x \in \mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$, we can define an equivalence relation on the set $\mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}} \setminus \{x\}$ as follows: $y \sim z$ if the paths $(x, y]$ and $(x, z]$ intersect. The tangent space T_x at x is the set of equivalences classes of $\mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}} \setminus \{x\}$ modulo $\sim$. See [Jon15, Section 2.5] for details. If x is an end point (a point of type 1 or 4), then $|T_x| = 1$. If x is of type 3, then $|T_x| = 2$. If x is of type 2, then $|T_x| \geq 3$. For a direction $v \in T_x$, let $U(v)$ be the set of all $y \in \mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$ such that the path $(x, y]$ presents v. Then, $U(v)$ is an open subset such that $\partial U(v) = x$.

4.3. *Dynamics on* $\mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$

Let $f : \mathbb{P}^1_{\mathbf{k}} \to \mathbb{P}^1_{\mathbf{k}}$ be an endomorphism of degree $d \geq 2$. We still denote by f the induced endomorphism on $\mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$.

4.3.1. The tangent map

For $x, y \in \mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$, if $f(x) = y$, then x, y have the same type. Moreover, f induces a tangent map $T_x f : T_x \to T_y$ sending $v \in T_x$ to the unique direction $w \in T_y$ such that for every $z \in U(v)$, $(y, f(z)] \cap U(w) \neq \emptyset$. We note that, in general, $f(U(v))$ may not be equal to $U(w)$. If $f(U(v)) = U(w)$, we say that v is a *good direction*. Otherwise, it is called a *bad direction*. If v is a bad direction, then $f(U(v)) = \mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$ [Ben, Theorem 7.34].

We may naturally identify T_{x_G} with $\mathbb{P}^1(\tilde{\mathbf{k}})$ as follows. Consider the standard model $\mathbb{P}^1_{\mathbf{k}^\circ}$ of $\mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$. There is a reduction map red : $\mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}} \to \mathbb{P}^1_{\tilde{\mathbf{k}}}$. The preimage of the generic point of $\mathbb{P}^1_{\tilde{\mathbf{k}}}$ is the Gauss point x_G, and for every $y \in \mathbb{P}^1(\tilde{\mathbf{k}})$, there is a unique $v_y \in T_{x_G}$ such that $U(v_y) = \mathrm{red}^{-1}(y)$. The map $\mathbb{P}^1(\tilde{\mathbf{k}}) \to T_{x_G}$ sending y to v_y is bijective. Let h be any endomorphism of $\mathbb{P}^1_{\mathbf{k}}$ such that $h(x_G) = x_G$, and it extends to a rational self-map $h_{\mathbf{k}^\circ}$ of $\mathbb{P}^1_{\mathbf{k}^\circ}$. We denote by $\tilde{h} : \mathbb{P}^1_{\tilde{\mathbf{k}}} \to \mathbb{P}^1_{\tilde{\mathbf{k}}}$ the restriction of h to the special fiber of $\mathbb{P}^1_{\mathbf{k}^\circ}$ and call it the reduction of h. Then, $T_{x_G} h : T_{x_G} = \mathbb{P}^1(\tilde{\mathbf{k}}) \to T_{x_G}$ is induced by $\tilde{h}$. We define $\deg T_{x_G} h$ to be the degree of $\tilde{h}$. We note that $\deg \tilde{h} \leq \deg h$. The equality holds if and only if $h_{\mathbf{k}^\circ}$ is an endomorphism. In this case, we say that h has *explicit good reduction*.

More generally, for every $x, y \in \mathrm{PGL}_2(\mathbf{k}) \cdot x_G$ with $f(x) = y$, we may define

$$\deg T_x f := \deg T_{x_G}(h^{-1} f g) = \deg \widetilde{h^{-1} f g},$$

where $h, g \in \mathrm{PGL}_2(\mathbf{k})$ with $g(x_G) = x$ and $h(x_G) = y$. Then, $1 \leq \deg T_{x_G} f \leq \deg f$ and $\deg T_{x_G} f$ does not depend on the choices of g, h.

Remark 4.4. Assume that $\mathbf{k}$ is algebraically closed. By Lemma 4.2, the set of type 2 points in $\mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$ is exactly $\mathrm{PGL}_2(\mathbf{k}) \cdot x_G$.

4.3.2. Periodic points

Assume that $\mathbf{k}$ is algebraically closed. For $n \geq 1$, a n-periodic point of f is a point $x \in \mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$ such that $f^n(x) = x$. They can be divided into three types: attracting, indifferent and repelling. A type 1 periodic point $x \in \mathbb{P}^1(\mathbf{k})$ of period $n \geq 1$ is called *attracting* if $|d(f^n)(x)| < 1$; *indifferent* if $|d(f^n)(x)| = 1$; and *repelling* if $|d(f^n)(x)| > 1$. A n-periodic point $x \in \mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$ of type 2 is called *indifferent* if $\deg T_x f = 1$; *repelling* if $\deg T_x f \geq 2$. Every n-periodic point $x \in \mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$ of type 3 or 4 are *indifferent* [RL03b, Lemma 5.3, 5.4].

4.3.3. Fatou and Julia sets

Assume that $\mathbf{k}$ is algebraically closed.

The *Julia set* of f is the set $\mathcal{J}(f)$ of points $z \in \mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$ with the following property: for every neighborhood U of z, the union of iterates $\bigcup_{n\geq 0} f^n(U)$ omits only finitely many points of $\mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}}$. Its complement $\mathcal{F}(f) := \mathbb{P}^{1,\mathrm{an}}_{\mathbf{k}} \setminus \mathcal{J}(f)$ is the *Fatou set* of *f*.

We list some basic properties of the Julia and Fatou sets of *f*.

Proposition 4.5 [Ben, Chapter 8 and Section 12.2].

(i) *The Fatou set $\mathcal{F}(f)$ is open and the Julia set $\mathcal{J}(f)$ is closed.*
(ii) *All attracting periodic points of f are contained in $\mathcal{F}(f)$.*
(iii) *All repelling periodic points of f are contained in $\mathcal{J}(f)$.*
(iv) *We have $\mathcal{J}(f) = f(\mathcal{J}(f)) = f^{-1}(\mathcal{J}(f))$ and $\mathcal{F}(f) = f(\mathcal{F}(f)) = f^{-1}(\mathcal{F}(f))$.*
(v) *Both $\mathcal{J}(f)$ and $\mathcal{F}(f)$ are nonempty.*
(vi) *For every $z \in \mathcal{J}(f)$, $\cup_{n\geq 0} f^{-n}(z)$ is dense in $\mathcal{J}(f)$.*
(vii) *Repelling periodic points are dense in $\mathcal{J}(f)$.*

4.3.4. Good reduction

We say *f* has *good reduction* if, after some coordinate change $h \in \mathrm{PGL}_2(\mathbf{k})$, the map $h^{-1} \circ f \circ h$ has explicit good reduction.

Theorem 4.6 [FRL10, Theorem E]. *The endomorphism f has explicit good reduction if and only if $\mathcal{J}(f) = x_G$. Moreover, if* $\mathbf{k}$ *is algebraically closed, f has good reduction if and only if $\mathcal{J}(f)$ is a single point.*

Remark 4.7. Assume that $\mathbf{k}$ is algebraically closed. If $\mathcal{J}(f)$ is a single point, then by Theorem 4.6 and (vii) of Proposition 4.5, it is a type 2 repelling point.

5. Rescaling limits of holomorphic families

5.1. Holomorphic families

Recall that $\Psi : \mathrm{Rat}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ is the quotient morphism, where $\mathcal{M}_d(\mathbb{C}) := \mathrm{Rat}_d(\mathbb{C})/\mathrm{PGL}_2(\mathbb{C})$ is the moduli space.

Let Λ be a complex manifold. We denote by $\mathcal{O}^{\mathrm{an}}(\Lambda)$ the ring of holomorphic functions on Λ. Moreover, if Λ is a complex algebraic variety, we denote by $\mathcal{O}(\Lambda)$ the ring of algebraic functions on Λ.

A *holomorphic (resp. meromorphic) family* on Λ is an endomorphism (resp. meromorphic self-map) f_Λ on $\mathbb{P}^1 \times \Lambda$ such that $\pi_\Lambda \circ f_\Lambda = \pi_\Lambda$, where $\pi_\Lambda : \mathbb{P}^1(\mathbb{C}) \times \Lambda \to \Lambda$ is the projection to Λ. More concretely, one may write $f_\Lambda([x : y], t) = ([P_t(x, y) : Q_t(x, y)], t)$ where $P_t(x, y), Q_t(x, y)$ are homogenous polynomials of same degree d in $\mathcal{O}^{\mathrm{an}}(\Lambda)[x, y]$ without a common divisor. We say that f_Λ is of degree d. Then, f_Λ is holomorphic if there is no $(t, x, y) \in \Lambda \times \mathbb{C}^* \times \mathbb{C}^*$ such that $P_t(x, y) = Q_t(x, y) = 0$.

For $t \in \Lambda$, we denote by f_t the restriction of f_Λ to the fiber above t. We denote by $I(f_\Lambda)$ the indeterminacy locus of f_Λ and $B(f_\Lambda) := \pi_\Lambda(I(f_\Lambda))$. Then, $I(f_\Lambda)$ and $B(f_\Lambda)$ are proper closed analytic subspaces of $\mathbb{P}^1 \times \Lambda$ and Λ, respectively. For every $t \in \Lambda \setminus B(f_\Lambda)$, we have $\deg f_t = d$. When Λ is connected, this is equivalent to say that $\deg f_t = d$ for one $t \in \Lambda \setminus B(f_\Lambda)$. A meromorphic family is holomorphic if and only if $B(f_\Lambda) = \emptyset$. So, giving a degree d holomorphic family f_Λ on Λ is equivalent to giving a holomorphic morphism $t \mapsto f_t = P_t/Q_t$ from Λ to $\mathrm{Rad}_d(\mathbb{C})$. We say that f_Λ is algebraic if Λ is a complex algebraic variety and $f_\Lambda : \mathbb{P}^1 \times \Lambda \to \mathbb{P}^1 \times \Lambda$ is algebraic (i.e., $P_t, Q_t \in \mathcal{O}(\Lambda)[x, y]$). In other words, it means that the induced morphism $\Lambda \to \mathrm{Rad}_d(\mathbb{C})$ is algebraic.

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For a degree d holomorphic family f_Λ on Λ, let $\Psi_\Lambda : \Lambda \to \mathcal{M}_d$ be the holomorphic morphism sending $t \in \Lambda$ to the class of f_t in $\mathcal{M}_d(\mathbb{C})$. We say that f_Λ is *isotrivial* if $\Psi_\Lambda : \Lambda \to \mathcal{M}_d$ is locally constant. More generally for degree d meromorphic family f_Λ, we say that f_Λ is *isotrivial* if $f|_{\Lambda\setminus B(f_\Lambda)}$ is isotrivial.

5.2. *Potentially good reduction*

Assume that Λ is a Riemann surface and f_Λ is a meromorphic family of degree d.

For $b \in \Lambda$, we say that f_Λ has *potentially good reduction* at b if $\Phi_{\Lambda\setminus(B(f_\Lambda)\cup\{b\})} : \Lambda \to \mathcal{M}_d$ extends to a holomorphic morphism on $(\Lambda \setminus B(f_\Lambda)) \cup \{b\}$. In particular, f_Λ has potentially good reduction at every $b \in \Lambda \setminus B(f_\Lambda)$.

Lemma 5.1. *Assume that Λ is an irreducible smooth projective curve. Let f_Λ be a meromorphic family of degree d. If f_Λ has potentially good reduction at every point in Λ, then f_Λ is isotrivial.*

Proof. Since f_Λ has potentially good reduction at every point in $B(f_\Lambda)$, $\Psi_{\Lambda\setminus B(f_\Lambda)} : \Lambda \setminus B(f_\Lambda) \to \mathcal{M}_d$ extends to a holomorphic morphism $\Psi_\Lambda : \Lambda \to \mathcal{M}_d$. Recall that $\mathcal{M}_d(\mathbb{C}) = \mathrm{Spec}\,(\mathcal{O}(\mathrm{Rat}_d(\mathbb{C})))^{\mathrm{PGL}_2(\mathbb{C})}$ is affine [Sil07, Theorem 4.36(c)]. This follows from the fact that $\mathrm{Rat}_d(\mathbb{C})$ is affine and the geometric invariant theory [MF82, Chapter 1]. Since Λ is projective, Ψ_Λ is a constant map. This concludes the proof. □

Having potentially good reduction is a local property at b (i.e., for every open neighborhood U of b in Λ, f_Λ has potentially good reduction at b if and only if $f_U := f_\Lambda|_{\mathbb{P}^1(\mathbb{C})\times U}$ has potentially good reduction at b). Note that there is an open neighborhood U of b which is isomorphic to a disk $\mathbb{D}$ such that $f_{U\setminus\{b\}}$ is holomorphic. So we can focus on the case that $f_\mathbb{D}$ is a meromorphic family that is holomorphic on $\mathbb{D}^*$. We will give another characterization of potentially good reduction via non-Archimedean dynamics.

5.3. *Holomorphic family on puncture disk*

Let $f_\mathbb{D}$ be a a meromorphic family of degree $d \geq 2$ that is holomorphic on $\mathbb{D}^*$. Let t be the standard coordinate on $\mathbb{D}$. We can relate $f_\mathbb{D}$ to some non-Archimedean dynamics on the field of Laurent's series $\mathbb{C}((t))$.

Recall that on $\mathbb{C}((t))$, there is a t-adic norm $|\cdot|$: Given an element $z = \sum_{n\geq n_0} a_n t^n \neq 0$, where $n_0 \in \mathbb{Z}$, $a_n \in \mathbb{C}$ and $a_{n_0} \neq 0$, the t-adic norm of z is $|z| := e^{-n_0}$. This norm is non-Archimedean and $\mathbb{C}((t))$ is complete for $|\cdot|$. Set $\mathbb{L} := \widehat{\overline{\mathbb{C}((t))}}$.

Write

$$f([x:y],t) = ([P_t(x,y) : Q_t(x,y)], t)$$

where $P_t(x,y), Q_t(x,y)$ are homogenous polynomials of degree d in $\mathcal{O}^{\mathrm{an}}(\mathbb{D})[1/t][x,y]$ without common divisors. Since $\mathcal{O}^{\mathrm{an}}(\mathbb{D})[1/t] \subseteq \mathbb{C}((t))$, $f_\mathbb{D}$ defines an endomorphism $f_{\mathbb{C}((t))} : [x,y] \mapsto [P_t(x,y) : Q_t(x,y)]$ on $\mathbb{P}^1_{\mathbb{C}((t))}$ of degree d. Set $f_\mathbb{L} := f_{\mathbb{C}((t))}\hat{\otimes}_{\mathbb{C}((t))}\mathbb{L} : \mathbb{P}^1_\mathbb{L} \to \mathbb{P}^1_\mathbb{L}$.

Recall that

$$\overline{\mathbb{C}((t))} = \cup_{n\geq 1}\mathbb{C}((t^{1/n})). \tag{5.1}$$

To get endomorphisms over $\mathbb{C}((t^{1/n}))$, we introduce some base changes of $f_\mathbb{D}$ as follows. Consider the morphism $\phi_n : U_n := \mathbb{D} \to \mathbb{D}$ sending t to t^n. There is $u_n \in \mathcal{O}^{\mathrm{an}}(U_n)$ such that $u_n^n = \phi^* t$. Then, u_n is a coordinate on U_n, and we may identify $\mathbb{C}[u_n]$ with $\mathbb{C}[t^{1/n}]$ (hence, we may identify $\mathbb{C}((u_n))$ with $\mathbb{C}((t^{1/n})))$. Let $o \in U_n$ be the point defined by $u_n = 0$. The endomorphism on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}((u_n))}$ induced by f_{U_n} is $f_{\mathbb{C}((u_n))} = f_{\mathbb{C}((t))}\hat{\otimes}_{\mathbb{C}((t))}\mathbb{C}((t^{1/n}))$.

Lemma 5.2. *If $f_\mathbb{L}$ has good reduction, then $f_\mathbb{D}$ has potentially good reduction at* 0.

Remark 5.3. The inverse statement of Lemma 5.2 is also true. However, we do not need that direction in this paper. So we leave it to readers.

Proof of Lemma 5.2. By Theorem 4.6, there is $h \in \mathrm{PGL}_2(\mathbb{L})$ such that $\mathcal{J}(f_{\mathbb{L}}) = \{h(x_G)\}$. Then, $h^{-1} \circ f_{\mathbb{L}} \circ h$ has explicit good reduction. By (5.1) and Remark 4.3, we may assume that $h \in \mathrm{PGL}_2(\mathbb{C}((t^{1/n})))$ for some $n \geq 1$. Since $\mathbb{C}(u_n)$ is dense in $\mathbb{C}((u_n)) = \mathbb{C}((t^{1/n}))$, by Remark 4.3 again, we may assume that $h \in \mathrm{PGL}_2(\mathbb{C}(u_n))$. There is an open neighborhood V of o such that h and h^{-1} are holomorphic on $V \setminus \{o\}$ (i.e., they define holomorphic families $h_{V\setminus\{o\}}$ and $h^{-1}_{V\setminus\{o\}}$). We may assume further that $V \simeq \mathbb{D}$. Consider the family $f_V := h_V^{-1} \circ f_{U_n}|_V \circ h_V$. Observe that

$$\Psi_{\mathbb{D}^*} \circ \phi|_{V\setminus\{o\}} = \Psi_{V\setminus\{o\}}. \tag{5.2}$$

Then, f_V induces an endomorphism $f_{\mathbb{C}((u))} = f_{\mathbb{C}((t))} \hat{\otimes}_{\mathbb{C}((t))} \mathbb{C}((u))$ on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}((u))}$, which has good reduction. So f_V is an endomorphism on $\mathbb{P}^1 \times V$. So $\Psi_{V\setminus\{o\}}$ extends to a holomorphic morphism $\Psi_V : V \to \mathcal{M}_d$. By (5.2), $\Psi_{\mathbb{D}^*}$ is bounded in some neighborhood of o. So $\Psi_{\mathbb{D}^*}$ extends to a holomorphic morphism on $\mathbb{D}$, which means that $f_{\mathbb{D}}$ has potentially good reduction at 0. □

The following definition was introduced by Kiwi.

Definition 5.4. [Kiw15] Let $f_{\mathbb{D}}$ be a meromorphic family of degree $d \geq 2$ which is holomorphic on $\mathbb{D}^*$. We say an endomorphism g is a rescaling limit of $f_{\mathbb{D}}$ (or $f_{\mathbb{D}^*}$) (via $(q, M_{\mathbb{D}})$) if there is an integer $q \geq 1$, a finite set $S \subset \mathbb{P}^1(\mathbb{C})$ and a meromorphic family $M_{\mathbb{D}}$ of degree 1, such that $M_{\mathbb{D}}$ and $M_{\mathbb{D}}^{-1}$ are holomorphic on $\mathbb{D}^*$ and

$$M_t^{-1} \circ f_t^q \circ M_t(z) \to g(z)$$

when $t \to 0$, uniformly on compact subsets of $\mathbb{P}^1(\mathbb{C}) \setminus S$.

The following result was proved by Kiwi.

Proposition 5.5 [Kiw15, Proposition 3.4]. *Let $f_{\mathbb{D}}$ be a meromorphic family of degree $d \geq 2$ which is holomorphic on $\mathbb{D}^*$. Let $M_{\mathbb{D}}$ be a meromorphic family of degree 1, such that $M_{\mathbb{D}}$ and $M_{\mathbb{D}}^{-1}$ are holomorphic on $\mathbb{D}^*$. Then, for all $q \geq 1$, the following are equivalent:*

(i) *There exist an endomorphism g on $\mathbb{P}^1$ and a finite set $S \subset \mathbb{P}^1(\mathbb{C})$ satisfying*

$$M_t^{-1} \circ f_t^q \circ M_t(z) \to g(z)$$

when $t \to 0$, uniformly on compact subsets of $\mathbb{P}^1(\mathbb{C}) \setminus S$.

(ii) *The point $x = M_{\mathbb{L}}(x_G)$ is fixed by $f_{\mathbb{L}}^q$ and $M_{\mathbb{L}}^{-1} \circ f_{\mathbb{L}}^q \circ M_{\mathbb{L}} = g$.*

In the case where (i) and (ii) hold, $T_x f^q : T_x \to T_x$ can be identified with g after identifying T_x to $T_{x_G} = \mathbb{P}^1(\mathbb{C})$ via $T_{x_G} M_{\mathbb{L}} : T_{x_G} \to T_x$. Under this identification, S is a finite subset of T_x, which contains all the bad directions of $T_x f^q$.

Remark 5.6. One may rewrite Definition 5.4 in the following more geometric way. Let $h_{\mathbb{D}}$ be the meromorphic family $h_{\mathbb{D}} := M_{\mathbb{D}}^{-1} \circ f_{\mathbb{D}}^q \circ M_{\mathbb{D}}$ on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{D}$. Then, $h_0 = g$. Moreover, S can be any finite subset containing S_0 where $I(h_{\mathbb{D}}) = S_0 \times \{0\} \subseteq \mathbb{P}^1(\mathbb{C}) \times \mathbb{D}$.

Corollary 5.7. *Let $x \in \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ be a type 2 fixed point of $f_{\mathbb{L}}$. Assume that $T_x f_{\mathbb{L}}$ is conjugate to some endomorphism $g : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. Then, there is $n \geq 1$, such that g is a rescaling limit of $f_{U_n}|_V$ where f_{U_n} is the base change of $f_{\mathbb{D}}$ by the morphism $U_n := \mathbb{D} \to \mathbb{D}$ sending t to t^n as in Section 5.3, and V is an open neighborhood of $o \in U_n$ isomorphic to $\mathbb{D}$.*

Proof. There is $M_{\mathbb{L}} \in \mathrm{PGL}_2(\mathbb{L})$ such that $x = M_{\mathbb{L}}(x_G)$. By (5.1) and Remark 4.3, we may assume that $M_{\mathbb{L}} \in \mathrm{PGL}_2(\mathbb{C}((t_n^{1/n})))$ for some $n \geq 1$. Let f_{U_n} be the base change of $f_{\mathbb{D}}$ by the morphism

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$\phi_n : U_n := \mathbb{D} \to \mathbb{D}$, sending t to t^n, and pick u_n with $u_n^n = \phi_n^{-1}(t)$ as in Section 5.3. Since $\mathbb{C}(u_n)$ is dense in $\mathbb{C}((u_n)) = \mathbb{C}((t^{1/n}))$, by Remark 4.3 again, we may assume that $M_{\mathbb{L}} \in \mathrm{PGL}_2(\mathbb{C}(u_n))$. There is an open neighborhood V of o such that $M_{\mathbb{L}}$ and $M_{\mathbb{L}}^{-1}$ are holomorphic on $V \setminus \{o\}$ (i.e., they define holomorphic families $M_{V\setminus\{o\}}$ and $M_{V\setminus\{o\}}^{-1}$). Then, we conclude the proof by Proposition 5.5. □

5.4. Endomorphisms without repelling type I periodic points

In general, the Julia set of an endomorphism $f_{\mathbb{L}}$ on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ is a complicated object. The following theorem due to Favre-Rivera Letelier [FRL], and independently by Luo [Luo22, Proposition 11.4], classifies the case when $f_{\mathbb{L}}$ has no repelling type I periodic points.

Theorem 5.8. *Let $f_{\mathbb{L}} : \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}} \to \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ be an endomorphism. Assume $f_{\mathbb{L}}$ has no type 1 repelling periodic points. Then, the Julia set of $f_{\mathbb{L}}$ is contained in a segment.*

By (v) of Proposition 4.5, $\mathcal{J}(f_{\mathbb{L}}) \neq \emptyset$. In the above theorem, if $f_{\mathbb{L}}$ does not have good reduction, then the segment cannot be a point. As a corollary, we get the following lemma.

Lemma 5.9. *Let $f_{\mathbb{L}} : \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}} \to \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ be an endomorphism of degree $d \geq 2$, which does not have good reduction. Assume that $\mathcal{J}(f_{\mathbb{L}})$ is contained in a minimal segment $[a, b]$. Let x be a repelling type 2 periodic point in (a, b) with period $q \geq 1$. Then, the tangent map $T_x f^q$ is conjugate to $z \mapsto z^m$ for some $|m| = \deg T_x f^q \geq 2$. Moreover, every bad direction of $T_x f^q$ is presented by $(x, a]$ or $(x, b]$ and under the above conjugacy, it is identified to 0 or ∞.*

Proof. Since $[a, b]$ is the minimal segment that contains $\mathcal{J}(f)$, a and b are contained in the Julia set. Since $\deg f_{\mathbb{L}} \geq 2$ and $f_{\mathbb{L}}$ does not have good reduction, the Julia set is not a single point. Hence, $a \neq b$. Let v_1 (resp. v_2) be the direction in T_x represented by the segment $(x, a]$ (resp. $(x, b]$). Since $\mathcal{J}(f_{\mathbb{L}}) \subseteq [a, b]$, $\{v \in T_x |\, U(v) \cap \mathcal{J}(f_{\mathbb{L}}) \neq \emptyset\} = \{v_1, v_2\}$. Since $\mathcal{J}(f_{\mathbb{L}})$ is totally invariant, for $v \in T_x$, if $f^q(U(v)) \cap \mathcal{J}(f_{\mathbb{L}}) \neq \emptyset$, then $U(v) \cap \mathcal{J}(f_{\mathbb{L}}) \neq \emptyset$. Hence, $v \in \{v_1, v_2\}$. This implies $\{v_1, v_2\}$ is totally invariant by $T_x f^q$. Actually, let $w \in (T_x f^q)^{-1}(v_i)$ for some $i = 1, 2$. Then, we have $U(v_i) \subset f^q(U(w))$. This implies $f^q(U(w)) \cap \mathcal{J}(f_{\mathbb{L}}) \neq \emptyset$. Thus, $w = v_i$. Bad directions of $T_x f^q$ are contained in $\{v_1, v_2\}$. Actually, if w is a bad direction, then we have $f^q(U(w)) = \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$. Hence, $f^q(U(w)) \cap \mathcal{J}(f_{\mathbb{L}}) \neq \emptyset$, which implies $w = v_1$ or v_2. Finally, an endomorphism of degree $\deg T_x f^q$ on $\mathbb{P}^1(\mathbb{C})$ has a totally invariant set with two elements that must conjugate to $z \mapsto z^m$ for some $|m| = \deg T_x f^q$. This conjugacy maps $\{v_1, v_2\}$ to $\{0, \infty\}$, which concludes the proof. □

The following theorem is the main result of this section.

Theorem 5.10. *Let $f_{\mathbb{D}}$ be a meromorphic family of degree $d \geq 2$ which is holomorphic on $\mathbb{D}^*$. Assume that $f_{\mathbb{D}}$ does not have potentially good reduction at 0. For every $n \geq 1$, assume that the multipliers of the n-periodic points of f_t are uniformly bounded in t. Then, there is $n \geq 1, m \geq 2$, such that $g : z \mapsto z^m$ is a rescaling limit of $f_{U_n}|_V$ where f_{U_n} is the base change of $f_{\mathbb{D}}$ by the morphism $U_n := \mathbb{D} \to \mathbb{D}$, sending t to t^n as in Section 5.3, and V is an open neighborhood of $o \in U_n$ isomorphic to $\mathbb{D}$. Moreover, we may ask the finite set S in Definition 5.4 to be contained in $\{0, \infty\}$.*

Proof. Let $f_{\mathbb{L}} : \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}} \to \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ be the endomorphism induced by $f_{\mathbb{D}}$. The multipliers of the n-periodic points of f_t are uniformly bounded in t, which implies $f_{\mathbb{L}}$ has no repelling type 1 periodic points. By Theorem 5.8, $\mathcal{J}(f_{\mathbb{L}})$ is contained in a minimal segment $[a, b]$. Since $f_{\mathbb{D}}$ does not have potentially good reduction at 0, by Lemma 5.2, $f_{\mathbb{L}}$ does not have good reduction. By a result of Rivera-Letelier [BR10, Theorem 10.88], there are infinitely many repelling type 2 periodic points. By (iii) of Proposition 4.5, they are necessarily contained in $\mathcal{J}(f_{\mathbb{L}})$. Pick a repelling type 2 periodic point x that is contained in (a, b) of period $q \geq 1$. By Lemma 5.9, replace q by $2q$ if necessary. The tangent map $T_x f^q$ is conjugate to $z \mapsto z^m$ for some $m \geq 2$. Moreover, the bad directions of $T_x f^q$ can be identified with a subset of $\{0, \infty\}$ by the conjugacy. The proof is finished by using Corollary 5.7. □

6. A new proof of McMullen's theorem

We can now give a new proof of Theorem 1.2.

Proof of Theorem 1.2. Let f_Λ be a non-isotrivial stable irreducible algebraic family of endomorphisms of degree $d \geq 2$. Since Λ is covered by affine open subsets, we may assume that Λ itself is affine. Cutting Λ by hyperplanes and removing the singular points, we can reduce to the case that Λ is a connected Riemann surface of finite type. Since the only non-isotrivial family of exceptional endomorphisms of degree d is the flexible Lattès family, we only need to show that there is a nonempty open subset W of Λ such that, for $t \in W$, f_t, is exceptional.

Write $\Lambda = M \setminus B$, where M is a compact Riemann surface and B is a finite subset. Since f_Λ is algebraic, it extends to a meromorphic family of degree d. We have $B(f_M) \subseteq B$. Since f_Λ is not isotrivial, by Lemma 5.1, there is $b \in B$ such that f_M does not have potentially good reduction at b. Reparametrize our family near $b \in M$, and we get a meromorphic family $f_{\mathbb{D}}$ of degree $d \geq 2$, which is holomorphic on $\mathbb{D}^*$ and preserves the multiplier spectrum.

By Theorem 5.10, after replacing $f_{\mathbb{D}}$ by the family f_V in Theorem 5.10, we may assume that $z \mapsto z^m$ for some $m \geq 2$ is a rescaling limit of $f_{\mathbb{D}}$ with $S = \{0, \infty\}$. Using the reformulation of the rescaling limit in Remark 5.6, there is an integer $q \geq 1$ and a meromorphic family $M_{\mathbb{D}}$ of degree 1, such that $M_{\mathbb{D}}$ and $M_{\mathbb{D}}^{-1}$ are holomorphic on $\mathbb{D}^*$, and h_0 is $z \to z^m$ where $h_{\mathbb{D}} := M_{\mathbb{D}}^{-1} \circ f_{\mathbb{D}}^q \circ M_{\mathbb{D}}$ on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{D}$. Moreover, $I(h_{\mathbb{D}}) \subseteq \{0, \infty\} \times \{0\} \subseteq \mathbb{P}^1(\mathbb{C}) \times \mathbb{D}$. We may replace $f_{\mathbb{D}}$ by $h_{\mathbb{D}}$ and assume that $f_0 : z \mapsto z^m$ and $I(f_{\mathbb{D}}) \subseteq \{0, \infty\} \times \{0\} \subseteq \mathbb{P}^1(\mathbb{C}) \times \mathbb{D}$.

The Julia set of f_0 is the unit circle S^1, and f_0 is expanding on S^1. We need the following classical lemma of holomorphic motions of expanding sets. A proof can be found (without using quasiconformal maps) in Jonsson [Jon98], which is also valid in higher dimension. Let $K \subset \mathbb{P}^1(\mathbb{C})$ be a compact set. We say $f : K \to K$ is *expanding* if there exist $C > 0$ and $\rho > 1$ such that $|df^n(x)| \geq C\rho^n$ for every $n \geq 0$ and $x \in K$.

Lemma 6.1. *Let $(f_t)_{t \in \mathbb{D}}$ be a family of endomorphisms on $\mathbb{P}^1(\mathbb{C})$. Suppose f_0 has an expanding set K, $f(K) = K$. Assume (f_t) is a holomorphic family in a neighborhood of K (i.e., there exists an open set V, $K \subset V$ such that for every $z \in V$, $t \mapsto f_t(z)$ is holomorphic in $\mathbb{D}$). Then, there exist $r > 0$ and a continuous map $h : \mathbb{D}_r \times K \to \mathbb{P}^1(\mathbb{C})$ such that for each $t \in \mathbb{D}_r$:*

(i) *$K_t := h(t, K)$ is an expanding set of f_t.*
(ii) *the map $h_t := h(t, \cdot) : K \to K_t$ is a homeomorphism and $f_t \circ h_t = h_t \circ f_0$.*

We set $f_0 : z \mapsto z^m$ and $K := S^1$ in the above lemma. The endomorphism f_0 has the following properties:

(1) $f_0^{-1}(K) = f_0(K) = K$;
(2) all periodic points outside the exceptional set $\{0, \infty\}$ are contained in K;
(3) for every n-periodic point $z \in K$, we have $df_0^n(z) = m^n$.

Since the family $(f_t)_{t \in \mathbb{D}^*}$ has the same multiplier spectrum, the multiplier of the periodic point $h_t(z)$ of f_t does not change in the family $t \in \mathbb{D}_r^*$. Hence, for every $t \in \mathbb{D}_r$ we have $df_t^n(h_t(z)) = m^n$. We choose a homoclinic orbit o_i, $i \geq 0$ of f_0 with $o_0 = 1$. By (1), all $o_i, i \geq 1$ are contained in K. Hence, $h_t(o_i)$, $i \geq 0$ is a homoclinic orbit of f_t at $z = h_t(1)$, for $t \in \mathbb{D}_r$. Let q_i, $i \geq 0$ be an adjoint sequence of o_i, $i \geq 0$. For every $t \in \mathbb{D}_r^*$, we need to show $h_t(q_i)$, $i \geq 0$ is an adjoint sequence of $h_t(o_i)$, $i \geq 0$. In fact, let U_t be a linearization domain of f_t at $h_t(1)$. Let $U_{t,i}$ be the connected component of $f_t^{-i}(U_t)$ containing $h_t(o_i)$. Let l be a good return time of U_t. For every $n \geq l$, $f_t^n : U_{t,n} \to U_t$ is an isomorphism, with a unique fixed point p_n. Let V be the connected component of $h_t^{-1}(U_t \cap K_t)$ containing 1. It is an open arc in S^1. Let V_n be the connected component of $f_0^{-n}(V)$ containing o_n. Since K is totally invariant by f_0 and V contains some linearization domain at 1, after enlarging l if necessary, for every $n \geq l$ we have $q_n \in V_n \cap K$. Hence, $h_t(q_n) \in U_{t,n} \cap K_t$, which is fixed by $f_t^n : U_{t,n} \to U_t$. By the uniqueness of p_n we have $p_n = h_t(q_n)$. Hence, $h_t(q_i)$, $i \geq 0$ is an adjoint sequence of $h_t(o_i)$, $i \geq 0$.

For every $t \in \mathbb{D}_r^*$, we consider the dynamics of f_t. The fixed point $h_t(1)$ has multiplier m and the adjoint sequence $h_t(q_i)$, $i \geq 0$ of the homoclinic orbit $h_t(o_i)$, $i \geq 0$ has multiplier m^i when i large enough. By Theorem 2.11, f_t is exceptional, which concludes the proof. □

7. Conformal expanding repellers and applications

7.1. Definition, examples and rigidity of CER

The following definition was introduced by Sullivan [Sul86].

Definition 7.1. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism over $\mathbb{C}$. A compact set $K \subset \mathbb{P}^1(\mathbb{C})$ is called a CER of f if

(i) there exist $m \geq 1$ and a neighborhood V of K such that $f^m(K) = K$ and $K = \cap_{n\geq 0} f^{-mn}(V)$.
(ii) $f^m : K \to K$ is expanding (i.e., there are constants $C > 0$ and $\lambda > 1$ such that $|df^{nm}(x)| \geq C\lambda^n$ for every $x \in K$ and $n \geq 1$);
(iii) $f^m : K \to K$ is topologically exact (i.e., for every open set $U \subset K$ there exists $n \geq 0$ such that $f^{mn}(U) = K$).

Remark 7.2. Condition (i)+(ii) is equivalent to f^m expanding on K and $f^m : K \to K$ is an open map [PU10, Lemma 6.1.2].

The following is an important class of examples of CER.

Example 7.3. Assume V, U_i, $1 \leq i \leq k$ are connected open sets in $\mathbb{P}^1(\mathbb{C})$, $k \geq 2$ such that $\overline{U_i} \subset V$, and there exists $m \geq 1$ such that $f^m : U_i \to V$ is an isomorphism. Then, we call

$$K := \left\{ z \in \bigcup_{i=1}^{k} U_i \,\middle|\, f^{mn}(z) \in \bigcup_{i=1}^{k} U_i \text{ for every } n \geq 0 \right\}$$

a *horseshoe* of f. We check that K satisfies the three conditions in Definition 7.1. Let $V_0 := \cup_{i=1}^k U_i$.

Condition (i): It follows from the definition of K;
Condition (ii): $f^m : V_0 \to V$ strictly expands the hyperbolic metric of V. This implies $f^m : K \to K$ is expanding;
Condition (iii): Again, using $f^m : V_0 \to V$ strictly expands the hyperbolic metric of V. The maximal diameter of the connected components of $f^{-nm}(V_0) \cap V_0$ shrinks to 0 when $n \to \infty$. For each open set $W \subset K$, there exist integer $n \geq 0$ and a connected component B of $f^{-nm}(V_0) \cap V_0$ such that $B \cap K \subset W$. Since $f^{(n+1)m}(B \cap K) = K$, we have $f^{(n+1)m}(W) = K$. Hence, $f^m : K \to K$ is topologically exact.

Moreover, K is a Cantor set. In particular, K is not a finite set.

When f has degree at least 2, there are plenty of horseshoes. Following the terminology in section 2, we can construct a horseshoe associated to finite numbers of homoclinic orbits at o. We prove the following lemma which will be used in the proof of Theorem 1.8.

Lemma 7.4. *Let o be a repelling fixed point. Let $k \geq 1$ be an integer. Assume for each fixed $1 \leq j \leq k$, o_i^j, $i \geq 0$ is a homoclinic orbit of o such that $o_i^j \notin C(f)$. Then, there exist an integer $m \geq 1$ and a horseshoe $f^m : K \to K$ such that $o_{mi}^j \in K$ for every $i \geq 0$ and $1 \leq j \leq k$. Moreover, for each $0 \leq q \leq m-1$, $f^q(K)$ is a CER.*

Proof. By Lemma 2.6, there exist a linearization domain U of o and an integer m such that, for every $1 \leq j \leq k$, m is a common good return time of U for the homoclinic orbits o_i^j, $i \geq 0$. Let U_m^j be the connected component of $f^{-m}(U)$ containing o_m^j. Let

$$V_0 := \left(\bigcup_{j=1}^{k} U_m^j \right) \cup g^m(U).$$

Then, the set

$$K := \{z \in V_0 |\ f^{mn}(z) \in V_0 \text{ for } n \geq 0\}$$

is a horseshoe of f. Clearly, we have $o^j_{mi} \in K$ for every $i \geq 0$ and $1 \leq j \leq k$.

For each $0 \leq q \leq m-1$, let $K_q := f^q(K)$. We know that $f^q : U^j_m \to U^j_{m-q}$ is an isomorphism, and $f^q : g^m(U) \to g^{m-q}(U)$ is an isomorphism. This implies $f^q : V_0 \to f^q(V_0)$ is a finite holomorphic covering (the image of f^q of two components of V_0 may coincide). We let ϕ_q denote this map. Then we have

$$\phi_q \circ f^m|_{V_0} = f^m|_{f^q(V_0)} \circ \phi_q$$

on $f^{-m}(V_0) \cap V_0$, which implies that $f^m : K \to K$ and $f^m : K_q \to K_q$ are holomorphically semi-conjugated by ϕ_q on the corresponding neighborhoods of K and K_q. We check that K_q satisfies the three conditions in Definition 7.1. Since ϕ_q is a covering and $f^m : K \to K$ is an open map, $f^m : K_q \to K_q$ is an open map. Since $f^m : K \to K$ is expanding and $|d\phi_q| > c$ on K for some constant $c > 0$, $f^m : K_q \to K_q$ is expanding. By Remark 7.2, conditions (i) and (ii) hold. Since $f^m : K \to K$ is topologically exact and $\phi_q : K \to K_q$ is a semi-conjugacy, $f^m : K_q \to K_q$ is topologically exact. This implies Condition (iii). Hence, $K_q = f^q(K)$ is a CER. □

The following definition of linear CER was introduced by Sullivan [Sul86].

Definition 7.5. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism over $\mathbb{C}$. Let K be a CER of f. $f(K) = K$. We call K linear if one of the following conditions holds.

(i) The function $\log |df|$ is cohomologous to a locally constant function on K (i.e., there exists a continuous function u on K such that $\log |df| - (u \circ f - u)$ is locally constant on K).
(ii) there exists an atlas $\{\phi_i\}_{1 \leq i \leq k}$ that is a family of holomorphic injections $\phi_i : V_i \to \mathbb{C}$ such that $K \subset \cup_{i=1}^k V_i$, and all the maps $\phi_i \circ \phi_j^{-1}$ and $\phi_i \circ f \circ \phi_j^{-1}$ are affine.

A proof that these two conditions are actually equivalent can be found in Przytycki-Urbanski [PU10, section 10.1].

The following Sullivan's rigidity theorem [Sul86] will be used in the proof of Theorem 1.5 and Theorem 1.8. A proof can be found in [PU10, section 10.2].

Theorem 7.6 (Sullivan). *Let (f, K_f), (g, K_g) be two CERs such that K_f is nonlinear, $f(K_f) = K_f$, $g(K_g) = K_g$. Let $h : K_f \to K_g$ be a homeomorphism such that $h \circ f = g \circ h$ on K_f. Then, the following two conditions are equivalent*

(i) *for every periodic point $x \in K_f$, we have $|df^n(x)| = |dg^n(h(x))|$, where n is the period of x;*
(ii) *there exist a neighborhood U of K_f and a neighborhood V of K_g such that h extends to a conformal map $h : U \to V$.*

Here, as in Theorem 1.8, a conformal map may change the orientation of $\mathbb{P}^1(\mathbb{C})$.

7.2. Having a linear CER implies exceptional

Now we give a proof of Theorem 1.1.

Proof of Theorem 1.1. Let K be a linear CER of f, which is not a finite set. By [PU10, Proposition 4.3.6], there exists a repelling periodic point $o \in K$ of f. Passing to an iterate of f, we may assume $f(K) = K$ and $f(o) = o$. Topological exactness of f on K implies for every $a \in K$, the preimages of $f|_K$ are dense in K. Let U be a linearization domain U of f at o. Since $K \neq \{o\}$, there exist $l \geq 1$ and a point $p_l \in K$ such that $p_l \neq o$, $f^l(p_l) = o$. Then, there exists a (unique) homoclinic orbit o_i, $i \geq 0$ such that $o_l = p_l$ and $o_i \in U$ for every $i \geq l$. Clearly, $o_i \in K$ when $i \leq l$. By the definition of CER, there exists a neighborhood V of K such that $K = \cap_{n\geq 0} f^{-n}(V)$. Shrink U if necessary. We assume $U \subset V$. Hence, for every $i \geq l$, we have $o_i \in V$. This implies for every fixed $i \geq 0$, for every $n \geq 0$ we have $f^n(o_i) \in V$. Hence, $o_i \in K$ for every $i \geq 0$.

Let $\{V_j\}_{1\leq j\leq k}$ be an affine atlas in Definition 7.5. Shrink the linearization domain U, if necessary. We may assume for every $i \geq 0$, U_i (the connected component of $f^{-i}(U)$ containing o_i) is contained in some affine chart, say $V_{j(i)}$. In particular, $U \subset V_{j(0)}$ and $U_i \subset V_{j(0)}$ for every $i \geq l$. Let $\{q_i\}$, $i \geq 0$ be the adjoint sequence of o_i, $i \geq 0$. For every large enough integer n, we have $q_n \in U_n$. For such fixed n, for every $1 \leq i \leq n$, we have $f^{n-i}(q_n) \in U_i \subset V_{j(i)}$. Let $\lambda_i \in \mathbb{C}^*$ be the derivatives of the affine map $\phi_{j(i+1)} \circ f \circ \phi^{-1}_{j(i)}$, where $0 \leq i \leq l-1$. Let $\lambda \in \mathbb{C}^*$ be the derivatives of the affine map $\phi_{j(0)} \circ f \circ \phi^{-1}_{j(0)}$. Then, we have $df(o) = \lambda$, and for every n large enough, we have

$$df^n(q_n) = \left(\prod_{i=0}^{l-1} \lambda_i\right) \lambda^{n-l}.$$

By Theorem 2.11, f is exceptional. The proof is finished. □

7.3. Marked length spectrum rigidity

We now prove Theorem 1.8 by using Theorem 1.1 and Lemma 7.4.

Proof of Theorem 1.8. It is clear that (ii) implies (i). We need to show (i) implies (ii). Assume that h preserves the marked length spectrum on Ω. If h extends to a global conformal map $\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, since $h \circ f = g \circ h$ on $\mathcal{J}(f)$, the same equality holds on $\mathbb{P}^1(\mathbb{C})$. So we may replace f by its iterate. Passing to an iterate of f, we assume f has a repelling fixed point $o \in \Omega$ and $o \notin PC(f)$. A result of Eremenko-van Strien [EVS11] says that if a non-Lattès endomorphism f has the property that all the multipliers are real for periodic points contained in a nonempty open set of $\mathcal{J}(f)$, then $\mathcal{J}(f)$ is contained in a circle. By this result, there are two cases:

(i) we can further choose o such that $df(o) \notin \mathbb{R}$;
(ii) $\mathcal{J}(f)$ is contained in a circle C.

By our choice of o, $h(o)$ is a repelling fixed point of g. Moreover, we have $h(o) \notin PC(g)$ since h preserves critical points in the Julia set. This can be proved using the total invariance of the Julia sets and the fact that critical means locally, not injective. Let o_i $i \geq 0$ be a homoclinic orbit of o. Then, $h(o_i)$, $i \geq 0$ is a homoclinic orbit of $h(o)$. Let U be a linearization domain of o such that $U \cap \mathcal{J}(f) \subset \Omega$. Let W be a connected open neighborhood of $h(o)$ such that $h(U \cap \mathcal{J}(f)) \subset W$ and $W \cap \mathcal{J}(g) \subset h(\Omega)$. By Lemma 2.6, shrink U and W, if necessary. There exists $m \geq 1$ such that m is a good return time of U (resp. W) for $o_i, i \geq 0$ (resp. $h(o_i), i \geq 0$). By Lemma 7.4, there exist two horseshoes, $f^m : K_f \to K_f$ (resp. $g^m : X_g \to X_g$) such that $o_{im} \in K_f$, $i \geq 0$ (resp. $h(o_{im}) \in X_g$, $i \geq 0$). We let $K_g := h(K_f)$. By our construction, we have $h : K_f \to K_g$ is a homeomorphism and $h \circ f^m = g^m \circ h$ on K_f. Moreover, $K_g \subset X_g$. We check that K_g is a CER of g: $g^m : K_g \to K_g$ is open and topologically exact since $f^m : K_f \to K_f$ is; $g^m : K_g \to K_g$ is expanding since K_g is contained in an expanding set X_g. Hence, K_g is a CER of g. Passing to an iterate we may assume $f(K_f) = K_f$ and $g(K_g) = K_g$. To simplify the notation, for $i \geq 0$, we let o_i be the unique point in $f^{-i}(o)$ which is contained in the previous homoclinic orbit.

Since f is not exceptional, K_f is a nonlinear CER by Theorem 1.1. Moreover, by our construction, we have $K_f \subset \Omega$. Hence, for every n-periodic point $x \in K_f$, we have $|df^n(x)| = |dg^n(h(x))|$. By Theorem 7.6, h can be extended conformally to a neighborhood V of K_f such that $V \cap \mathcal{J}(f) \subset \Omega$. We denote this extension by $\tilde{h}$. In case (ii), we can further assume that $\tilde{h}$ is in fact holomorphic. If $\tilde{h}$ is antiholomorphic on some connected component B of V, let ϕ be a nonidentity conformal map (necessarily antiholomorphic) on $\mathbb{P}^1(\mathbb{C})$ such that ϕ fixes every point in C, then on B. We may replace

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$\tilde{h}$ by $\tilde{h} \circ \phi$, which is holomorphic. We have $\tilde{h} = h$ on K_f. Since $\tilde{h} \circ f = g \circ \tilde{h}$ on K_f and K_f is a perfect set, by the conformality of $\tilde{h}$, we have $\tilde{h} \circ f = g \circ \tilde{h}$ on V.

Next, we show that $\tilde{h} = h$ on $U_0 \cap \mathcal{J}(f)$, where $U_0 \subset V$ is a linearization domain of o. Let E be the set of all f-preimages of o. For every $a \in E \cap U_0$, $f^q(a) = o$, there exists a homoclinic orbit o'_i of o such that $a = o'_q$, and $o'_i \in U_0$ for every $i \geq q$.

Choose $m' \geq q$ by similar construction as in the first paragraph. We get two CERs, $f^{m'} : K''_f \to K''_f$ (resp. $g^{m'} : K''_g \to K''_g$) such that $o_{im'} \in K''_f$ and $o'_{im'} \in K''_f$ (resp. $h(o_{im'}) \in K''_g$ and $h(o'_{im'}) \in K''_g$) for $i \geq 0$. Moreover, K''_f is a horseshoe and K''_g is contained in a horseshoe X''_g. By Lemma 7.4, $K'_f := f^{m'-q}(K''_f)$ and $f^{m'-q}(X''_f)$ are CERs. Since $K'_g := f^{m'-q}(K''_g) \subseteq f^{m'-q}(X''_f)$, $g^m : K'_g \to K'_g$ is expanding. Since $h : K'_f \to K'_g$ is a homeomorphism and $h \circ f^{m'} = g^{m'} \circ h$ on K'_f, $g^{m'} : K'_g \to K'_g$ is open and topologically exact. By Remark 7.2, K'_g is a CER. Moreover, we have $o_{q+im'} \in K'_f$ and $o'_{q+im'} \in K'_f$ (resp. $h(o_{q+im'}) \in K'_g$ and $h(o'_{q+im'}) \in K'_g$) for $i \geq 0$. Since f is not exceptional, K'_f is a nonlinear CER by Theorem 1.1. Moreover, every periodic point x of $f^{m'} : K'_f \to K'_f$ has the form $x = f^{m'-q}(y)$, where y is a periodic point x of $f^{m'} : K''_f \to K''_f$. Since $K''_f \subset \Omega$, we get that the f-orbit of x has nonempty intersection with Ω. This implies for every n-periodic point x of $f^{m'} : K'_f \to K'_f$, we have $|df^{m'n}(x)| = |dg^{m'n}(h(x))|$. By Theorem 7.6, h can be extended conformally to a neighborhood V' of K'_f. Denote this extension by $\tilde{h}'$. In case (ii), we further assume that $\tilde{h}'$ is holomorphic. We have $\tilde{h}'(o_{q+im'}) = \tilde{h}(o_{q+im'}) = h(o_{q+im'})$, $i \geq 0$. The set $\{o_{q+im'}, i \geq 0\}$ is a set with accumulation point o. We claim that $\tilde{h}' = \tilde{h}$ on V_0, where V_0 is the connected component of $V \cap V'$ containing o. In case (i), since $df(o) \notin \mathbb{R}$, $\tilde{h}'$ and $\tilde{h}$ are both holomorphic or both antiholomorphic on V_0, hence $\tilde{h}' = \tilde{h}$ on V_0. In case (ii), by our choices $\tilde{h}'$ and $\tilde{h}$ are both holomorphic. Hence $\tilde{h}' = \tilde{h}$ on V_0.

There exists $b \in V_0 \cap K'_f$ such that $f^{q+nm'}(b) = a$ for some $n \geq 0$ and $\{b, f(b), \cdots, f^{q+nm'}(b)\} \subset U_0$. We also have $\tilde{h}(b) = \tilde{h}'(b) = h(b)$. Since $\tilde{h} \circ f = g \circ \tilde{h}$ on U_0, we have

$$\tilde{h}(a) = \tilde{h}(f^q(b)) = g^q(\tilde{h}(b)) = g^q(h(b)) = h(f^q(b)) = h(a).$$

This implies $\tilde{h} = h$ on $E \cap U_0$. Since E is dense in $\mathcal{J}(f)$, we get that $\tilde{h} = h$ on $U_0 \cap \mathcal{J}(f)$.

In summary, we have shown that the homeomorphism $h : \mathcal{J}(f) \to \mathcal{J}(g)$ conjugates f to g and can be extended conformally to a disk intersecting $\mathcal{J}(f)$. By a lemma due to Przytycki-Urbanski [PU99, Proposition 5.4, Lemma 5.5], h extends to a conformal map $h : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ such that $h \circ f = g \circ h$ on $\mathbb{P}^1(\mathbb{C})$. □

7.4. *Marked multiplier spectrum rigidity*

Combining Theorem 1.8 and Eremenko-van Strien's theorem [EVS11], we now prove Theorem 1.7.

Proof of Theorem 1.7. It is clear that (ii) implies (i). We need to show that (i) implies (ii). Assume h preserves the marked multiplier spectrum on Ω. By Theorem 1.8, h can be extended to a conformal map on $\mathbb{P}^1(\mathbb{C})$. If h is holomorphic, then we are done. If h is antiholomorphic, then the multipliers of all periodic points in Ω are real. By the main theorem in [EVS11], $\mathcal{J}(f)$ is contained in a circle C. Let ϕ be a nonidentity conformal map on $\mathbb{P}^1(\mathbb{C})$ such that ϕ fixes every point in C. Let $\tilde{h} := h \circ \phi$. Then $\tilde{h} \in \mathrm{PGL}_2(\mathbb{C})$, and we have $\tilde{h} \circ f = g \circ \tilde{h}$ on $\mathbb{P}^1(\mathbb{C})$. This finishes the proof. □

7.5. *Another proof of McMullen's theorem*

Now we can give another proof of Theorem 1.2 using λ-Lemma and Theorem 1.7.

Proof of Theorem 1.2. By using λ-Lemma [McM16, Theorem 4.1], it is well known that two endomorphisms in a stable family are quasiconformally conjugate on thier Julia sets. Assume by contradiction

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the conclusion is not true. Since exceptional endomorphisms that are not flexible Lattès are isolated in the moduli space $\mathcal{M}_d$, there is at least one f in the familly that is not exceptional. Let g be another endomorphism in the family. Let $h : \mathcal{J}(f) \to \mathcal{J}(g)$ be the quasicoformal conjugacy. Since multiplier spectrum is preserved in this family and the conjugacy h moves continuously in the family, for every n-periodic point x of f, we have $df^n(x) = dg^n(h(x))$. By Theorem 1.7, h extends to an automorphism on $\mathbb{P}^1(\mathbb{C})$. This contradicts the assumption that the family is non-isotrivial. □

7.6. Milnor's conjecture on Lyapunov exponent

We now prove Theorem 1.14 using Theorem 1.1.

Proof of Theorem 1.14. Let S be the finite exceptional set of periodic points in Theorem 1.14. Passing to an iterate of f, there exists a repelling fixed point o of f such that $o \notin S$. Choose a linearization domain U of o such that $U \cap S = \emptyset$. By the discussion in Lemma 7.4, there exists a horseshoe $K \subset U$. Passing to an iterate of f, we assume that $f(K) = K$. For every n-periodic point $x \in K$, we have $|df^n(x)| = b^n$ for some $b > 0$. Consider the function $\phi := \log|df|$. We have shown that, for every n-periodic point $x \in K$, $\sum_{i=0}^{n-1} \phi(f^i(x)) = n \log b$. Recall the following classical Livsic Theorem [Liv72].

Lemma 7.7. *Let K be a CER of f, $f(K) = K$. Let ϕ be a Hölder continuous function on K. Assume there exists a constant C such that for every n-periodic point $x \in K$ of f, we have*

$$\sum_{i=0}^{n-1} \phi(f^i(x)) = nC.$$

Then, there exists a continuous function u on K such that $\phi - C = u \circ f - u$.

Applying the above theorem to $\phi := \log|df|$, we get that ϕ is cohomologous to a constant function on K in the sense of Definition 7.5. Hence, K is a linear CER, which is not a finite set. By Theorem 1.1, f is exceptional. The proof is finished. □

Next, we prove Corollary 1.16. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism over $\mathbb{C}$ of degree at least 2. By Gelfert-Przytycki-Rams [GPR10], there is a forward invariant finite set $\Sigma \subset \mathcal{J}(f)$ with cardinality at most 4 (possibly empty), such that for every finite set $F \subset \mathcal{J}(f) \setminus \Sigma$, we have $f^{-1}(F) \setminus C(f) \neq F$. Let $\Delta'(f)$ be the closure of the Lyapunov exponents of periodic points contained in $\mathcal{J}(f) \setminus \Sigma$. The following theorem was proved by Gelfert-Przytycki-Rams-Rivera Letelier. Be aware that the definition of 'exceptional' in [GPR10] and [GPRRL13] has a different meaning.

Theorem 7.8 [GPR10, Theorem 2], [GPRRL13, Theorem 1, Proposition 10]. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism over $\mathbb{C}$ of degree at least 2. Then, $\Delta'(f)$ is a closed interval (possibly a singleton).*

Proof of Corollary 1.16. If $\Delta'(f)$ is not a singleton, then we are done by Theorem 7.8. If $\Delta'(f)$ is a singleton, then by Theorem 1.14, f is exceptional, contradicting our assumption. This finishes the proof. □

7.7. A simple proof of Zdunik's theorem

Next, we give a simple proof of Theorem 1.11, using Theorem 1.1.

Proof of Theorem 1.11. It is easy to observe that if f is exceptional, then μ is absolutely continuous with respect to Λ_α. We only need to show the converse is true.

Let $\phi := \alpha \log|df|$. Following Zdunik [Zdu90], we say ϕ is cohomologous to $\log d$ if there exists a function $u \in L^2(\mathcal{J}(f), \mu)$ such that $\phi - \log d = u \circ f - u$ holds for almost every point, where $\mathcal{J}(f)$ is the Julia set. By a result of Przytycki-Urbanski-Zdunik [PUZ89, Theorem 6], ϕ is not cohomologous to $\log d$, implying μ is singular with respect to Λ_α. So we only need to show that ϕ is cohomologous to $\log d$ implying f is exceptional. Now we assume $\phi - \log d = u \circ f - u$ for some $u \in L^2(\mathcal{J}(f), \mu)$. By a lemma due to Zdunik [Zdu90, Lemma 2], for every $p \notin PC(f)$, there exists a neighborhood U of p such that u equals to a continuous function almost everywhere. We observe that if $\phi_f := \alpha \log |df|$ satisfy $\phi_f - \log d = u \circ f - u$, then $\phi_{f^n} := \alpha \log |df^n|$ satisfies

$$\phi_{f^n} - n \log d = u \circ f^n - u. \tag{7.1}$$

Passing to an iterate of f, there exists a repelling fixed point $o \notin PC(f)$. Let U be a linearization domain of o such that u is continous on U. Let K be a horseshoe of f contained in U. Passing to an iterate of f, we may assume $f(K) = K$. Since u is continuous on K, by (7.1), the function $\log |df|$ is cohomologous to a constant on K in the sense of Definition 7.5. This implies K is a linear CER. Since K is not a finite set, by Theorem 1.1, f is exceptional. The proof is finished. □

8. Length spectrum as moduli

For $N \geq 1$, the symmetric group S_N acts on $\mathbb{C}^N$ (resp. $\mathbb{R}^N$) by permuting the coordinates. Using symmetric polynomials, one can show that $\mathbb{C}^N/S_N \simeq \mathbb{C}^N$. For every element $(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ (resp. $\mathbb{R}^N$), we denote by $\{\lambda_1, \ldots, \lambda_N\}$ its image in $\mathbb{C}^N/S_N$ (resp. $\mathbb{R}^N/S_N$). We may view the elements in $\mathbb{C}^N/S_N$ as multisets. [4]

For $d \geq 2$, let $f_{\mathrm{Rat}_d} : \mathrm{Rat}_d \times \mathbb{P}^1$ be the endomorphism sending (t, z) to $(t, f_t(z))$ where f_t is the endomorphism associated to $t \in \mathrm{Rat}_d$. For $t \in \mathrm{Rat}_d$, f_t^n has $N_n := d^n + 1$ fixed points counted with multiplicities. Let $\lambda_1, \ldots, \lambda_{d^n+1}$ be the multipliers of such fixed points. Define $s_n(t) = s_n(f_t) := \{\lambda_1, \ldots, \lambda_{d^n+1}\} \in \mathbb{A}^{N_n}/S_{N_n}$ the *n-th multiplier spectrum of* f_t. Similarly, define $L_n(t) = L_n(f_t) := \{|\lambda_1|, \ldots, |\lambda_{d^n+1}|\} \in \mathbb{R}^{N_n}/S_{N_n}$ the *n-th length spectrum of* f_t. Both $s_n(f_t)$ and $L_n(f_t)$ only depend on the conjugacy class of f_t.

For every $n \geq 1$, let $\mathrm{Per}_n(f_{\mathrm{Rat}_d})$ be the closed subvariety of $\mathrm{Rat}_d \times \mathbb{P}^1$ of the n-periodic points of f_{Rat_d}. Let $\phi_n : \mathrm{Per}_n(f_{\mathrm{Rat}_d}) \to \mathrm{Rat}_d$ be the first projection. It is a finite map of degree $d^n + 1$. Let $\lambda_n : \mathrm{Per}_n(f_{\mathrm{Rat}_d}) \to \mathbb{A}^1$ be the algebraic morphism $(f_t, x) \mapsto df_t^n(x) \in \mathbb{A}^1$. Let $|\lambda_n| : \mathrm{Per}_n(f_{\mathrm{Rat}_d(\mathbb{C})})(\mathbb{C}) \to [0, +\infty)$ be the composition of λ_n to the norm map $z \in \mathbb{C} \mapsto |z| \in [0, +\infty)$. A fixed point x of f_t^n has multiplicity > 1 if and only if $df_t^n(x) = 1$. This shows that the map ϕ_n is étale at every point $x \in \mathrm{Per}_n(f_{\mathrm{Rat}_d}) \setminus \lambda_n^{-1}(1)$.

We may view $\mathrm{Per}_n(f_{\mathrm{Rat}_d})$ as the moduli space of endomorphisms of degree d with a marked n-periodic point. So we may also denote it by $\mathrm{Rat}_d[n]$ or $\mathrm{Rat}_d^1[n]$. More generally, for every $s = 1, \ldots, d^n + 1$, one may construct the moduli space $\mathrm{Rat}_d^s[n]$ of endomorphisms of degree d with s marked n-periodic point as follows: For $s = 2, \ldots, d^n + 1$, consider the fiber product $(\mathrm{Rat}_d[n])^s_{/\mathrm{Rat}_d}$ of s copies of $\mathrm{Rat}_d[n]$ over Rat_d. For $i \neq j \in \{1, \ldots, d^n + 1\}$, let $\pi_{i,j} : (\mathrm{Rat}_d[n])^s_{/\mathrm{Rat}_d} \to (\mathrm{Rat}_d[n])^2_{/\mathrm{Rat}_d}$ be the projection to the i, j coordinates. The diagonal $\Delta \subseteq (\mathrm{Rat}_d[n])^2_{/\mathrm{Rat}_d}$ is an irreducible component of $(\mathrm{Rat}_d[n])^2_{/\mathrm{Rat}_d}$. One may define $\mathrm{Rat}_d^s[n]$ to be the Zariski closure of

$$(\mathrm{Rat}_d[n])^s_{/\mathrm{Rat}_d} \setminus (\cup_{i \neq j \in \{1,\ldots,d^n+1\}} \pi_{i,j}^{-1}(\Delta))$$

in $(\mathrm{Rat}_d[n])^s_{/\mathrm{Rat}_d}$. Denote by $\phi_n^s : \mathrm{Rat}_d^s[n] \to \mathrm{Rat}_d$ the morphism induced by ϕ_n. Let $\lambda_n^s : \mathrm{Rat}_d^s[n] \to \mathbb{A}^s$ be the morphism defined by $(t, x_1, \ldots, x_s) \mapsto (df^n(x_1), \ldots, df^n(x_s))$ and $|\lambda_n^s| : \mathrm{Rat}_d^s[n](\mathbb{C}) \to \mathbb{R}^s$ the map defined by $(t, x_1, \ldots, x_s) \mapsto (|df^n(x_1)|, \ldots, |df^n(x_s)|)$. Since ϕ_n is étale at every point $x \in \mathrm{Per}_n(f_{\mathrm{Rat}_d}) \setminus \lambda_n^{-1}(1)$, ϕ_n^s is étale at every point $x \in (\lambda_n^s)^{-1}((\mathbb{A}^1 \setminus \{1\})^s)$.

To prove Theorem 1.5, we need to study the subsets taking the form $\Lambda_n(a) := L_n^{-1}(a)$ where $a \in \mathbb{R}^{N_n}/S_{N_n}$. Since L_n is not holomorphic (hence, not algebraic), in general, the above set is not algebraic. The problem is that one projects a real algebraic set under a finite map, but it may not be

[4] A multiset is a set allowing multiple instances for each of its elements. The number of the instances of an element is called the multiplicity. For example, $\{a, a, b, c, c, c\}$ is a multiset of cardinality 6, and the multiplicities for a, b, c are 2,1,3.

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real algebraic. To get some algebricity of $\Lambda_n(a)$, one can view $\mathrm{Rat}_d(\mathbb{C})$ as a real algebraic variety by splitting a complex variable z into two real varieties x, y via $z = x + iy$. A more theoretic way to do this is using the notion of Weil restriction. See Section 8.1.1 for a brief introduction. However, even when we view $\mathrm{Rat}_d(\mathbb{C})$ as a real algebraic variety, $\Lambda_n(a)$ is not real algebraic in general (c.f. Theorem 8.10). Here, *real algebraic* means Zariski closed when viewing $\mathrm{Rat}_d(\mathbb{C})$ as a real algebraic variety. See Section 8.1.1 for the precise definition. This is one of the main difficulties in the proof of Theorem 1.5. To solve this problem, we introduce a class of closed subsets of $\mathrm{Rat}_d(\mathbb{C})$ that are images of algebraic subsets under étale morphisms. We will study such subsets in Section 8.2.

8.1. *An example of a length level set which is not real algebraic*

The main result of this section is Theorem 8.10, in which we give an example to show that the subsets $\Lambda_n(a)$ may not be real algebraic in $\mathrm{Rat}_d(\mathbb{C})$[5].

Except Definition 8.1, in which we give a precise definition of the notion *real algebraic* using Weil restriction, this section will not be used in the rest of the paper.

8.1.1. Weil restriction

We briefly recall the notion of Weil restriction. See [Poo17, Section 4.6] and [BLR90, Section 7.6] for more information.

Denote by $Var_{/\mathbb{C}}$ (resp. $Var_{/\mathbb{R}}$) the category of varieties over $\mathbb{C}$ (resp. $\mathbb{R}$). For every variety X over $\mathbb{C}$, there is a unique variety $R(X)$ over $\mathbb{R}$ representing the functor $Var_{/\mathbb{R}} \to Sets$, sending $V \in Var_{/\mathbb{R}}$ to $\mathrm{Hom}(V \otimes_{\mathbb{R}} \mathbb{C}, X)$. It is called the *Weil restriction of* X. The functor $X \mapsto R(X)$ is called the Weil restriction. One has the canonical morphism $\tau_X : X(\mathbb{C}) \to R(X)(\mathbb{R})$, which is a real analytic diffeomorphism. One may view $X(\mathbb{C})$ as a real algebraic variety via τ_X.

Definition 8.1. The *real Zariski topology* on $X(\mathbb{C})$ is the restriction of the Zariski topology on $R(X)$ via τ_X. A subset Y of $X(\mathbb{C})$ is *real algebraic* if it is closed in the real Zariski topology.

By (iii) of Proposition 8.3 below, the real Zariski topology is stronger than the Zariski topology on $X(\mathbb{C})$.

Roughly speaking, the Weil restriction is just constructed by splitting a complex variable z into two real variables x, y via $z = x + iy$. For the convenience of the reader, in the following example, we show the concrete construction of $R(X)$ when X is affine.

Example 8.2. First assume that $X = \mathbb{A}^N_{\mathbb{C}}$. Then $R(X) = \mathbb{A}^{2N}_{\mathbb{R}}$. The map

$$\tau_X : \mathbb{A}^N_{\mathbb{C}}(\mathbb{C}) = \mathbb{C}^N \to \mathbb{A}^{2N}_{\mathbb{R}}(\mathbb{R}) = \mathbb{R}^{2N}$$

sends $(z_1, \ldots, z_N)$ to $(x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ where $z_j = x_j + iy_j$.

Consider the algebra $\mathbb{B} := \mathbb{C}[I]/(I^2+1) \simeq \mathbb{C} \oplus I\mathbb{C}$. Every $f \in \mathbb{C}[z_1, \ldots, z_N]$ defines an element

$$F := f(x_1 + Iy_1, \ldots, x_N + Iy_N) \in \mathbb{B}[x_1, y_1, \ldots, x_N, y_N].$$

Since

$$\mathbb{B}[x_1, y_1, \ldots, x_N, y_N] = \mathbb{C}[x_1, y_1, \ldots, x_N, y_N] \oplus I\mathbb{C}[x_1, y_1, \ldots, x_N, y_N],$$

F can be uniquely decomposed to $F = r(f) + Ii(f)$ where $r(f), i(f) \in \mathbb{C}[x_1, y_1, \ldots, x_N, y_N]$.

If X is the closed subvariety of $\mathbb{A}^N_{\mathbb{C}} = \mathrm{Spec}\,\mathbb{C}[z_1, \ldots, z_M]$ defined by the ideal $(f_1, \ldots, f_s)$, then $R(X)$ is the closed subvariety of $R(\mathbb{A}^N_{\mathbb{C}}) = \mathbb{A}^{2N}_{\mathbb{R}} = \mathrm{Spec}\,\mathbb{R}[x_1, y_1, \ldots, x_N, y_N]$ defined by the ideal generated by $r(f_1), i(f_1), \ldots, r(f_s), i(f_s)$.

We list some basic properties of the Weil restriction without proof.

[5]In our example, we will take $d = 2$ and $n = 1$.

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Proposition 8.3. *Let $X, Y \in Var_{/\mathbb{C}}$. Then, we have the following properties:*

(i) *if X is irreducible, then $R(X)$ is irreducible;*
(ii) $\dim R(X) = 2\dim X$;
(iii) *if $f : Y \to X$ is a closed (resp. open) immersion, then the induced morphism $R(f) : R(Y) \to R(X)$ is a closed (resp. open) immersion.*

Then, we get the following easy consequence.

Lemma 8.4. *Let $Y \in Var_{\mathbb{C}}$ and X be a closed subset Y. Then, $R(X)$ is the Zariski closure of $X(\mathbb{C}) = R(X)(\mathbb{R})$ in $R(Y)$.*

Proof. We may assume that X and Y are irreducible. It is clear that $R(X)(\mathbb{R}) \subseteq R(X)$. So $\overline{R(X)(\mathbb{R})}^{\text{zar}} \subseteq R(X)$. Since

$$\dim_{\mathbb{R}} \overline{R(X)(\mathbb{R})}^{\text{zar}} \geq \dim_{\mathbb{R}} R(X)(\mathbb{R}) = 2\dim X = \dim R(X)$$

and $R(X)$ is irreducible, we get $\overline{R(X)(\mathbb{R})}^{\text{zar}} = R(X)$. □

We denote by $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{R})$ the complex conjugation $z \mapsto \bar{z}$. For every complex variety X, one denotes by X^σ the base change of X by the field extension $\sigma : \mathbb{C} \to \mathbb{C}$. This induces a morphism of schemes (over $\mathbb{Z}$) $\sigma : X^\sigma \to X$. It is not a morphism of schemes over $\mathbb{C}$. It is clear that $(X^\sigma)^\sigma = X$.

Example 8.5. If X is the subvariety of $\mathbb{A}^N_{\mathbb{C}} = \text{Spec}\,\mathbb{C}[z_1, \ldots, z_N]$ defined by the equations $\sum_I a_{i,I} z^I = 0, i = 1, \ldots, s$, then X^σ is the subvariety of $\mathbb{A}^N_{\mathbb{C}}$ defined by $\sum_I \overline{a_{i,I}} z^I = 0, i = 1, \ldots, s$. The map $\sigma : X = (X^\sigma)^\sigma \to X^\sigma$ sends a point $(z_1, \ldots, z_N) \in X(\mathbb{C})$ to $(\overline{z_1}, \ldots, \overline{z_N}) \in X^\sigma(\mathbb{C})$.

The following result due to Weil is useful for computing the Weil restriction.

Proposition 8.6 [Poo17, Exercise 4.7]. *We have a canonical isomorphism*

$$R(X) \otimes_{\mathbb{R}} \mathbb{C} \simeq X \times X^\sigma.$$

Under this isomorphism,

$$R(X)(\mathbb{R}) = \{(z_1, z_2) \in X(\mathbb{C}) \times X^\sigma(\mathbb{C}) |\ z_2 = \sigma(z_1)\}$$

and τ_X sends $z \in X(\mathbb{C})$ to $(z, \sigma(z)) \in R(X)(\mathbb{R})$.

8.1.2. The norm map

For $N \geq 1$, let $\nu_N : \mathbb{C}^N/S_N \to \mathbb{R}^N/S_N$ be the real analytic map sending $\{z_1, \ldots, z_N\}$ to $\{|z_1|^2, \ldots, |z_N|^2\}$. We view $\mathbb{C}^N/S_N$ as a real algebraic variety via the identification

$$\mathbb{C}^N/S_N = (\mathbb{A}^N_{\mathbb{C}}/S_N)(\mathbb{C}) = R(\mathbb{A}^N_{\mathbb{C}}/S_N)(\mathbb{R}) \subseteq R(\mathbb{A}^N_{\mathbb{C}}/S_N)(\mathbb{C}).$$

The following result is the aim of this section. We postpone its proof to the end of this section.

Proposition 8.7. *For $a := \{a_1, \ldots, a_N\} \in \mathbb{R}^N_{>0}/S_N$, $\nu_N^{-1}(a)$ is real Zariski closed if and only if $N = 1$ or $N = 2$ and $a_1 \neq a_2$.*

Set $X := R(\mathbb{A}^N_{\mathbb{C}}/S_N) \otimes_{\mathbb{R}} \mathbb{C} = (\mathbb{A}^N_{\mathbb{C}}/S_N) \times (\mathbb{A}^N_{\mathbb{C}}/S_N)$. (Since $\mathbb{A}^N_{\mathbb{C}}/S_N$ is defined over $\mathbb{R}$, we have $\mathbb{A}^N_{\mathbb{C}}/S_N = (\mathbb{A}^N_{\mathbb{C}}/S_N)^\sigma$.) Consider the quotient morphisms $q_1 : \mathbb{A}^N_{\mathbb{C}} \twoheadrightarrow \mathbb{A}^N_{\mathbb{C}}/S_N$ defined by

$$(z_1, \ldots, z_N) \mapsto \{z_1, \ldots, z_N\}$$

and $q_2 : \mathbb{A}^N_{\mathbb{C}} \times \mathbb{A}^N_{\mathbb{C}} \twoheadrightarrow X$ defined by

$$(u_1, \ldots, u_N; v_1, \ldots, v_N) \mapsto (\{u_1, \ldots, u_N\}, \{v_1, \ldots, v_N\}).$$

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Consider the morphism $\mu_N : \mathbb{A}^N_{\mathbb{C}} \times \mathbb{A}^N_{\mathbb{C}} \to \mathbb{A}^N_{\mathbb{C}}$ defined by

$$(u_1, \ldots, u_N; v_1, \ldots, v_N) \mapsto (u_1 v_1, \ldots, u_N v_N).$$

Let Γ_{μ_N} be the graph of μ_N in $(\mathbb{A}^N_{\mathbb{C}} \times \mathbb{A}^N_{\mathbb{C}}) \times \mathbb{A}^N_{\mathbb{C}}$. Set $\Gamma_N = (q_2 \times q_1)(\Gamma_{\mu_N}) \subseteq X \times (\mathbb{A}^N_{\mathbb{C}}/S_N)$. Since $q_2 \times q_1$ is finite, Γ_N is an irreducible closed subvariety of $X \times (\mathbb{A}^N_{\mathbb{C}}/S_N)$. We view it as a correspondence between X and $\mathbb{A}^N_{\mathbb{C}}/S_N$.

Let $\pi_1 : X \times (\mathbb{A}^N_{\mathbb{C}}/S_N) \to X$ and $\pi_2 : X \times (\mathbb{A}^N_{\mathbb{C}}/S_N) \to (\mathbb{A}^N_{\mathbb{C}}/S_N)$ be the first and the second projection. Then, $\pi_1|_{\Gamma_N}$ is a finite morphism of degree $N!$. For every $x \in X$, the image of x under Γ_N is $\Gamma_N(x) := \pi_2(\Gamma_N \cap \pi_1^{-1}(x))$. For a general $x \in X(\mathbb{C})$, $\Gamma_N(x)$ has $N!$ points. Similarly, for every $y \in \mathbb{A}^N_{\mathbb{C}}/S_N$, the preimage of y under Γ_N is $\Gamma_N^{-1}(y) := \pi_1(\Gamma_N \cap \pi_2^{-1}(y))$.

Lemma 8.8. *For every $a = \{a_1, \ldots, a_N\} \in (\mathbb{A}^N_{\mathbb{C}}/S_N)(\mathbb{C})$ with $a_i \neq 0, i = 1, \ldots, N$, $\Gamma_N^{-1}(a)$ is irreducible and of dimension N.*

Proof. Consider the actions of $g \in S_N$ on $\mathbb{A}^N_{\mathbb{C}} \times \mathbb{A}^N_{\mathbb{C}}$ by

$$g.(u_1, \ldots, u_N; v_1, \ldots, v_N) = (u_{g(1)}, \ldots, u_{g(N)}; v_{g(1)}, \ldots, v_{g(N)})$$

and on $\mathbb{A}^N_{\mathbb{C}}$ by $g.(z_1, \ldots, z_N) = (z_{g(1)}, \ldots, z_{g(N)})$. Then, we have

$$q_1(g.x) = q_1(x), q_2(g.x) = q_2(x).$$

Since

$$\Gamma_N^{-1}(a) = q_2(\mu_N^{-1}(q_1^{-1}(\{a_1, \ldots, a_N\})))$$

and

$$q_1^{-1}(\{a_1, \ldots, a_N\}) = \{g.(a_1, \ldots, a_N)|\ g \in S_N\},$$

we get $\Gamma_N^{-1}(a) = q_2(\mu_N^{-1}((a_1, \ldots, a_N)))$. Since $\mu_N^{-1}((a_1, \ldots, a_N))$ is defined by $u_i v_i = a_i, i = 1, \ldots, N$, it is isomorphic to $(\mathbb{A}^1 \setminus \{0\})^N$, which is irreducible. Since q_2 is finite, $\Gamma_N^{-1}(a)$ is irreducible of dimension N. □

For $a = \{a_1, \ldots, a_N\} \in \mathbb{R}^N_{>0}/S_N \subseteq (\mathbb{A}^N_{\mathbb{C}}/S_N)(\mathbb{R})$, we have

$$\Gamma_N^{-1}(a)(\mathbb{R}) = \Gamma_N^{-1}(a) \cap X(\mathbb{R}) = \cup_{g \in S_N} V_{N,g}(a)$$

where

$$V_{N,g}(a) = q_2(\{(u_1, \ldots, u_N; \overline{u_1}, \ldots, \overline{u_N}) \in \mathbb{C}^{2N}|\ u_i \overline{u_{g(i)}} = a_i, 1 \le i \le N\})$$

$$= \{(\{u_1, \ldots, u_N\}, \{\overline{u_1}, \ldots, \overline{u_N}\}) \in R(X)(\mathbb{R})|\ u_i \overline{u_{g(i)}} = a_i, 1 \le i \le N\}$$

We note that, if $g_1, g_2 \in S_N$ are conjugate, then $V_{N,g_1}(a) = V_{N,g_2}(a)$. For every $g \in S_N$, it can be uniquely written as a product of disjoint cycles (i.e., there is a partition $\{1, \ldots, N\} = \sqcup_{i=1}^s I_i$ such that $g = \sigma_1 \cdots \sigma_s$ where σ_i acts trivially outside I_i and transitively on I_i). Set

$$Z_{N,g}(a) := \{(u_1, \ldots, u_N; \overline{u_1}, \ldots, \overline{u_N}) \in \mathbb{C}^{2N}|\ u_i \overline{u_{g(i)}} = a_i, i = 1, \ldots, N\}.$$

Then, $V_{N,g}(a) = q_2(Z_{N,g}(a))$.

For $i = 1, \ldots, s$, set $m_i := \# I_i$ and write $I_i = \{j_1, \ldots, j_{m_i}\}$ with $\sigma(j_n) = j_{n+1}$. Here, the index n is viewed in $\mathbb{Z}/m_i\mathbb{Z}$. We define $Z_i, i = 1, \ldots, s$ as follows:

(E_0) : If m_i is even and $\sum_{n=1}^{m_i}(-1)^n \log a_{j_n} \neq 0$, $Z_i := \emptyset$.

(E_1) : If m_i is even and $\sum_{n=1}^{m_i}(-1)^n \log a_{j_n} = 0$, then Z_i is the set of points taking forms $(U, \overline{U}) \in \mathbb{C}^{I_i} \times \mathbb{C}^{I_i}$ where

$$U = (r_{j_1}e^{i\theta}, a_1 r_{j_1}^{-1}e^{i\theta}, a_2 a_1^{-1} r_{j_1}e^{i\theta}, \dots, a_{j_{m_i-1}}a_{j_{m_i-2}}^{-1}\dots a_1 r_{j_i}^{-1}e^{i\theta})$$

for some $r_{j_1} \in \mathbb{R}_{>0}$ and $\theta \in \mathbb{R}$. Hence, $Z_i \simeq \mathbb{R}_{>0} \times (\mathbb{R}/\mathbb{Z})$.

(O) : If m_i is odd, then Z_i is the set of points taking forms $(U, \overline{U}) \in \mathbb{C}^{I_i} \times \mathbb{C}^{I_i}$ where

$$U = (r_{j_1}e^{i\theta}, \dots, r_{j_{m_i}}e^{i\theta}), r_{j_n} = \left(\prod_{l=0}^{m_i-1} a_{j_{n+l}}^{(-1)^l}\right)^{1/2}$$

for some $\theta \in \mathbb{R}$. Hence, $Z_i \simeq \mathbb{R}/\mathbb{Z}$.

It is easy to show that

$$Z_{N,g}(a) = \prod_{i=1}^{s} Z_i.$$

Let $e_0(g), e_1(g)$ and $o(g)$ be the numbers of the index i that falls into the cases $(E_0), (E_1)$ and (O) respectively. Then, $Z_{N,g}(a) = \emptyset$ if $e_0(g) > 0$. Otherwise,

$$Z_{N,g}(a) \simeq \mathbb{R}_{>0}^{e_1(g)} \times (\mathbb{R}/\mathbb{Z})^{e_1(g)+o(g)}.$$

Lemma 8.9. *We have $V_{N,\mathrm{id}}(a) = \nu_N^{-1}(a)$, and it is Zariski dense in $\Gamma_N^{-1}(a)$.*

Proof. It is clear that $V_{N,\mathrm{id}}(a) = \nu_N^{-1}(a)$. By Lemma 8.8, $\Gamma_N^{-1}(a)$ is irreducible and of dimension N. Since $Z_{N,\mathrm{id}}(a) \simeq (\mathbb{R}/\mathbb{Z})^N$, $V_{N,\mathrm{id}}(a) = q_2(Z_{N,\mathrm{id}}(a))$ is of dimension N. Then, it is Zariski dense in $\Gamma_N^{-1}(a)$. □

Proof of Proposition 8.7. By Lemma 8.9, $\nu_N^{-1}(a) = V_{N,\mathrm{id}}(a)$ is Zariski closed if and only if $V_{N,g}(a) \subseteq V_{N,\mathrm{id}}(a)$ for every $g \in S_N$.

The case $N = 1$ is trivial. If $N = 2$ and $a_1 \neq a_2$, then $e_0(g) > 0$ for $g \in S_2 \setminus \{\mathrm{id}\}$. Hence, $V_{N,\mathrm{id}}(a)$ is Zariski closed. If there is $i \neq j$ with $a_i = a_j$, let $g := (i, j) \in S^N$. Then

$$Z_{N,g}(a) \simeq \mathbb{R}_{>0} \times (\mathbb{R}/\mathbb{Z})^{N-1}$$

which is not compact. Since q_2 is finite, $q_2(Z_{N,g}(a))$ is closed but not compact. Hence, it is not contained in $V_{N,\mathrm{id}}(a)$.

Now we may assume that $N \geq 3$ and $a_i \neq a_j$ for every $i \neq j$. We may assume that $a_1 > a_2 > a_3$ and $a_1 = \max\{a_i, i = 1, \dots, N\}$. Then, for every $(\{u_1, \dots, u_N\}, \{\overline{u_1}, \dots, \overline{u_N}\}) \in V_{N,\mathrm{id}}(a)$, we have

$$\max\{|u_i|, i = 1, \dots, N\} = a_1^{1/2}.$$

Pick $g = (1, 2, 3) \in S_N$. Then $Z_{N,\mathrm{id}}(a) \neq \emptyset$ and for every point $(u_1, \dots, u_N; \overline{u_1}, \dots, \overline{u_N}) \in Z_{N,g}(a)$, we have

$$\max\{|u_i|, i = 1, \dots, N\} \geq |u_2| = (a_1 a_2 a_3^{-1})^{1/2} > a_1^{1/2}.$$

Since $V_{N,\mathrm{id}}(a) = q_2(Z_{N,\mathrm{id}}(a))$, $V_{N,g}(a) \cap V_{N,\mathrm{id}}(a) = \emptyset$. Hence, $V_{N,\mathrm{id}}(a)$ is not Zariski closed. □

8.1.3. The example

In this section, we focus on the first length spectrum map $L_1 : \mathrm{Rat}_2(\mathbb{C}) \to \mathbb{R}^3_{>0}/S_3$. We view $\mathrm{Rat}_2(\mathbb{C})$ as a real algebraic variety via identifying $\mathrm{Rat}_2(\mathbb{C})$ with $R(\mathrm{Rat}_2)(\mathbb{R})$

Theorem 8.10. *For $a \in (1, \sqrt{2})$, $L_1^{-1}(\{a, a, a\})$ is not real algebraic in* $\mathrm{Rat}_2(\mathbb{C})$.

Proof. We follow the notations in Section 8.1.2.

Recall the first multiplier spectrum map $s_1 : \mathrm{Rat}_2(\mathbb{C}) \to (\mathbb{A}^3/S_3)(\mathbb{C})$. Then, $L_1^{-1}(\{a, a, a\}) = s_1^{-1}(v_3^{-1}(\{a^2, a^2, a^2\}))$. Set $b := \{a^2, a^2, a^2\}$. Since s_1 factors through the moduli space $\mathcal{M}_2(\mathbb{C})$, there is a morphism $[s_1] : \mathcal{M}_2(\mathbb{C}) \to (\mathbb{A}^3/S_3)(\mathbb{C})$ such that $[s_1] \circ \Psi_2 = s_1$. It was proved by Milnor[Mil93] that $[s_1]$ is an isomorphism to its image M (see also [Sil12, Theorem 2.4.5]). Moreover, by [Sil12, Theorem 2.4.5 and Lemma 2.4.6], $M = q_1(Y_0)$ and $R(M) = q_2(R(Y_0))$, where

$$Y_0 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 |\ z_1 z_2 z_3 = z_1 + z_2 + z_3 - 2, z_1 z_2 \neq 1\} \cup \{(1, 1, z_3)\}.$$

Set $Y := \{(z_1, z_2, z_3) \in \mathbb{C}^3 |\ z_1 z_2 z_3 = z_1 + z_2 + z_3 - 2\}$, which is the Zariski closure of Y_0. The Zariski closure of $R(M)$ in $R(\mathbb{A}^3_{\mathbb{C}}/S_3)$ is $q_2(R(Y))$.

Lemma 8.11. *The intersection $q_2(R(Y)) \cap \Gamma_3^{-1}(b)$ is irreducible of dimension* 1.

Proof. Observe that $(q_2(R(Y)) \cap \Gamma_3^{-1}(b)) \otimes_{\mathbb{R}} \mathbb{C} = q_2(Z)$ where Z is the closed subset of $R(\mathbb{A}^3_{\mathbb{C}}) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{A}^3_{\mathbb{C}} \times \mathbb{A}^3_{\mathbb{C}} = \mathrm{Spec}\, \mathbb{C}[u_1, u_2, u_3, v_1, v_2, v_3]$ defined by the following equations:

(i) $u_1 u_2 u_3 = u_1 + u_2 + u_3 - 2$;
(ii) $v_1 v_2 v_3 = v_1 + v_2 + v_3 - 2$;
(iii) $u_1 v_1 = a$;
(iv) $u_2 v_2 = a$;
(v) $u_3 v_3 = a$.

Using symmetric polynomials, one may write

$$R(\mathbb{A}^3_{\mathbb{C}}/S_3) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{A}^3_{\mathbb{C}}/S_3 \times \mathbb{A}^3_{\mathbb{C}}/S_3$$

as

$$\mathbb{A}^3_{\mathbb{C}} \times \mathbb{A}^3_{\mathbb{C}} = \mathrm{Spec}\, \mathbb{C}[x, y, z, x', y', z']$$

and in this coordinate, q_2 is given by $x \mapsto u_1 + u_2 + u_3$, $y \mapsto u_1 u_2 + u_1 u_3 + u_2 u_3$, $z \mapsto u_1 u_2 u_3$, $x' \mapsto v_1 + v_2 + v_3$, $y' \mapsto v_1 v_2 + v_1 v_3 + v_2 v_3$ and $z' \mapsto v_1 v_2 v_3$. One may compute that $q_2(Z)$ is defined by the following equations:

(i) $z \neq 0$;
(ii) $x = z + 2$;
(iii) $y = (2z + a^3)/a$;
(iv) $x' = a^3/z + 2$;
(v) $y' = a^2(z + 2)/z$;
(vi) $z' = a^3/z$.

Then, it is irreducible of dimension 1 since it is parametrized by a single variable z. □

Then, $R(M) \cap \Gamma_3^{-1}(b)$ is irreducible, and if this intersection is nonempty, it is of dimension 1. We note that

$$v_3^{-1}(b) = M \cap q_2(Z_{3,\mathrm{id}}(b)).$$

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Let $g = (1,2) \in S_3$. We have

$$M \cap (q_2(Z_{3,\mathrm{id}}(b)) \cup q_2(Z_{3,g}(b))) \subseteq (R(M) \cap \Gamma_3^{-1}(b))(\mathbb{R}).$$

Lemma 8.12. *Both* $M \cap q_2(Z_{3,\mathrm{id}}(b))$ *and* $M \cap q_2(Z_{3,g}(b))$ *are infinite and* $M \cap q_2(Z_{3,g}(b)) \not\subseteq M \cap q_2(Z_{3,\mathrm{id}}(b))$.

Proof. Since q_2 is finite, we only need to show that $Y_0 \cap Z_{3,\mathrm{id}}(b)$ and $Y \cap Z_{3,g}(b)$ are infinite and $M \cap q_2(Z_{3,g}(b)) \not\subseteq M \cap q_2(Z_{3,\mathrm{id}}(b))$.

Since $a > 1$, one may compute that $Y_0 \cap Z_{3,\mathrm{id}}(b) = Y \cap Z_{3,\mathrm{id}}(b)$ and it is the set of points $(u_1, u_2, u_3) \in \mathbb{C}^3$ satisfying the following equations:

$$u_1u_2u_3 = u_1 + u_2 + u_3 - 2 \text{ and } |u_1| = |u_2| = |u_3| = a. \tag{8.1}$$

Consider the function $F : [0,\pi]^2 \to [0,+\infty)$ given by

$$F : (\theta_1, \theta_2) \mapsto \left| \frac{a(e^{i\theta_1} + e^{i\theta_2}) - 2}{a^3 e^{i(\theta_1+\theta_2)} - a} \right|.$$

Since $a > 1$, it is well-defined and continuous. We have

$$F(0,0) = |(2a-2)/(a^3 - a)| = \frac{2}{a(a+1)} < 1$$

and

$$F(\pi,\pi) = |(-2a-2)/(a^3 - a)| = \frac{2}{a(a-1)} > 1.$$

There is $\beta \in (0, \pi/2)$ such that for every $\alpha \in [0,\beta]$, we have

$$F(0,\alpha) < 1 \text{ and } F(\pi - \alpha, \pi) > 1.$$

Hence, for every $\alpha \in [0,\beta]$, there is $\theta(\alpha) \in [0, \pi - \alpha]$ such that

$$F(\theta(\alpha), \theta(\alpha) + \alpha) = 1.$$

One may check that

$$u_1 = ae^{i\theta(\alpha)}, u_2 = ae^{i\theta(\alpha)+\alpha}, u_3 = a\frac{a(e^{i\theta(\alpha)} + e^{i\theta(\alpha)+\alpha}) - 2}{a^3 e^{i(2\theta(\alpha)+\alpha)} - a}, \alpha \in [0,\beta]$$

are infinitely many distinct solutions of (8.1). So $Y_0 \cap Z_{3,\mathrm{id}}(b)$ is infinite.

Since $a > 1$, one may compute that $Y_0 \cap Z_{3,g}(b) = Y \cap Z_{3,g}(b)$, and it is the set of points $(u_1, u_2, u_3) \in \mathbb{C}^3$ satisfying the following equations:

$$u_1u_2u_3 = u_1 + u_2 + u_3 - 2 \text{ and } u_1\overline{u_2} = |u_3|^2 = a^2. \tag{8.2}$$

Consider the function $G : \mathbb{R}_{>0} \times [0,\pi] \to [0,+\infty)$ given by

$$G : (r, \theta) \mapsto \left| \frac{a(r + 1/r)e^{i\theta} - 2}{a^3 e^{2i\theta} - a} \right|.$$

Since $a > 1$, it is well-defined and continuous. We note that $G(1,\theta) = F(\theta,\theta)$ for $\theta \in [0,\pi]$. So $G(1,0) < 1$ and $G(1,\pi) > 1$. There is $R > 1$ such that for every $r \in [1,R]$, $G(r,0) < 1$ and $G(r,\pi) > 1$. Then, for every $r \in [1,R]$, there is $\theta_r \in [0,\pi]$ such that $G(r,\theta_r) = 1$.

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One may check that

$$u_1(r) = are^{i\theta_r}, u_2(r) = ar^{-1}e^{i\theta_r}, u_3(r) = a\frac{a(r+1/r)e^{i\theta_r} - 2}{a^3e^{2i\theta_r} - a}, r \in [1, R]$$

are infinitely many distinct solutions of (8.1). So $Y_0 \cap Z_{3,g}(b)$ is infinite. Moreover, if $r > 1$, then $\max\{|u_1(r)|, |u_2(r)|, |u_3(r)|\} = ar > a$, so $\{u_1(r), u_2(r), u_3(r)\} \in (M \cap q_2(Z_{3,g}(b))) \setminus (M \cap q_2(Z_{3,\mathrm{id}}(b)))$. This concludes the proof. □

Since $M \cap q_2(Z_{3,\mathrm{id}}(b))$ is infinite and $\dim R(M) \cap \Gamma_3^{-1}(b) = 1$, the Zariski closure of $M \cap q_2(Z_{3,\mathrm{id}}(b))$ in $R(M)$ is $R(M) \cap \Gamma_3^{-1}(b)$ but $M \cap q_2(Z_{3,\mathrm{id}}(b)) \subsetneq (R(M) \cap \Gamma_3^{-1}(b))(\mathbb{R})$. So $L_1^{-1}(\{a, a, a\}) = s_1^{-1}(M \cap q_2(Z_{3,\mathrm{id}}(b)))$ is Zariski dense in $R(s_1)^{-1}(R(M) \cap \Gamma_3^{-1}(b))$, where $R(s_1) : R(\mathrm{Rat}_2) \to R(M)$ is induced by s_1. Since $M \cap q_2(Z_{3,\mathrm{id}}(b)) \subsetneq (R(M) \cap \Gamma_3^{-1}(b))(\mathbb{R})$ and M is the image of s_1, $L_1^{-1}(\{a, a, a\}) \subsetneq R(s_1)^{-1}(R(M) \cap \Gamma_3^{-1}(b))$. This concludes the proof. □

8.2. *Images of algebraic subsets under étale morphisms*

Let X be a variety over $\mathbb{R}$. A closed subset V of $X(\mathbb{R})$ is called *admissible* if there is a morphism $f : Y \to X$ of real algebraic varieties and a Zariski closed subset $V' \subseteq Y$ such that $V = f(V'(\mathbb{R}))$ and f is étale at every point in $V'(\mathbb{R})$.

Every algebraic subset of $X(\mathbb{R})$ is admissible.

Remark 8.13. Denote by J the non-étale locus for f in V. We have $J \cap V(\mathbb{R}) = \emptyset$. Since we may replace V by $V \setminus J$ in the above definition, we may further assume that f is étale.

Remark 8.14. Let Y be a Zariski closed subset of X. Since étale morphisms are preserved under base changes, if V is admissible as a subset of $X(\mathbb{R})$, it is admissible as a subset of $Y(\mathbb{R})$.

Remark 8.15. An admissible subset is semialgebraic. So it has finitely many connected components.

Proposition 8.16. *Let V_1, V_2 be two admissible closed subsets of $X(\mathbb{R})$. Then $V_1 \cap V_2$ is admissible.*

Proof. There is a morphism $f_i : Y_i \to X, i = 1, 2$ of algebraic varieties and a Zariski closed subset $V_i' \subseteq Y_i$ such that $V_i = f(V_i'(\mathbb{R}))$, and f_i is étale. Then, the fiber product $f : Y_1 \times_X Y_2 \to X$ is étale. Since

$$V_1 \cap V_2 = f_1(V_1'(\mathbb{R})) \cap f_2(V_2'(\mathbb{R})) = f((V_1' \times_X V_2')(\mathbb{R})),$$

$V_1 \cap V_2$ is admissible. □

The key result in this section is the following, which shows that admissible subsets satisfy the descending chain condition.

Theorem 8.17. *Let $V_n, n \geq 0$ be a sequence of decreasing admissible subsets of $X(\mathbb{R})$. Then, there is $N \geq 0$ such that $V_n = V_N$ for all $n \geq N$.*

We need the following lemma.

Lemma 8.18. *Let V be an admissible closed subset of $X(\mathbb{R})$. Assume that X and $\overline{V}^{\mathrm{zar}}$ are smooth. Then, V is a finite union of connected components of $\overline{V}^{\mathrm{zar}}(\mathbb{R})$.*

Proof. Since $\overline{V}^{\mathrm{zar}}$ is smooth, different irreducible components of $\overline{V}^{\mathrm{zar}}$ do not meet. So we may assume that $\overline{V}^{\mathrm{zar}}$ is irreducible of dimension d. Hence, $\overline{V}^{\mathrm{zar}}(\mathbb{R})$ is smooth; it is of dimension d everywhere.

There is a morphism $f : Y \to X$ of algebraic varieties and a Zariski closed subset $V' \subseteq Y$ such that $V = f(V'(\mathbb{R}))$, and f is étale at every point in $V'(\mathbb{R})$. After replacing V' by $\overline{V'(\mathbb{R})}^{\mathrm{zar}}$, we may assume that $V'(\mathbb{R})$ is Zariski dense in V'.

For $x \in V$, there is $y \in V'(\mathbb{R})$ such that $V'(\mathbb{R})$ has dimension d at y. Since f is étale, $f^{-1}(\overline{V}^{\mathrm{zar}}(\mathbb{R}))$ is smooth and of dimension d. Hence, V' coincides with $f^{-1}(\overline{V}^{\mathrm{zar}})$ in some Zariski open neighborhood of y. So $V'(\mathbb{R})$ is smooth at y. It follows that f maps some Euclidean neighborhood of y in $V'(\mathbb{R})$ to some Euclidean neighborhood of x in $\overline{V}^{\mathrm{zar}}(\mathbb{R})$. This shows that V is open in $\overline{V}^{\mathrm{zar}}(\mathbb{R})$. Then V is a finite union of connected components of $\overline{V}^{\mathrm{zar}}(\mathbb{R})$. □

Proof of Theorem 8.17. We do the proof by induction on $\dim X$. When $\dim X = 0$, Theorem 8.17 is trivial.

There is $N \geq 0$ such that $\overline{V_n}^{\mathrm{zar}}$ are the same for $n \geq N$. After removing $V_n, n = 1, \ldots, N$, we may assume that $\overline{V_n}^{\mathrm{zar}}, n \geq 0$ are the same variety. After replacing X by this variety, we may assume that $\overline{V_n}^{\mathrm{zar}} = X$ for all $n \geq 0$. Let X_0, X_1 be the smooth and singular part of X. We only need to show that both $V_n \cap X_0(\mathbb{R}), n \geq 0$ and $V_n \cap X_1(\mathbb{R}), n \geq 0$ are stable for n large. Since $\dim X_1 < \dim X$, $V_n \cap X_1(\mathbb{R}), n \geq 0$ is stable for n large by the induction hypothesis. Since X_0 is smooth, by Lemma 8.18, every V_n is a union of connected components of $X_0(\mathbb{R})$. Since $X_0(\mathbb{R})$ has at most finitely many connected components, we conclude the proof. □

Remark 8.19. Theorem 8.17 does not hold for general semialgebraic subsets. The following example shows that it does not hold even for images of algebraic subsets under finite morphisms. For $n \geq 0$, set $Z_n := [n, \infty) \subseteq \mathbb{A}^1(\mathbb{R})$. They are the images of $\mathbb{A}^1(\mathbb{R})$ under the finite morphisms $z \mapsto z^2 + n, n \geq 0$. We have $Z_{n+1} \subset Z_n$ but $\cap_{n \geq 0} Z_n = \emptyset$.

Let $d \geq 2$. We now view $\mathrm{Rat}_d(\mathbb{C})$ as a real variety and study the locus in it with given length spectrum. For $n \geq 1$, $s = 1, \ldots, N_n$ and $a \in \mathbb{R}^s/S_s$, let $\Lambda_n^s(a)$ be the subset of $t \in \mathrm{Rat}_d(\mathbb{C})$ such that $a \subseteq L_n(t)$ (i.e., f_t^n has a subset of fixed points counting with multiplicity, such that the set of norms of multipliers of these fixed points equals to a). It is a closed subset in $\mathrm{Rat}_d(\mathbb{C})$.

Remark 8.20. This notion generalizes the notion $\Lambda_n(a)$. When $s = N_n$, we get $\Lambda_n(a) = \Lambda_n^s(a)$.

Pick $(a_1, \ldots, a_s) \in \mathbb{R}^s$ representing $a \in [0, +\infty)^s/S_s$. We have

$$\Lambda_n^s(a) = \phi_n^s(|\lambda_n^s|^{-1}(a_1, \ldots, a_s)).$$

Even though $|\lambda_n^s|$ is not real algebraic, its square $|\lambda_n^s|^2$ is real algebraic. So $|\lambda_n^s|^{-1}(a_1, \ldots, a_s) = (|\lambda_n^s|^2)^{-1}(a_1^2, \ldots, a_s^2)$ is real algebraic. Hence, $\Lambda_n^s(a)$ is semialgebraic. Moreover, if $a_i \neq 1$ for every $i = 1, \ldots, s$,

$$|\lambda_n^s|^{-1}(a_1, \ldots, a_s) \subseteq (\lambda_n^s)^{-1}((\mathbb{A}^1 \setminus \{1\})^s).$$

So ϕ_n^s is étale along $|\lambda_n^s|^{-1}(a_1, \ldots, a_s)$. This shows the following fact.

Proposition 8.21. *For $a \in ([0, +\infty) \setminus \{1\})^s/S_s$, $\Lambda_n^s(a)$ is admissible.*

8.3. Length spectrum

Let f be an endomorphism of $\mathbb{P}^1(\mathbb{C})$ of degree $d \geq 2$. Recall that the *length spectrum* $L(f) = \{L(f)_n, n \geq 1\}$ of f is a sequence of finite multisets, where $L(f)_n := L_n(f)$ is the multiset of norms of multipliers of all fixed points of f^n. In particular, $L(f)$ is a multiset of positive real numbers of cardinality $d^n + 1$. For every $n \geq 0$, let $RL(f)_n$ be the sub-multiset of $L(f)_n$ consisting of all elements > 1. We call $RL(f) := \{RL(f)_n, n \geq 1\}$ the *repelling length spectrum* of f and $RL^*(f) := \{RL^*(f)_n := RL(f)_{n!}, n \geq 1\}$ the *main repelling length spectrum* of f. We have $d^n + 1 \geq |RL(f)_n| \geq d^n + 1 - M$ for some $M \geq 0$. It is clear that the difference $d^{n!} + 1 - |RL^*(f)_n|$ is increasing and bounded.

Let Ω be the set of sequences $A_n, n \geq 0$ of multisets consisting of real numbers of norm strictly larger than 1 satisfying $|A_n| \leq d^{n!} + 1$ and for every $a \in A_n$ with multiplicity m, $a^{n+1} \in A_{n+1}$ with multiplicity at least m. For $A, B \in \Omega$, we write $A \subseteq B$ if $A_n \subseteq B_n$ for every $n \geq 0$. An element $A = (A_n) \in \Omega$ is called *big* if $d^{n!} + 1 - |A_n|$ is bounded. For every endomorphism f of $\mathbb{P}^1(\mathbb{C})$ of degree d, we have $RL^*(f) \in \Omega$ and it is big.

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For $A \subseteq RL^*(f)$, by induction, we can show that there is a sequence of sub-multisets $P_n \subseteq \mathrm{Fix}_{n!}(f), n \geq 1$ (here we view $\mathrm{Fix}_{n!}(f)$ as a multiset of cardinal $d^{n!}+1$) such that $P_n \subseteq P_{n+1}$ and $A_n = \{|df^{n!}(x)|| \, x \in P_n\}$. Such $P := (P_n)$ is called a *realization of* A, which may not be unique. Further, assume that A is big. Then, for every realization of A, $|\mathrm{Fix}_{n!}(f) \setminus P_n|$ is bounded. It follows that $\mathrm{Per}\,(f) \setminus (\cup_{n\geq 0} P_n)$ is finite.

Let $A \in \Omega$. Define $\Lambda(A) := \cap_{n\geq 1}\Lambda_{n!}^{|A_n|}(A_n)$, which is the locus of $t \in \mathrm{Rat}_d$ satisfying $A \subseteq RL^*(f_t)$. It is clear that $\Lambda_{n!}^{|A_n|}(A_n), n \geq 1$ is decreasing, and by Proposition 8.21, each of them is admissible. Hence, by Theorem 8.17, we get the following result.

Proposition 8.22. *There is $N(A) \geq 0$ such that*

$$\Lambda(A) = \Lambda_{N(A)!}^{|A_{N(A)}|}(A_{N(A)}),$$

which is admissible.

Let $\gamma \simeq [0,1]$ be a real analytic curve in $\mathrm{Rat}_d(\mathbb{C})$. We view $\gamma \times \mathbb{P}^1(\mathbb{C})$ as a subset of $\mathrm{Rat}_d(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let f_γ be the restriction of $f_{\mathrm{Rat}_d(\mathbb{C})}$ to $\gamma \times \mathbb{P}^1(\mathbb{C})$. For every n-periodic point $x = (t,y) \in \gamma \times \mathbb{P}^1(\mathbb{C})$, let γ_x^n be the connected component of

$$(\gamma \times \mathbb{P}^1(\mathbb{C})) \cap \mathrm{Rat}_d(\mathbb{C})[n] = \phi_n^{-1}(\gamma)$$

containing x.

Remark 8.23. If x is repelling for f_t, then ϕ_n is étale at (t,x). Hence, it induces an isomorphism from some neighborhood of (x,t) in γ_x^n to its image in γ.

Moreover, if $|\lambda_n|(\gamma_x^n) \subseteq (1,+\infty)$, then ϕ_n is étale along γ_x^n. In particular, $\phi_n|_{\gamma_x^n} : \gamma_x^n \to \gamma$ is a covering map. Since γ is simply connected, $\phi_n|_{\gamma_x^n} : \gamma_x^n \to \gamma$ is an isomorphism. If $n|m$, then $\gamma_x^n \subseteq \gamma_x^m$. However, for every $(u,y) \in \gamma_x^n$, the multiplicity of y in $\mathrm{Fix}(f_u^m)$ is 1. So γ_x^m coincide with γ_x^n in a neighborhood of y. Hence, $\gamma_x^m = \gamma_x^n$. This implies that every $y \in \gamma_x$ has the same minimal period and for every period l of y, $\gamma_y^l = \gamma_x^n$.

Lemma 8.24. *Fix $A \in \Omega$. Assume that for every $t \in \gamma$, $A \subseteq RL^*(f_t)$. Then, there is a realization P of A for f_0, such that the following holds:*

(i) *For every $x \in \cup_{n\geq 0} P_n$, $\gamma_{(0,x)}^m$ does not depend on the choice of period m of x. We denote by $\gamma_x = \gamma_{(0,x)}^m$ for some (then every) period m of x. Then $\phi_m|_{\gamma_x} : \gamma_x \to \gamma$ is a homeomorphism and it is étale along γ_x. In particular, for different points x, γ_x are disjoint.*
(ii) *For every $x \in \cup_{n\geq 0} P_n$, with a period m, $|\lambda_m|$ is a constant on γ_x.*

Proof. For every $n \geq 1$, let B_n be the subset of $\mathrm{Fix}(f_0^n)$ such that $|\lambda_n|$ is a constant > 1 on $\gamma_{(0,x)}^n$. If $x \in B_n$ for some $n \geq 1$, by Remark 8.23, $x \in B_m$ for every period m of x and $\gamma_x := \gamma_{(0,x)}^m$ does not depend on the choice of period m. Moreover, $\phi_m|_{\gamma_x} : \gamma_x \to \gamma$ is a homeomorphism and it is étale along γ_x. In particular, for different points x, γ_x are disjoint.

It is clear that $B = (B_{n!})$ realizes an element $C \in \Omega$. We only need to show that $A \subseteq C$. Let a be an element in A_n of multiplicity $l \geq 1$. Then, for every $t \in \gamma$, since $|a| > 1$, $|\lambda_{n!}|^{-1}(a) \cap \phi_{n!}^{-1}(t)$ contains at least l distinct points. Let $x_1, \ldots, x_s$ be the elements in $x \in B_{n!}$ with $\lambda_{n!}((0,x)) = a$. We only need to show that $s \geq l$. For every $i = 1, \ldots, s$, γ_{x_i} is a connected component of $\phi_{n!}^{-1}(\gamma)$. Set $Z := \phi_{n!}^{-1}(\gamma) \setminus \cup_{i=1}^s \gamma_{x_i}$. If $s < l$, then for every $t \in \gamma$, $Z \cap |\lambda_{n!}|^{-1}(a) \cap \phi_{n!}^{-1}(t)$ has at least one point. So there is $y \in Z$ such that $\gamma_z^{n!} \cap |\lambda_{n!}|^{-1}(a)$ is infinite. Since both $\gamma_z^{n!}$ and $|\lambda_{n!}|^{-1}(a)$ are real analytic in $\gamma \times \mathbb{P}^1(\mathbb{C})$, $\gamma_z^{n!} \subseteq |\lambda_{n!}|^{-1}(a)$. By Remark 8.23, $\gamma_z^{n!}$ meets $\phi_{n!}^{-1}(0)$ at some point $(0,x)$ for some $x \in B_n$. So $\gamma_z^{n!} = \gamma_x$, which is a contradiction. ◻

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8.4. *Length spectrum as moduli*

Let $\Psi : \mathrm{Rat}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C}) = \mathrm{Rat}_d(\mathbb{C})/\mathrm{PGL}_2(\mathbb{C})$ be the quotient map. Let $FL_d(\mathbb{C}) \subseteq \mathrm{Rat}_d(\mathbb{C})$ be the locus of Lattès maps, which is Zariski closed in $\mathrm{Rat}_d(\mathbb{C})$. We now prove Theorem 1.5 via proving the following stronger statement.

Theorem 8.25. *If $A \in \Omega$ is big, then $\Phi(\Lambda(A) \setminus FL_d(\mathbb{C})) \subseteq \mathcal{M}_d$ is finite.*

Proof. By Proposition 8.22, $\Lambda(A)$ is admissible in $\mathrm{Rat}_d(\mathbb{C})$. Hence $\Lambda(A) \setminus FL_d(\mathbb{C})$ is admissible in $\mathrm{Rat}_d(\mathbb{C}) \setminus FL_d(\mathbb{C})$. In particular, $\Lambda(A) \setminus FL_d(\mathbb{C})$ and $\Phi(\Lambda(A) \setminus FL_d(\mathbb{C}))$ are semialgebraic.

To get a contradiction, assume that $\Phi(\Lambda(A)\setminus FL_d(\mathbb{C}))$ is not finite. By Nash Curve Selection Lemma [BCR98, Proposition 8.1.13], there is a real analytic curve $\gamma \simeq [0, 1]$ in $\Lambda(A) \setminus FL_d(\mathbb{C})$ whose image in $\mathcal{M}_d$ is not a point. Since non-flexible Lattès exceptional endomorphisms are isolated in the moduli space $\mathcal{M}_d$, there is at least one f_t that is not exceptional. Without loss of generality we assume f_0 is not exceptional. We now apply Lemma 8.24 for γ and A, and follows the notation there. Set $Q := \cup_{n\geq 0}P_n$. Then $S := \mathrm{Per}\,(f_0) \setminus Q$ is finite.

Pick any $z_0 \in Q$. By the discussion in Example 7.3, there exists a horseshoe K of f_0 containing z_0 and $K \cap S = \emptyset$. There is $m \geq 0$ such that $f_0^m(K) = K$ and $f_0^m(z_0) = z_0$. By Lemma 6.1, there exists $\varepsilon > 0$ and a continuous map $h : [0, \varepsilon] \times K \to \mathbb{P}^1(\mathbb{C})$ such that for each $t \in [0, \varepsilon]$:

(i) $K_t := h(t, K)$ is an expanding set of f_t^m.
(ii) the map $h_t := h(t, \cdot) : K \to K_t$ is a homeomorphism and $f_t^m \circ h_t = h_t \circ f_0^m$.

For every $t \in [0, \varepsilon]$ and for every $w_0 \in K$ satisfying $f_0^{nm}(w_0) = w_0$, we have $f_t^{nm}(h_t(w_0)) = h_t(w_0)$. It follows that $h_t(w_0) = \gamma_{w_0}(t)$. Since $|\lambda_{nm}|$ is a constant on γ_{w_0}, we get $|df_0^{nm}(w_0)| = |df_t^{mn}(h_t(w_0))|$. We claim that K_t is a CER of f_t. We check that (f_t, K_t) satisfies Definition 7.1: since K_t is expanding by Lemma 6.1, (ii) holds; since topological exactness and openness are preserved by topological conjugacy, by Remark 7.2), (i) and (iii) hold.

Since f_0 is not exceptional, by Theorem 1.1, K is a nonlinear CER for f_0. By Theorem 7.6, for every fixed $t \in [0, \varepsilon]$, the conjugacy h_t can be extended to a conformal map $h_t : U \to V$ where U is a neighborhood of K and V is a neighborhood of K_t. This implies $df_0^m(z_0) = df_t^m(\gamma_{z_0}(t))(= df_t^m(h_t(z_0)))$ or $df_0^m(z_0) = \overline{df_t^m(\gamma_{z_0}(t))}$. Since $df_t^m(\gamma_{z_0}(t))$ depends continuously on t, we must have $df_0^m(z_0) = df_t^m(\gamma_{z_0}(t))$ when $t \in [0, \varepsilon]$. Since γ_{z_0} is real analytic, the map $t \mapsto df_t^m(\gamma_{z_0}(t))$ is real analytic on $\gamma = [0, 1]$. It is a constant on $[0, \varepsilon]$. Hence, it is a constant on γ. Let n be any period of z_0. The above argument shows that $(\lambda_n|_{\gamma_{z_0}})^m$ is a constant. Hence, $\lambda_n|_{\gamma_{z_0}}$ is a constant.

Since our choice of $z_0 \in Q$ is arbitrary, for every $z_0 \in Q$, of period n, the map $t \mapsto df_t^n(\phi(t))$ is a constant on $[0, 1]$. Since S is finite, all f_t have the same multiplier spectrum for periodic points of sufficiently high period.

The set of all endomorphisms in $\mathrm{Rat}_d(\mathbb{C})$ with the same multiplier spectrum of f_0 for periodic points with period at least $N \geq 1$ is an algebraic variety. We denote it by V_N. There exists $N \geq 1$ such that $\gamma \subseteq V_N$. Furthermore, there exists an irreducible component X of V_N which contains γ. The irreducible variety X forms a stable family (see [McM16, Chapter 4]), since the period of attracting cycles are bounded in V_N. The variety X is not isotrivial since $\Psi(\gamma)$ is not a point. By Theorem 1.2, $\gamma \subseteq X$ is contained in the flexible Lattès family, which is a contradiction. □

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