

STOCHASTIC INTEGRALS IN ABSTRACT WIENER SPACE II: REGULARITY PROPERTIES

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Introduction

This paper continues the study of stochastic integrals in abstract Wiener space previously given in [14]. We will present, among other things, the detailed discussion and proofs of the results announced in [16]. Let $H \subset B$ be an abstract Wiener space. Consider the following stochastic integral equation in $H \subset B$,

$$(1) \quad X(t) = x + \int_0^t A(s, X(s))dW(s) + \int_0^t \sigma(s, X(s))ds ,$$

where $W(t)$ is a Wiener process in B . Under certain assumptions on A and σ we showed in [14] that (1) has a unique non-anticipating continuous solution and that this solution is a Markov process. If A and σ are differentiable in the second variable we can differentiate the above equation "formally" with respect to the starting point x to obtain the formal operator-valued stochastic integral equation

$$(2) \quad Y(t) = I + \int_0^t A_x(s, X(s))Y(s)dW(s) + \int_0^t \sigma_x(s, X(s))Y(s)ds ,$$

where A_x and σ_x are derivatives of A and σ in the second variable, respectively. (2) is a linear integral equation and obviously has a unique solution which qualifies to be called the derivative of $X(t)$ in some sense. If A and σ are furthermore twice differentiable we can differentiate (2) formally in the same manner to obtain another stochastic integral equation whose solution is the second derivative of $X(t)$. Thus roughly speaking, the solution $X(t)$ of (1), regarded as a function of its starting point, is as smooth as A and σ .

Let f be a real-valued continuous function in B . Let $\theta(x) =$

Received July 25, 1972.

* This work was supported in part by NSF Grant GU-3784.

$E_x[f(X(t))]$. If f is differentiable then formally by the "chain rule" we have $\theta'(x) = E_x[Y(t)^*(f'(X(t)))]$, where $Y(t)$ is the solution of (2) and $*$ denotes the adjoint of operators of H . If f is twice differentiable then so is θ and a formal expression for $\theta''(x)$ can be written by using also the second derivative of $X(t)$. Thus if A and σ are C^∞ -functions then θ is as smooth as f . Furthermore, if $f''(x)$ is a Hilbert-Schmidt operator then $\theta''(x)$ is also a Hilbert-Schmidt operator.

The above approach of discussing the regularity properties of $X(t)$ and $\theta(x)$ was first introduced by Gikhman [3; 4]. It was carried over to infinite dimensional Hilbert spaces by Dalec'kii [1; 2]. See also [18; 23]. We generalize it to Banach spaces (§ 2) and, furthermore, study the related operator-valued stochastic integrals and prove the corresponding versions of Ito's formula and Girsanov-Skorokhod-McKean's formula (§ 1). In case A and σ are time-independent we show in the end of the paper that $X(t)$ generates a semi-group on the Banach space of bounded continuous functions on B vanishing at infinity. The proof is due to K. Ito.

Recently, Kannan and Bharucha-Reid [10; 11] have defined several operator-valued stochastic integrals and proved some generalizations of Ito's formula. However, there is no apparent relation between their work and ours.

This paper is closely related to Piech's. In a series of papers [19; 20; 21; 22] she studies the corresponding parabolic equation of (1) with $\sigma \equiv 0$ and A satisfying stronger assumptions. In particular, A is non-degenerate. She constructs a fundamental solution $\{q_t(s, dy)\}$ which is related to the process $X(t)$ by $\int_B f(y)q_t(x, dy) = E_x[f(X(t))]$ for bounded Lip-1 functions f [17]. Her conclusions about the regularity properties of the function $\theta(x) = E_x[f(X(t))]$ are stronger than ours in this particular case.

Notation

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|------------------------------|--|
| 1. E | expectation |
| 2. $H \subset B$ | abstract Wiener space |
| 3. $B^* \subset H \subset B$ | (through identifications) |
| 4. $ \cdot $ | H -norm (see 7) |
| 5. $\ \cdot\ $ | B -norm (see 8) |
| 6. $L^n(X; Y)$ | continuous n -linear maps from $\underbrace{X \times X \times \cdots \times X}_{n \text{ times}}$ into Y |

- 7. $|\cdot|$ norm of $L^n(H; R)$
- 8. $\|\cdot\|$ norm of $L^n(B; R)$
- 9. $L_{(2)}(H; H)$ Hilbert-Schmidt operators of H (see 12)
- 10. $|\cdot|_2, \langle \cdot, \cdot \rangle_2$ norm, inner product of $L_{(2)}(H; H)$. (see 13)
- 11. \sim $\tilde{T}(x) = T(x, \cdot, \dots, \cdot)$. $T \in L^n(X; R), \tilde{T} \in L(X; L^{n-1}(X; R))$
(cf. 33).
- 12. $L_{(2)}^n(H; R)$ Hilbert-Schmidt type n -linear forms of H .
- 13. $|\cdot|_2, \langle \cdot, \cdot \rangle_2$ norm, inner product of $L_{(2)}^n(H; R)$
- 14. \circ_j $S \circ_j T, S \in L^n(X; R), T \in L(X; X); S \circ_j T \in L^n(X; R)$.
($S \circ_j T(x_1, \dots, x_j, \dots, x_n) = S(x_1, \dots, Tx_j, \dots, x_n)$)
- 15. $\|\cdot\|_X$ norm of $L(X; X)$.
- 16. $W(t)$ Wiener process in B .
- 17. \mathcal{M}_t σ -field generated by $\{W(s); s \leq t\}$
- 18. $\mathcal{L}[L_{(2)}^n(H; R)]$ non-anticipating stochastic processes ξ with state space $L_{(2)}^n(H; R)$ such that $\int_0^\tau E|\xi(t)|^2 dt < \infty$ for each finite τ . (see 20)
- 19. $\mathcal{L}[L^n(B; R)]$
- 20. $\mathcal{L}[X]$ non-anticipating stochastic processes ξ with state space X such that $\int_0^\tau E|\xi(t)|_X^2 dt < \infty$ for each finite τ . (cf. 27)
- 21. $\mathcal{S}(\mathcal{H}; \mathcal{K})$ trace-class type bilinear form from $\mathcal{H} \times \mathcal{H}$ into \mathcal{K} .
- 22. TRACE S trace of $S \in \mathcal{S}(\mathcal{H}; \mathcal{K})$.
- 23. Δ a) $T \in L^n(H; R), S \in L(L^{n-1}(H; R); L^{n-1}(H; R))$
 $S \Delta T \in L^n(H; R)$.
 $((S \Delta T)^\sim = S \circ \tilde{T})$
c) $S \in L^n(H; R), T \in L(L^{n-1}(H; R); R); S \Delta T \in H$.
 $(\langle S \Delta T, h \rangle = T(\tilde{S}(h)))$.
- 24. $L_1(H; H)$ trace class operators of H .
- 25. $X_x(t)$ diffusion process starting at x .
- 26. C_H^n n -smooth functions in H -directions.
- 27. $\mathcal{L}(D)$ square integrable random variables taking values in D . (cf. 20)
- 28. $\delta\xi_x$ MS - H -derivative of a random variable ξ at x .
- 29. $MS-C_H^n$ MS - n -smooth random variables in H -directions.
- 30. $\delta Z, \delta^n Z$ MS - H -derivative of a diffusion process Z .

- 31. $\hat{}$ $S \in L^n(H; R), \hat{S} \in L^{n-1}(H; H)$
 $(\langle \hat{S}(h_1, \dots, h_{n-1}), \hat{h} \rangle = S(h, h_1, \dots, h_{n-1}))$
- 32. $:$ $T \in L^3(H; R), S \in L^n(H; R); S: T \in L^{n+1}(H; R).$
 $(S: T(h_1, h_2, \dots, h_n, h_{n+1}) = T(h_1, \hat{S}(h_2, \dots, h_n), h_{n+1})).$
- 33. $\check{}$ $S \in L^n(H; R), \check{S} \in L(H; L^{n-1}(H; R))$
 $(\check{S}(h) = S(\cdot, \cdot, \dots, h)). \quad (cf. 11)$

1. Operator-Valued Stochastic Integrals

Let $H \subset B$ be an abstract Wiener space. $|\cdot|$ and $\|\cdot\|$ denote the H -norm and B -norm, respectively. We will regard $B^* \subset H^* \approx H \subset B$ in the natural way. As in [14] we assume that there is a sequence Q_n of finite dimensional projections such that (i) $Q_n(B) \subset B^*$ and (ii) Q_n converges strongly to the identity both in B and in H . Furthermore, we will assume that there exists an orthonormal basis $\{e_n\}$ of H such that $\sum_{n=1}^\infty \|e_n\|^2 < \infty$. This additional assumption is satisfied by all of the presently known abstract Wiener spaces.

Notation:

- (i) $L^n(X; Y)$ = the Banach space of all continuous n -linear maps from X^n into Y , where X and Y are Banach spaces. L^1 will be written as L .
- (ii) $L^{n-1}(X; X^*) \approx L^n(X; R)$
- (iii) $\|\cdot\|$ and $|\cdot|$ denote the norms of $L^n(B; R)$ and $L^n(H; R)$, respectively. Clearly $L^n(B; R) \subset L^n(H, R)$ and $|\cdot|$ is dominated by $\|\cdot\|$ with some constant depending on n .
- (iv) $L_{(2)}(H; H) (\equiv L^2_{(2)}(H; R))$ denotes the Hilbert space of all Hilbert-Schmidt operators of H with H - S -norm $|\cdot|_2 = \langle \cdot, \cdot \rangle_2^{1/2}$. It can be shown easily that $|S|_2 \leq (\sum_{j=1}^\infty \|e_j\|^2)^{1/2} \|S\|$ for all $S \in L^2(B; R)$, where $\{e_j\}$ is given in the additional assumption. Thus we have $L^2(B; R) \subset L^2_{(2)}(H; R)$.
- (v) Let $T \in L^n(X; R)$. Define $\tilde{T} \in L(X; L^{n-1}(X; R))$ by $\tilde{T}(x) = T(x, \cdot, \cdot, \dots, \cdot)$.

Now we want to define inductively a sequence of Hilbert spaces $L^n_{(2)}(H; R), n \geq 1$, with $L^1_{(2)}(H; R) = H$ by convention and $L^2_{(2)}(H; R)$ given above.

DEFINITION 1. Let $T \in L^n(H; R), n \geq 3$. T is said to be of *Hilbert-Schmidt type* if (i) $\tilde{T}(H) \subset L^{n-1}_{(2)}(H; R)$ and (ii) \tilde{T} is a Hilbert-Schmidt operator from H into $L^{n-1}_{(2)}(H; R)$.

Let $L_{(2)}^n(H; R)$ denote the space of all Hilbert-Schmidt type n -linear forms of H . It is a Hilbert space with the inner product $\langle S, T \rangle_2 =$ the Hilbert-Schmidt inner product of \tilde{S} and \tilde{T} , $S, T \in L_{(2)}^n(H; R)$. Clearly,

$$\langle S, T \rangle_2 = \sum_{i_1, i_2, \dots, i_n} S(v_{i_1}, v_{i_2}, \dots, v_{i_n}) T(v_{i_1}, v_{i_2}, \dots, v_{i_n}),$$

where $\{v_k\}$ is any orthonormal basis of H . Let $|S|_2 = \langle S, S \rangle_2^{1/2}$. Note that we have used the same notation $|\cdot|_2$ and \langle, \rangle_2 to denote the norm and the inner product of $L_{(2)}^n(H; R)$ for all $n \geq 2$ since there is no confusion. For example, the meaning of the following equality is clear, when $S, T \in L_{(2)}^n(H; R)$,

$$(3) \quad \langle S, T \rangle_2 = \sum_k \langle \tilde{S}(v_k), \tilde{T}(v_k) \rangle_2.$$

LEMMA 1.1. (a) $|S| \leq |S|_2$ for all S in $L_{(2)}^n(H; R)$.

(b) $|T|_2 \leq c^n \|T\|$ for all T in $L^n(B; R)$, where c is a constant. Thus we have the relation $L^n(B; R) \subset L_{(2)}^n(H; R) \subset L^n(H; R)$, $n \geq 1$.

(c) $L^n(B; R)$ is dense in $L_{(2)}^n(H; R)$.

Proof. Let $\{v_j\}$ be an orthonormal basis of H . Then

$$\begin{aligned} S(h_1, h_2, \dots, h_n)^2 &= \{\sum_j S(h_1, v_j) S(v_j, h_2, \dots, h_n)\}^2 \\ &\leq \{\sum_j S(h_1, v_j)^2\} \{\sum_j S(v_j, h_2, \dots, h_n)^2\} \\ &= |h_1|^2 \sum_j S(v_j, h_2, \dots, h_n)^2 \\ &\leq |h_1|^2 |h_2|^2 \dots |h_n|^2 \sum_{i_1, i_2, \dots, i_n} S(v_{i_1}, v_{i_2}, \dots, v_{i_n})^2, \end{aligned}$$

whence (a) follows. To prove (b) and (c) let $\{Q_n\}$ and $\{e_k\}$ be given in the beginning of this section. Then

$$\begin{aligned} |T|_2^2 &= \sum_{i_1, i_2, \dots, i_n} T(e_{i_1}, e_{i_2}, \dots, e_{i_n})^2 \\ &\leq \sum_{i_1, i_2, \dots, i_n} (\|T\| \|e_{i_1}\| \|e_{i_2}\| \dots \|e_{i_n}\|)^2 \\ &= (\sum_i \|e_i\|^2)^n \|T\|^2. \end{aligned}$$

Moreover, if $U \in L_{(2)}^n(H; R)$, let $U_j = U(Q_j(\cdot), Q_j(\cdot), \dots, Q_j(\cdot))$. Then $U_j \in L^n(B; R)$ and $|U_j - U|_2 \rightarrow 0$.

EXAMPLE 1. Let $H = L^2(0, 1)$ (real-valued). Suppose ϕ is a measurable function on $(0, 1)^n$ such that

$$\int_0^1 \int_0^1 \dots \int_0^1 |\phi(t_1, t_2, \dots, t_n)|^2 dt_1 dt_2 \dots dt_n < \infty.$$

Define $K: H^n \rightarrow R$ by

$$K(f_1, f_2, \dots, f_n) = \int_0^1 \int_0^1 \cdots \int_0^1 \phi(t_1, t_2, \dots, t_n) f_1(t_1) f_2(t_2) \cdots f_n(t_n) dt_1 dt_2 \cdots dt_n.$$

Then K is a Hilbert-Schmidt type n -form on H and $|K|_2 = \left[\int_0^1 \int_0^1 \cdots \int_0^1 |\phi(t_1, t_2, \dots, t_n)|^2 dt_1 dt_2 \cdots dt_n \right]^{1/2}$.

EXAMPLE 2. Let C consist of all real-valued continuous functions on $[0, 1]$ which vanish at the origin. C is a Banach space with the sup norm. Let $C' = \{f \in C; f \text{ is absolutely continuous and } f' \in L^2(0, 1)\}$. C' is a Hilbert space with the inner product $\langle f, g \rangle = \int_0^1 f'(t)g'(t)dt$. $C' \subset C$ is an abstract Wiener space [5; 6 pp. 388–390]. Define $K: C'^n \rightarrow R$ by

$$K(f_1, f_2, \dots, f_n) = \int_0^1 f_1'(t) f_2(t) \cdots f_n(t) dt.$$

Then K is a Hilbert-Schmidt n -form on C' and it can be checked easily that $|K|_2 = n^{-1/2}$. However, K can not be extended to C^n . This example shows that $L^n(C; R) \subsetneq L^n_{(2)}(C'; R)$.

Notation. Let X be a Banach space. Let $S \in L^n(X; R)$ and $T \in L(X; X)$. Define the composition $S \circ_j T$ of S and T in the j -th factor by: $S \circ_j T(x_1, x_2, \dots, x_j, \dots, x_n) = S(x_1, x_2, \dots, Tx_j, \dots, x_n)$, $x_k \in X$, $k = 1, 2, \dots, n$. Thus $S \circ_j T \in L^n(X; R)$. $\|T\|_X$ denotes the operator norm of T .

LEMMA 1.2. (a) $\|S \circ_j T\| \leq \|S\| \|T\|_B$, $S \in L^n(B; R)$, $T \in L(B; B)$.

(b) $|S \circ_j T| \leq |S| \|T\|_H$, $S \in L^n(H; R)$, $T \in L(H; H)$.

(c) If $S \in L^n_{(2)}(H; R)$ and $T \in L(H; H)$ then $S \circ_j T \in L^n_{(2)}(H; R)$ and $|S \circ_j T|_2 \leq |S|_2 \|T\|_H$, $j = 1, 2, \dots, n$.

Proof. (a) and (b) are trivial. We use induction to prove (c). The cases with $n = 1, 2$ are well-known. Assume we have the lemma for $n-1$. Let $S \in L^n_{(2)}(H; R)$ and $T \in L(H; H)$. Clearly $(S \circ_j T)^\sim(h) = S^\sim(h) \circ_{j-1} T$ for $j = 2, 3, \dots, n$. Hence by induction $S \circ_j T \in L^n_{(2)}(H; R)$, $j = 2, 3, \dots, n$. Furthermore, let $\{v_k\}$ be an orthonormal basis of H ,

$$\begin{aligned} |S \circ_j T|_2^2 &= \sum_k |(S \circ_j T)^\sim(v_k)|_2^2 \\ &= \sum_k |S^\sim(v_k) \circ_{j-1} T|_2^2 \\ &\leq \sum_k |S^\sim(v_k)|_2^2 \|T\|_H^2 \quad \text{by induction} \\ &= |S|_2^2 \|T\|_H^2. \end{aligned}$$

It remains to show the conclusion for $S \circ_1 T$. But $(S \circ_1 T)^\sim = S^\sim \circ T$. Thus $(S \circ_1 T)^\sim(H) \subset S^\sim(H) \subset L_{(2)}^{n-1}(H; R)$. Moreover by definition $|S \circ_1 T|_2 =$ the H - S -norm of $(S \circ_1 T)^\sim =$ the H - S -norm of $S^\sim \circ T \leq$ the product of H - S -norm of S^\sim and $\|T\|_H = |S|_2 \|T\|_H$. Hence $|S \circ_1 T|_2 \leq |S|_2 \|T\|_H$.

We have now various spaces $L^n(B; R)$, $L_{(2)}^n(H; R)$ and $L^n(H; R)$, $n \geq 1$. Each such space has three topologies, namely, the uniform topology, strong topology and weak topology. However, it can be shown, by a similar argument used in [9], that these topologies generate the same Borel field. Thus we do not need to specify the Borel field corresponding to a particular topology when we talk about the measurability of a random variable with values in those spaces.

Let $W(t)$ be a Wiener process in B . Let \mathcal{M}_t be the σ -field generated by $\{W(s); 0 \leq s \leq t\}$. A stochastic process $\zeta(t, \omega)$, $0 \leq t$ and $\omega \in \Omega$, is *non-anticipating* if it is (t, ω) -jointly measurable and $\zeta(t, \cdot)$ is \mathcal{M}_t -measurable for each t . Let $\mathcal{L}[L_{(2)}^n(H; R)]$ denote the space consisting of all non-anticipating stochastic processes $\xi(t)$ with state space $L_{(2)}^n(H; R)$ such that $\int_0^\tau E \|\xi(t)\|_2^2 dt < \infty$ for each $0 < \tau < \infty$. We will define a linear operator J from $\mathcal{L}[L_{(2)}^n(H; R)]$ into $\mathcal{L}[L_{(2)}^{n-1}(H; R)]$, $n \geq 3$. (The cases $n = 1, 2$ have been defined in [14], $L_{(2)}^0(H; R) = R$ by convention). In order to do this, we prove first a lemma about the space $\mathcal{L}[L^n(B; R)]$ consisting of all non-anticipating stochastic processes $\zeta(t)$ with state space $L^n(B; R)$ such that $\int_0^\tau E \|\zeta(t)\|^2 dt < \infty$ for each $0 < \tau < \infty$. By Lemma 1.1 $\mathcal{L}[L^n(B; R)] \subset \mathcal{L}[L_{(2)}^n(H; R)]$. Moreover, $\mathcal{L}[L^n(B; R)]$ is dense in $\mathcal{L}[L_{(2)}^n(H; R)]$ in the following sense:

LEMMA 1.3. *If $\xi \in \mathcal{L}[L_{(2)}^n(H; R)]$ then there exists a sequence $\xi_n \in \mathcal{L}[L^n(B; R)]$ such that $\int_0^\tau E \|\xi_n(t) - \xi(t)\|_2^2 dt \rightarrow 0$ as $n \rightarrow \infty$ for each $0 < \tau < \infty$.*

LEMMA 1.4. *If $\zeta \in \mathcal{L}[L^n(B; R)]$ then*

- (a) *for $s < t$, $E \|\tilde{\zeta}(s)(W(t) - W(s))\|_2^2 = (t - s)E \|\zeta(s)\|_2^2$*
- (b) *for $s < t < u < v$, $E \langle \tilde{\zeta}(s)(W(t) - W(s)), \tilde{\zeta}(u)(W(v) - W(u)) \rangle_2 = 0$.*

Remark. The special cases $n = 1, 2$ appeared in [14].

Proof. Let $\{Q_k\}$ be the projections given in the beginning of this section. Let

$$\phi = \|\tilde{\zeta}(s)(W(t) - W(s))\|_2^2$$

and

$$\phi_k = |\tilde{\zeta}(s)(Q_k(W(t) - W(s)))|_2^2.$$

Since Q_k converges strongly to the identity in B , $\phi_k \rightarrow \phi$ almost surely. Furthermore,

$$\begin{aligned} \phi_k &\leq c^{2n} \|\tilde{\zeta}(s)(Q_k(W(t) - W(s)))\|^2 && \text{by Lemma 1.1,} \\ &\leq c^{2n} \|\zeta(s)\|^2 \|Q_k(W(t) - W(s))\|^2 \\ &\leq c^{2n} \|\zeta(s)\|^2 \|Q_k\|_B^2 \|W(t) - W(s)\|^2 \\ &\leq \text{constant} \|\zeta(s)\|^2 \|W(t) - W(s)\|^2. \end{aligned}$$

Recall that $\sup_k \|Q_k\|_B^2 < \infty$ by the Uniform Boundedness Principle. But since ζ is non-anticipating,

$$\begin{aligned} E(\|\zeta(s)\|^2 \|W(t) - W(s)\|^2) &= E(\|\zeta(s)\|^2)E(\|W(t) - W(s)\|^2) \\ &= E(\|\zeta(s)\|^2)(t - s) \int_B \|x\|^2 p_1(dx), \end{aligned}$$

where p_1 is Wiener measure with variance parameter 1. Therefore, by the Lebesgue dominated convergence theorem,

$$(4) \quad E |\tilde{\zeta}(s)(W(t) - W(s))|_2^2 = \lim_{k \rightarrow \infty} E |\tilde{\zeta}(s)(Q_k(W(t) - W(s)))|_2^2.$$

Without loss of generality, we may assume that Q_k is the orthogonal projection onto the span of $\{f_j; j = 1, 2, \dots, k\}$, where $\{f_j\}$ is an orthonormal basis of H . Then

$$\begin{aligned} &|\tilde{\zeta}(s)(Q_k(W(t) - W(s)))|_2^2 \\ &= \langle \tilde{\zeta}(s)(Q_k(W(t) - W(s))), \tilde{\zeta}(s)(Q_k(W(t) - W(s))) \rangle_2 \\ &= \sum_{j,m=1}^k (W(t) - W(s), f_j)(W(t) - W(s), f_m) \langle \tilde{\zeta}(s)(f_j), \tilde{\zeta}(s)(f_m) \rangle_2. \end{aligned}$$

Recall that ζ is non-anticipating and also that $E(W(t) - W(s), f_j)(W(t) - W(s), f_m) = (t - s)\delta_{jm}$. Hence we have

$$(5) \quad E |\tilde{\zeta}(s)(Q_k(W(t) - W(s)))|_2^2 = \sum_{j=1}^k (t - s)E |\tilde{\zeta}(s)(f_j)|_2^2.$$

It follows from (4) and (5) that

$$\begin{aligned} E |\tilde{\zeta}(s)(W(t) - W(s))|_2^2 &= \sum_{j=1}^{\infty} (t - s)E |\tilde{\zeta}(s)(f_j)|_2^2 \\ &= (t - s)E |\zeta(s)|_2^2 \quad \text{by (3).} \end{aligned}$$

Clearly, (b) can be shown in the same way.

Now, we are ready to define the linear operator J from $\mathcal{L}[L^2_{(2)}(H; R)]$ into $\mathcal{L}[L^{n-1}(H; R)]$. Let $\xi \in \mathcal{L}[L^n(B; R)]$ be simple with jumps at $0 < t_1 < t_2 < \dots < t_k$. Define, if $t_j \leq t < t_{j+1}$, $0 \leq j \leq k$,

$$J_\xi(t) = \sum_{i=0}^{j-1} \tilde{\xi}(t_i)(W(t_{i+1}) - W(t_i)) + \tilde{\xi}(t_j)(W(t) - W(t_j)).$$

Here $t_0 = 0$ and $t_{k+1} = \infty$ by convention. Clearly $J_\xi \in \mathcal{L}[L^{n-1}(B; R)] \subset \mathcal{L}[L^{n-1}(H; R)]$. Without loss of generality we may assume that $t = t_j$ for some j . Thus

$$J_\xi(t) = \sum_{i=0}^{j-1} \tilde{\xi}(t_i)(W(t_{i+1}) - W(t_i)).$$

Hence

$$|J_\xi(t)|_2^2 = \sum_{i,t=0}^{j-1} \langle \tilde{\xi}(t_i)(W(t_{i+1}) - W(t_i)), \tilde{\xi}(t_l)(W(t_{l+1}) - W(t_l)) \rangle_2.$$

It follows immediately from Lemma 1.4 that

$$(6) \quad \begin{aligned} E |J_\xi(t)|_2^2 &= \sum_{i=0}^{j-1} (t_{i+1} - t_i) E |\xi(t_i)|_2^2 \\ &= E \int_0^t |\xi(s)|_2^2 ds. \end{aligned}$$

Moreover, it is easy to see that

$$(7) \quad E(J_\xi(t) | \mathcal{M}_s) = J_\xi(s) \quad s \leq t.$$

From Lemma 1.3, (6), (7) and a standard argument in stochastic integral, we have

PROPOSITION 1.1. *There exists a linear operator J from $\mathcal{L}[L^2_{(2)}(H; R)]$ into $\mathcal{L}[L^{n-1}(H; R)]$, denoted by $J_\xi(t) = \int_0^t \xi(s) dW(s)$, such that*

- (a) J_ξ has continuous sample paths,
- (b) J_ξ is a martingale,
- (c) $\text{prob} \{ \sup_{0 \leq t \leq \tau} |J_\xi(t)|_2 > \delta \} \leq \delta^{-2} E |J_\xi(\tau)|_2^2$,
- (d) $E J_\xi(t) = 0$ and $E |J_\xi(t)|_2^2 = E \int_0^t |\xi(s)|_2^2 ds$.

DEFINITION 2. Let \mathcal{H} and \mathcal{K} be two Hilbert spaces. A continuous bilinear map S from $\mathcal{H} \times \mathcal{H}$ into \mathcal{K} is said to be of *trace-class-type* if (i) for each $x \in \mathcal{H}$, S_x is a trace class operator of \mathcal{H} , where $S_x(\cdot, \cdot) = \langle S(\cdot, \cdot), x \rangle_x$ and (ii) the linear functional $x \rightarrow \text{trace}_x S_x$ is continuous.

Notation. The definition implies obviously that there exists a unique element, denoted by $\text{TRACE } S$, of \mathcal{K} such that $\langle \text{TRACE } S, x \rangle_x = \text{trace}_x S_x$ for all $x \in \mathcal{K}$. $\mathcal{S}(\mathcal{H}; \mathcal{K})$ will denote the vector space of all trace-class-type bilinear maps from \mathcal{H} into \mathcal{K} .

PROPOSITION 1.2. (a) *If $S \in \mathcal{S}(\mathcal{H}; \mathcal{K})$ and $\{\phi_k\}$ is an orthonormal basis of \mathcal{H} then $\sum_{k=1}^\infty S(\phi_k, \phi_k)$ converges in \mathcal{K} to $\text{TRACE } S$,*

(b) *If $S \in \mathcal{S}(\mathcal{H}; \mathcal{K})$ and $T, U \in L(\mathcal{H}; \mathcal{H}), V \in L(\mathcal{K}; \mathcal{K})$ then $S \circ [T \times U]$ and $V \circ S$ belong to $\mathcal{S}(\mathcal{H}; \mathcal{K})$ and $\text{TRACE } V \circ S = V(\text{TRACE } S)$,*

(c) $L^2(B; L^n(B; R)) \subset \mathcal{S}(H; L^n_{(2)}(H; R))$.

Proof. (a) and (b) appeared in [15] in a similar form. (c) follows from the fact that $L^2(B; R) \approx L(B; B^*) \subset L_1(H; H)$, the Banach space of all trace class operators of H with the trace class norm $|\cdot|_1$. Actually, $|S|_1 \leq \|S\| \int_B \|x\|^2 p_1(dx)$ for all $S \in L^2(B; R)$.

Notation. 1) If $T \in L^n(H; R)$ and $S \in L(L^{n-1}(H; R); L^{n-1}(H; R))$ we define the composition $S \Delta T$ of S and T to be an element of $L^n(H; R)$ by $(S \Delta T)^\sim = S \circ \tilde{T}$. Thus $S \Delta T(h_1, h_2, \dots, h_n) = S(\tilde{T}(h_1))(h_2, \dots, h_n)$.

2) If $S \in L^n(H; R)$ and $T \in L(L^{n-1}(H; R); R)$ we define $S \Delta T$ to be an element of H by: $\langle S \Delta T, h \rangle = T(\tilde{S}(h)), h \in H$. Of course if $S \in L^n_{(2)}(H; R)$ and $T \in L^n_{(2)}(H; R)$ then define $\langle S \Delta T, h \rangle = \langle T, \tilde{S}(h) \rangle_2$.

Remarks. (1) If $T \in L^n_{(2)}(H; R)$ and $L^n_{(2)}(H; R)$ is invariant under S then $S \Delta T \in L^n_{(2)}(H; R)$.

(2) For the case $n = 2$ in Notation 2, it is easy to see that $S \Delta h = S^*h, h \in H$.

In [14] we proved an infinite dimensional analogue of well-known Ito's formula [8]. This formula was used in [17] to show the relation between the work of [14] and that of [19]. Later, in [15] we proved another version of Ito's formula and used it to construct diffusion processes in a Riemann-Wiener manifold. We will give three versions of Ito's formula for stochastic processes with state space $L^n(H; R), n \geq 2$. Let $\mathcal{L}[L^n(H; R)]$ consist of all non-anticipating processes $\zeta(t)$ with state space $L^n(H; R)$ such that $\int_0^\tau E |\zeta(t)|^2 dt < \infty$ for each $0 < \tau < \infty$.

THEOREM 1. (Ito's formula). *Let θ be a twice Frechet differentiable map from $L^n(H; R)$ into itself such that for all $S \in L^n(H; R)$ (i) $\theta'(S)(L^n_{(2)}(H; R)) \subset L^n_{(2)}(H; R)$, (ii) $\theta''(S)(L^n_{(2)}(H; R) \times L^n_{(2)}(H; R)) \subset L^n_{(2)}(H; R)$*

and (iii) $\theta''(S) \in \mathcal{L}(L^2_{(2)}(H; R); L^2_{(2)}(H; R))$. If $\Phi(t) = \Phi_0 + \int_0^t \xi(s)dW(s) + \int_0^t \zeta(s)ds$, where $\xi \in \mathcal{L}[L^2_{(2)}(H; R)]$ and $\zeta \in \mathcal{L}[L^n(H; R)]$. Then

$$\begin{aligned} \theta(\Phi(t)) &= \theta(\Phi_0) + \int_0^t \theta'(\Phi(s)) \triangle \xi(s)dW(s) + \int_0^t \left\{ \theta'(\Phi(s))(\zeta(s)) \right. \\ &\quad \left. + \frac{1}{2} \text{TRACE } \theta''(\Phi(s)) \circ [\xi(s) \times \xi(s)] \right\} ds . \end{aligned}$$

Proof. Kunita-Watanabe’s method [12; 13] can be employed here. We will sketch the outline only. Let $\varepsilon > 0$ and $\{\sigma_j\}$ be an increasing sequence of stopping time converging to ∞ such that $\sigma_0 = 0$ and for $\sigma_j \leq s$, $t < \sigma_{j+1}$, we have

$$\left| \int_s^t \xi(\tau)dW(\tau) \right|_2 < \varepsilon/2$$

and

$$\left| \int_s^t \zeta(s)ds \right| < \varepsilon/2 .$$

Thus, whenever $\sigma_j \leq s$, $t < \sigma_{j+1}$

$$|\Phi(t) - \Phi(s)| < \varepsilon .$$

Because θ is twice Frechet differentiable, we have, whenever x and y are near in $L^n(H; R)$,

$$\theta(x) - \theta(y) = \theta'(y)(x - y) + \frac{1}{2}\theta''(y)(x - y, x - y) + o(|x - y|^2) .$$

Time parameter will be also subscribed from now on. Let $\tau_j = t \wedge \sigma_j$. Thus

$$\begin{aligned} \theta(\Phi(t)) - \theta(\Phi_0) &= \sum_{j=1}^{\infty} [\theta(\Phi_{\tau_j}) - \theta(\Phi_{\tau_{j-1}})] \\ &= \sum_{j=1}^{\infty} \theta'(\Phi_{\tau_{j-1}})(\Phi_{\tau_j} - \Phi_{\tau_{j-1}}) \\ &\quad + \frac{1}{2} \sum_{j=1}^{\infty} \theta''(\Phi_{\tau_{j-1}})(\Phi_{\tau_j} - \Phi_{\tau_{j-1}}, \Phi_{\tau_j} - \Phi_{\tau_{j-1}}) \\ &\quad + o(|\Phi_{\tau_j} - \Phi_{\tau_{j-1}}|^2) . \end{aligned}$$

Putting $\Phi_{\tau_j} - \Phi_{\tau_{j-1}} = \int_{\tau_{j-1}}^{\tau_j} \xi(s)dW(s) + \int_{\tau_{j-1}}^{\tau_j} \zeta(s)ds$ into the above equation, we see that to finish the proof it is sufficient to show the following two equalities:

$$(8) \quad \theta'(\Phi_s) \left(\int_s^t \xi(\tau) dW(\tau) \right) = \int_s^t \theta'(\Phi_s) \Delta \xi(\tau) dW(\tau)$$

$$(9) \quad \theta''(\Phi_s) \left(\int_s^t \xi(\tau) dW(\tau), \int_s^t \xi(\tau) dW(\tau) \right) \\ = \int_s^t \text{TRACE } \theta''(\Phi_s) \circ [\xi(\tau) \times \xi(\tau)] d\tau$$

(8) is easily checked, while (9) follows from the following observation: If $s \leq u < v$ then

$$E\theta''(\Phi_s)(\xi(u)(W(v) - W(u)), \xi(u)(W(v) - W(u))) \\ = (v - u) E \text{ TRACE } \theta''(\Phi(s)) \circ [\xi(u) \times \xi(u)] .$$

If $s \leq u < v \leq u' < v'$ then

$$E\theta''(\Phi_s)(\xi(u)(W(v) - W(u)), \xi(u')(W(v') - W(u'))) = 0 .$$

THEOREM 2 (Ito's formula). *Let Γ be a twice differentiable map from $L^2_n(H; R)$ into itself such that $\Gamma''(S) \in \mathcal{L}(L^2_n(H; R); L^2_n(H; R))$. If $\Phi(t) = \Phi_0 + \int_0^t \xi(s) dW(s) + \int_0^t \zeta(s) ds$, where $\xi \in \mathcal{L}[L^{2n+1}(H; R)]$ and $\zeta \in \mathcal{L}[L^2_n(H; R)]$. Then*

$$\Gamma(\Phi(t)) = \Gamma(\Phi_0) + \int_0^t \Gamma'(\Phi(s)) \Delta \xi(s) dW(s) \\ + \int_0^t \left\{ \Gamma'(\Phi(s))(\zeta(s) + \frac{1}{2} \text{TRACE } \Gamma''(\Phi(s)) \circ [\xi(s) \times \xi(s)]) \right\} ds .$$

THEOREM 3 (Ito's formula). *Let f be a twice Frechet differentiable function from $L^n(H; R)$ (resp. $L^2_n(H; R)$) into R . If $\Phi(t) = \Phi_0 + \int_0^t \xi(s) dW(s) + \int_0^t \zeta(s) ds$, where ξ and ζ are same as Theorem 1 (resp. Theorem 2). Then*

$$f(\Phi(t)) = f(\Phi_0) + \int_0^t (\xi(s) \Delta f'(\Phi(s)), dW(s)) \\ + \int_0^t \{ f'(\Phi(s))(\zeta(s) + \frac{1}{2} \text{trace } f''(\Phi(s)) \circ [\xi(s) \times \xi(s)]) \} ds .$$

Remark. The proof of the above theorems goes in the same way as that of Theorem 1. We point out that $\xi(s) \Delta f'(\Phi(s))$ (see Notation 2) following Proposition 1.2) is a non-anticipating process with state space H and the stochastic integral $\int_0^t (\xi(s) \Delta f'(\Phi(s)), dW(s))$ was defined in the

previous paper [14]. Furthermore, $f''(\Phi(s)) \circ [\tilde{\xi}(s) \times \tilde{\xi}(s)]$ is a non-anticipating process with state space $L_1(H; H)$, the Banach space of all trace class operators of H . To see this, note that if $S \in L^2(L^n(H; R); R)$ and $T \in L^2_{(2)}(H; R)$ then $S \circ [\tilde{T} \times \tilde{T}]$ is a trace class operator of H .

THEOREM 4 (Girsanov-Skorokhod-McKean’s formula). *Suppose $\xi \in \mathcal{L}[L^3_{(2)}(H; R)]$ and $\eta \in \mathcal{L}[L^2(H; R)]$ and with probability 1, $\{\tilde{\xi}(t)(x), \eta(t); 0 \leq t < \infty, x \in H\}$ forms a commutative family of operators of $H(L^2(H; R) \cong L(H; H))$. Then the solution of*

$$(10) \quad Y(t) = I + \int_0^t Y(s) \circ_3 \xi(s) * dW(s) + \int_0^t Y(s) \circ \eta(s) ds$$

can be represented by

$$(11) \quad Y(t) = \exp \left\{ \int_0^t \tilde{\xi}(s) dW(s) + \int_0^t \left\{ \eta(s) - \frac{1}{2} \text{TRACE } \kappa \circ [\tilde{\xi}(s) \times \tilde{\xi}(s)] \right\} ds \right\},$$

where κ is the map from $L(H; H) \times L(H; H)$ into $L(H; H)$ given by $\kappa(S, T) = S \circ T$.

Remark. We will discuss stochastic integral equation below. Moreover, in [16] we define, for $S \in L(H; H)$ and $T \in L^3(H; R)$, $S \Delta T \in L^3(H; R)$ by $(S \Delta T)^\sim(x) = S \circ (\tilde{T}(x))$, $x \in H$. It is easy to see that $S \Delta T$ is nothing but $T \circ_3 S^*$. Thus equation (10) is the same as the equation in § 4 of [16].

Proof. As in one dimensional case, we can solve (10) directly by using log function. Here, we prove this theorem in the reverse direction. Theorem 5 below implies that (10) has a unique solution. Thus it suffices to check that (11) satisfies (10). Consider the function $\theta(x) = \exp(x)$, $x \in L(H; H)$. θ is a C^∞ -function from $L(H; H)$ into itself satisfying the hypothesis of Theorem 1 and, in particular, if x and y commute we have $\theta'(x)y = e^x y$ and $\theta''(x)(y, y) = e^x y^2$. Let

$$\Phi(t) = \int_0^t \tilde{\xi}(s) dW(s) + \int_0^t \left\{ \eta(s) - \frac{1}{2} \text{TRACE } \kappa \circ [\tilde{\xi}(s) \times \tilde{\xi}(s)] \right\} ds.$$

Then $Y(t) = \exp \{\Phi(t)\}$. By stochastic differentiation given in Theorem 1, we have

$$(12) \quad \begin{aligned} dY(t) = & \theta'(\Phi(t)) \Delta \xi(t) dW(t) + \theta'(\Phi(t)) \eta(t) \\ & - \frac{1}{2} \text{TRACE } \kappa \circ [\tilde{\xi}(t) \times \tilde{\xi}(t)] dt \\ & + \frac{1}{2} \text{TRACE } \theta''(\Phi(t)) \circ [\tilde{\xi}(t) \times \tilde{\xi}(t)] dt. \end{aligned}$$

Recall the notation 1) following Proposition 1.2. Let $h_1, h_2, h_3 \in H$

$$\begin{aligned}
 \theta'(\Phi(t)) \triangle \xi(t)(h_1, h_2, h_3) &= \theta'(\Phi(t))(\tilde{\xi}(t)h_1)(h_2, h_3) \\
 &= e^{\theta(\omega)\tilde{\xi}(t)}(\xi(t)(h_1, h_2, h_3)) \quad \text{by commutativity assumption,} \\
 &= \langle Y(t)\tilde{\xi}(t)(h_1)h_2, h_3 \rangle \\
 &= \langle \tilde{\xi}(t)(h_1)h_2, Y(t)^*h_3 \rangle \\
 &= \xi(t)(h_1, h_2, Y(t)^*h_3) \\
 &= \xi(t) \circ_3 Y(t)^*(h_1, h_2, h_3) .
 \end{aligned}$$

Therefore, we have

$$(13) \quad \theta'(\Phi(t)) \triangle \xi(t) = \xi(t) \circ_3 Y(t)^* .$$

Clearly,

$$(14) \quad \theta'(\Phi(t))(\eta(t)) = Y(t) \circ \eta(t) .$$

Moreover, it can be checked easily that

$$(15) \quad \theta'(\Phi(t))(\kappa \circ [\tilde{\xi}(t) \times \tilde{\xi}(t)]) = \theta''(\Phi(t)) \circ [\tilde{\xi}(t) \times \tilde{\xi}(t)] .$$

Putting (13), (14), and (15) into (12), we obtain

$$dY(t) = \xi(t) \circ_3 Y(t)^*dW(t) + Y(t) \circ \eta(t)dt ,$$

or

$$Y(t) = I + \int_0^t \xi(s) \circ_3 Y(s)^*dW(s) + \int_0^t Y(s) \circ \eta(s)ds .$$

THEOREM 5. *Let f and g be maps from $[t_0, \infty) \times L^n(H; R) \times \Omega$ ($t_0 \geq 0, n \geq 2$) into $L^{n+1}_{(2)}(H; R)$ and $L^n(H; R)$, respectively. Assume that f and g satisfy the following conditions:*

- (a) *for each $S \in L^n(H; R)$, $f(\cdot, S, \cdot)$ and $g(\cdot, S, \cdot)$ are non-anticipating,*
- (b) *there is a constant c such that with probability 1,*

$$|f(t, S) - f(t, T)|_2 + |g(t, S) - g(t, T)| \leq c|S - T| ,$$

and

$$|f(t, S)|_2 + |g(t, S)| \leq c(1 + |S|)$$

for all $t \in [t_0, \infty)$ and $S, T \in L^n(H; R)$.

Let $\zeta \in \mathcal{L}[L^n(H; R)]$ have continuous sample paths. Then the $L^n(H; R)$ -valued stochastic integral equation

$$Y(t) = \zeta(t) + \int_{t_0}^t f(s, Y(s))dW(s) + \int_{t_0}^t g(s, Y(s))ds$$

has a unique continuous solution $Y \in \mathcal{L}[L^n(H; R)]$. Moreover, $Y(t)$ is a Markov process if $\zeta(t)$ is so.

Proof. We may assume that $t_0 \leq t \leq t_1 < \infty$. Let \mathfrak{A} be the Banach space of all non-anticipating processes $Y(t)$ in $L^n(H; R)$ with norm

$$|||Y||| = \left\{ \int_{t_0}^{t_1} E|Y(t)|^2 dt \right\}^{1/2} < \infty .$$

Clearly, $\mathcal{L}[L^n(H; R)] \subset \mathfrak{A}$. Define a map Φ in \mathfrak{A} by

$$\Phi(Y)(t) = \zeta(t) + \int_{t_0}^t f(s, Y(s))dW(s) + \int_{t_0}^t g(s, Y(s))ds .$$

It is easy to see that Φ is a map from \mathfrak{A} into itself and $\Phi(Y)$ has continuous sample paths. Furthermore,

$$(16) \quad E|\Phi(Y)(t) - \Phi(Z)(t)|^2 \leq \alpha \int_{t_0}^t E|Y(s) - Z(s)|^2 ds ,$$

where α is a constant depending only on c, t_0 and t_1 . (16) implies that there exists an N such that whenever $m \geq N$,

$$|||\Phi^m(Y) - \Phi^m(Z)||| \leq \frac{1}{2} |||Y - Z||| .$$

The rest of the proof goes in the same way as Theorem 5.1 of [14].

THEOREM 6. *In the hypothesis of Theorem 5 replace $L^n(H; R)$ by $L^n_{(2)}(H; R)$ and $L^n(H; R)$ -norm $|\cdot|$ by $L^n_{(2)}(H; R)$ -norm $|\cdot|_2$. Then the $L^n_{(2)}(H; R)$ -valued stochastic integral equation*

$$Z(t) = \zeta(t) + \int_{t_0}^t f(s, Z(s))dW(s) + \int_{t_0}^t g(s, Z(s))ds$$

has a unique continuous solution $Z \in \mathcal{L}[L^n_{(2)}(H; R)]$. $Z(t)$ is a Markov process if $\zeta(t)$ is so.

Remark. Theorem 5 with $n = 2$ and Theorem 6 with $n \geq 3$ will be used in the next section. Proof of Theorem 6 is obvious.

2. Regularity Properties

We assume that A and σ satisfy the following conditions:

- (A – 1) A is of the form $A(t, x) = C + K(t, x)$, where $C \in L(B; B)$ and K is a continuous map from $[0, \infty) \times B$ into $L_{(2)}(H; H)$,
 (A – 2) There is a constant γ such that for all $t \geq 0$ and $x, y \in B$,

$$\|K(t, x) - K(t, y)\|_2 \leq \gamma \|x - y\| \quad \text{and} \quad \|K(t, x)\|_2 \leq \gamma(1 + \|x\|),$$

- (σ – 1) σ is continuous map from $[0, \infty) \times B$ into B such that for all $t \geq 0$ and $x, y \in B$, $\|\sigma(t, x) - \sigma(t, y)\| \leq \gamma \|x - y\|$ and $\|\sigma(t, x)\| \leq \gamma(1 + \|x\|)$.

Although the above conditions are weaker than those in Theorem 5.1 [14], it is easy to see that the proof there goes in the same way to conclude that under (A – 1), (A – 2) and (σ – 1) the stochastic integral equation (1) has a unique non-anticipating continuous solution. Moreover, this solution is a Markov process. In the sequel, we denote this solution by $X_x(t)$, where x is the starting point.

DEFINITION 3 [7]. A map f from B into a Banach space D is said to be *Frechet differentiable at x in H -directions* (briefly, *H -differentiable at x*) if there exists a linear operator $T \in L(H; D)$ such that $\|f(x + h) - f(x) - T(h)\|_D = o(\|h\|)$, $h \in H$. T is easily checked to be unique and will be denoted by $f'(x)$, called the *H -derivative of f at x* . f is said to be C_H^1 if $f'(x)$ exists for all $x \in B$ and f' is continuous from B into $L(H; D)$. Inductively, we can define *n -th H -differentiability* and C_H^n .

Notation. Let D be a Banach space. $\mathcal{L}(D)$ will denote the Banach space of square integrable random variables taking values in D . Note that in Section 1 we used $\mathcal{L}[D]$ to denote the space of all non-anticipating processes ζ such that $\int_0^\tau E \|\zeta(t)\|_D^2 dt < \infty$ for each $0 \leq \tau < \infty$.

DEFINITION 4. A function ξ from B into $\mathcal{L}(D)$ is said to be *mean-square differentiable at x in H -directions* (briefly, *MS- H -differentiable at x*) if there is $\theta \in \mathcal{L}(L(H; D))$ such that $E \|\xi(x + h) - \xi(x) - \theta(h)\|_D^2 = o(\|h\|^2)$, $h \in H$. θ is unique and will be denoted by $\delta \xi_x$, called the *MS- H -derivative of ξ at x* . *MS- H -differentiability* and *MS- C_H^n* ($n \geq 1$) are defined in an obvious way.

DEFINITION 5. A transformation Z from B into $\mathcal{L}[D]$ is said to be *MS- H -differentiable* if there is a transformation Y from B into

$\mathcal{L}[L(H; D)]$ such that for each $t \geq 0$, $Y(t)$ is the *MS-H*-derivative of $Z(t)$. Y is unique and will be denoted by δZ . Higher order *MS-H*-derivatives will be denoted by $\delta^n Z$, $n \geq 2$.

EXAMPLE 1. Let $X(t) = x + W(t)$, where W is Wiener process starting at the origin. Then $\delta X_x(t) = I$ for all x and $\delta^n X_x(t) \equiv 0$, $n \geq 2$.

EXAMPLE 2. Consider the Langevin equation $dU(t) = dW(t) - U(t)dt$. Its solution $U(t)$ is called Uhlenbeck-Ornstein process. We have $\delta U_x(t) = e^{-t}I$ for all x , $\delta^n U_x(t) \equiv 0$, $n \geq 2$.

EXAMPLE 3. Let $K \in L_{(2)}(H; H)$, $T \in L(B; H)$ and $x_0 \in H \cap \ker T^*$, where $*$ denotes the adjoint of operators of H . Consider the equation $dX(t) = (I + K)dW(t) + f(|TX(t)|^2)x_0 dt$, where f is a real-valued differentiable function with compact support. We have $\delta X_x(t) = e^{\zeta_x(t)}$, where $\zeta_x \in \mathcal{L}[L(H; H)]$ is given by $\zeta_x(t) = 2 \left[\int_0^t f'(|TX_x(s)|^2) \langle T^2 X_x(s), \cdot \rangle ds \right] x_0$.

Remark. Two transformations Z_1 and Z_2 from B into $\mathcal{L}[D]$ have the same *MS-H*-derivative if and only if there exists $\xi \in \mathcal{L}[D]$ such that $Z_1 - Z_2 \equiv \xi$. Moreover, if ξ is an *MS-H*-differential function from B into $\mathcal{L}(D)$ then $\xi_{x+h} - \xi_x = \int_0^1 \delta \xi_{x+\tau h}(h) d\tau$, $x \in B$ and $h \in H$.

PROPOSITION 2.1. Suppose $\xi \in \mathcal{L}[L_{(2)}^{n+1}(H; R)]$, $n \geq 0$. Then $E |J_\xi(t)|_2^4 = 2 \int_0^t E |J_\xi(s)|_2^2 |\xi(s)|_2^2 ds + 4 \int_0^t E |\xi(s) \Delta J_\xi(s)|^2 ds$.

Remark. $\xi \Delta J_\xi \in \mathcal{L}[H]$. See Notation 2) following Proposition 1.2.

Proof. Apply Ito's formula in Theorem 3 to the function $f(x) = |x|_2^4$, $x \in L_{(2)}^n(H; R)$ and to the process $J_\xi(t) = \int_0^t \xi(s) dW(s)$, $\xi \in \mathcal{L}[L_{(2)}^{n+1}(H; R)]$. Note that $f'(x) = 4|x|_2^2 \langle x, \cdot \rangle_2$ and $f''(x) = 4|x|_2^2 \langle \cdot, \cdot \rangle_2 + 8 \langle x, \cdot \rangle_2 \langle x, \cdot \rangle_2$. Hence we have

$$f(J_\xi(t)) = \int_0^t (\xi(s) \Delta f'(J_\xi(s)), dW(s)) + \frac{1}{2} \int_0^t \text{trace } f''(J_\xi(s)) \circ [\tilde{\xi}(s) \times \xi(s)] ds .$$

After taking expectation, we get

$$(17) \quad E |J_\xi(t)|_2^4 = \frac{1}{2} \int_0^t E \text{trace } f''(J_\xi(s)) \circ [\tilde{\xi}(s) \times \xi(s)] ds .$$

Let $\{e_i\}$ be an orthonormal basis of H , then

$$\begin{aligned}
 (18) \quad & \text{trace } |J_\xi(s)|_2^2 \langle \tilde{\xi}(s), \tilde{\xi}(s) \rangle_2 \\
 &= \sum_j |J_\xi(s)|_2^2 \langle \tilde{\xi}(s)e_j, \tilde{\xi}(s)e_j \rangle_2 \\
 &= |J_\xi(s)|_2^2 |\xi(s)|_2^2,
 \end{aligned}$$

and

$$\begin{aligned}
 (19) \quad & \text{trace } \langle J_\xi(s), \tilde{\xi}(s) \rangle_2 \langle J_\xi(s), \tilde{\xi}(s) \rangle_2 \\
 &= \sum_j \langle J_\xi(s), \tilde{\xi}(s)e_j \rangle_2^2 \\
 &= \sum_j \langle \xi(s) \Delta J_\xi(s), e_j \rangle^2 \\
 &= |\xi(s) \Delta J_\xi(s)|^2.
 \end{aligned}$$

Combining (17), (18), and (19), we obtain the conclusion.

PROPOSITION 2.2. *Suppose $\xi \in \mathcal{L}[L_{(2)}^{n+1}(H; R)]$, $n \geq 0$. Then $E|J_\xi(t)|_2^4 \leq 36t \int_0^t E|\xi(s)|_2^4 ds$.*

Proof. First note that from the previous proposition $E|J_\xi(t)|_2^4$ is an increasing function of t . Hence

$$\begin{aligned}
 E[|J_\xi(s)|_2^2 |\xi(s)|_2^2] &\leq \{E|J_\xi(s)|_2^4\}^{1/2} E|\xi(s)|_2^4 \\
 &\leq \{E|J_\xi(t)|_2^4\}^{1/2} \{E|\xi(s)|_2^4\}^{1/2}.
 \end{aligned}$$

Now, recall that for $S \in L_{(2)}^{n+1}(H; R)$ and $T \in L_{(2)}^n(H; R)$, $S \Delta T$ is an element in H defined by $\langle S \Delta T, h \rangle = \langle T, \tilde{S}(h) \rangle_2$. Thus we have $|S \Delta T| \leq |T|_2 |S|_2$. So,

$$\begin{aligned}
 E|\xi(s) \Delta J_\xi(s)|^2 &\leq E[|\xi(s)|_2^2 |J_\xi(s)|_2^2] \\
 &\leq \{E|J_\xi(t)|_2^4\}^{1/2} \{E|\xi(s)|_2^4\}^{1/2}.
 \end{aligned}$$

Therefore, by the previous proposition

$$\begin{aligned}
 E|J_\xi(t)|_2^4 &\leq 6\{E|J_\xi(t)|_2^4\}^{1/2} \int_0^t \{E|\xi(s)|_2^4\}^{1/2} ds \\
 &\leq 6\{E|J_\xi(t)|_2^4\}^{1/2} \left\{ t \int_0^t E|\xi(s)|_2^4 ds \right\}^{1/2},
 \end{aligned}$$

and

$$E|J_\xi(t)|_2^4 \leq 36t \int_0^t E|\xi(s)|_2^4 ds.$$

Notation. 1) Let $S \in L^n(H; R)$. $\hat{S} \in L^{n-1}(H; H)$ is defined by $\langle \hat{S}h, \cdot \rangle$

$h_2, \dots, h_{n-1}, h\rangle = S(h, h_1, h_2, \dots, h_{n-1})$. Note that for $n = 2$, $\tilde{S} = S$, while $\hat{S} = S^*$.

2) Let $T \in L^3(H; R)$ and $S \in L^n(H; R)$. Define $S : T \in L^{n+1}(H; R)$ by $S : T(h_1, h_2, \dots, h_n, h_{n+1}) = T(h_1, \hat{S}(h_2, \dots, h_n), h_{n+1})$. Note that if $T \in L^3_{(2)}(H; R)$ and $S \in L^n_{(2)}(H; R)$ then $S : T \in L^{n+1}_{(2)}(H; R)$ and $|S : T|_2 \leq |S|_2 |T|_2$. But for $n = 2$, $|S : T|_2 \leq |S| |T|_2$.

Remark. If $T \in L^3(H; R)$, $S \in L^2(H; R)$ and $\tilde{T}(h)$ commutes with S for all $h \in H$. Then $S : T = T \circ_3 S$.

THEOREM 7. Assume A and σ satisfy (A – 1), (A – 2), (σ – 1) and the following conditions :

(A – 3) $K(t, x)$ is C^2_H in x variable with $K'(t, x) \in L^3_{(2)}(H; R)$ and $K''(t, x) \in L^4_{(2)}(H; R)$. $K'(\cdot, \cdot)$ and $K''(\cdot, \cdot)$ are bounded continuous maps from $[0, \tau] \times B$ into $L^3_{(2)}(H; R)$ and $L^4_{(2)}(H; R)$, respectively, for each τ .

(A – 4) for all t and x , $K'(t, x) \in L^3_{(2)}(H; R)$ and $K''(t, x) \in L^4_{(2)}(H; R)$ are symmetric in the first two components,

(σ – 2) $\sigma(t, x)$ is C^2_H in x variable with $\sigma'(t, x) \in L(H; H)$ and $\sigma''(t, x) \in L^3_{(2)}(H; R)$. $\sigma'(\cdot, \cdot)$ and $\sigma''(\cdot, \cdot)$ are bounded continuous maps from $[0, \tau] \times B$ into $L(H; H)$ and $L^3_{(2)}(H; R)$ respectively, for each τ . Then the diffusion process given by the solution of the stochastic integral equation

$$(20) \quad X(t) = X(0) + \int_0^t A(s, X(s))dW(s) + \int_0^t \sigma(s, X(s))ds$$

is twice MS-H-differentiable. The first derivative at x is given by the solution of the operator-valued stochastic integral equation

$$(21) \quad Y(t) = I + \int_0^t Y(s) : K'(s, X_x(s))dW(s) + \int_0^t \sigma'(s, X_x(s)) \circ Y(s)ds .$$

The second derivative at x is given by the solution of the 3-form-valued stochastic integral equation

$$(22) \quad Z(t) = \phi(t) + \int_0^t Z(s) : K'(s, X_x(s))dW(s) + \int_0^t Z(s) \circ_3 \sigma''(s, X_x(s))*ds ,$$

where

$$\phi(t) = \int_0^t K''(s, X_x(s)) \circ_4 [\delta X_x(s)]^2 *dW(s) + \int_0^t \sigma''(s, X_x(s)) \circ [\delta X_x(s) \times \delta X_x(s)]ds .$$

Furthermore, $\delta X_x \in \mathcal{L}[L(H; H)]$ and $\delta^2 X_x \in \mathcal{L}[L^3_{(2)}(H; R)]$.

Proof. We need to show that $E \|X_{x+h}(t) - X_x(t) - Y(t)h\|^2 = o(|h|^2)$, $h \in H$. But it is easy to see that $X_{x+h}(t) - X_x(t)$ is in H . Thus we will show below a stronger statement, namely, $E |X_{x+h}(t) - X_x(t) - Y(t)h|^2 = o(|h|^2)$, $h \in H$. Assume $0 \leq t \leq \tau$. Let $\psi_h(t) = X_{x+h}(t) - X_x(t)$. Then

$$(23) \quad \psi_h(t) = h + \int_0^t \xi_h(s)(\psi_h(s))dW(s) + \int_0^t \zeta_h(s)(\psi_h(s))ds ,$$

where $\xi_h(s)$ and $\zeta_h(s)$ are given by

$$(24) \quad \xi_h(s) = \int_0^1 A'(s, X_x(s) + \tau\psi_h(s))d\tau = \int_0^1 K'(s, X_x(s) + \tau\psi_h(s))d\tau ,$$

$$(25) \quad \zeta_h(s) = \int_0^1 \sigma'(s, X_x(s) + \tau\psi_h(s))ds .$$

On the other hand,

$$(26) \quad Y(t)h = h + \int_0^t K'(s, X_x(s))(Y(s)h)dW(s) + \int_0^t \sigma'(s, X_x(s))(Y(s)h)ds .$$

Here we have used the condition (A - 4) to bring h into the integral sign.

Now, it can be shown with some computation that

$$(27) \quad \begin{aligned} E |\xi_h(s)(\psi_h(s)) - K'(s, X_x(s))(Y(s)h)|^2 \\ \leq c_1 E |\psi_h(s) - Y(s)h|^2 + c_2 E (|\psi_h(s)|^2 |Y(s)h|^2) , \end{aligned}$$

and

$$(28) \quad \begin{aligned} E |\zeta_h(s)(\psi_h(s)) - \sigma'(s, X_x(s))(Y(s)h)|^2 \\ \leq c_1 E |\psi_h(s) - Y(s)h|^2 + c_2 E (|\psi_h(s)|^2 |Y(s)h|^2) , \end{aligned}$$

where c_1 and c_2 are constants independent of s and t . From (23)–(28), we obtain for all $0 \leq t \leq \tau$,

$$E |\psi_h(t) - Y(t)h|^2 \leq c_3 \lambda(h) + c_4 \int_0^t E |\psi_h(s) - Y(s)h|^2 ds ,$$

where c_3 and c_4 are constants independent of t , and

$$\lambda(h) = \int_0^\tau E |\psi_h(s)|^2 |Y(s)h|^2 ds .$$

Hence by Gronwall’s Lemma,

$$E |\psi_h(t) - Y(t)h|^2 \leq c_3 \lambda(h) e^{c_4 t} \quad 0 \leq t \leq \tau .$$

But $\lambda(h) \leq |h|^2 \int_0^\tau E |\psi_h(s)|^2 \|Y(s)\|_H^2 ds$, hence we are remained to prove that

$$(29) \quad \lim_{|h| \rightarrow 0} \int_0^\tau E |\psi_h(s)|^2 \|Y(s)\|_H^2 ds = 0 .$$

By a complicated computation using Proposition 2.2 and Gronwall’s Lemma, we have

$$E |\psi_h(t)|^4 \leq \text{constant } |h|^4 , \quad 0 \leq t \leq \tau ,$$

and

$$E \|Y(t)\|_H^4 \leq \text{constant} , \quad 0 \leq t \leq \tau .$$

Hence (29) is evident and, in particular, we have also that $Y \in \mathcal{L}[L(H; H)]$.

We should not try to prove the second assertion. But we will show that ϕ given in (22) is in $\mathcal{L}[L^2_{(2)}(H; R)]$. ϕ is clearly non-anticipating.

$$(30) \quad \begin{aligned} |\phi(t)|_2^2 &\leq 2 \left| \int_0^t K''(s, X_x(s)) \circ_4 [\delta X_x(s)^2]^* dW(s) \right|_2^2 \\ &\quad + 2 \left| \int_0^t \hat{\sigma}''(s, X_x(s)) \circ [\delta X_x(s) \times \delta X_x(s)] ds \right|_2^2 . \end{aligned}$$

Apply Proposition 1.1 to get

$$(31) \quad \begin{aligned} E \left| \int_0^t K''(s, X_x(s)) \circ_4 [\delta X_x(s)^2]^* dW(s) \right|_2^2 &= \int_0^t E |K''(s, X_x(s)) \circ_4 [\delta X_x(s)^2]^*|_2^2 ds \\ &\leq \int_0^t E \|\delta X_x(s)\|_H^2 |K''(s, X_x(s))|_2^2 ds \\ &\leq \alpha \int_0^t E \|\delta X_x(s)\|_H^2 ds , \end{aligned}$$

where $\alpha = \sup_{0 \leq s \leq \tau, x \in B} |K''(s, x)|_2^2 < \infty$.

On the other hand,

$$(32) \quad \begin{aligned} E \left| \int_0^t \hat{\sigma}''(s, X_x(s)) \circ [\delta X_x(s) \times \delta X_x(s)] ds \right|_2^2 &\leq t \int_0^t E |\hat{\sigma}''(s, X_x(s)) \circ [\delta X_x(s) \times \delta X_x(s)]|_2^2 ds \\ &\leq t \int_0^t E |\sigma''(s, X_x(s))|_2^2 \|\delta X_x(s)\|_H^2 ds \\ &\leq \beta t \int_0^t E \|\delta X_x(s)\|_H^2 ds , \end{aligned}$$

where $\beta = \sup_{0 \leq s \leq \tau, x \in B} |\sigma''(s, X_x(s))|_2^2 < \infty$.

Note that we have used the property that if $S \in L^3_{(2)}(H; R)$ and $T \in L(H; H)$ then $\hat{S} \circ [T \times T] \in L^3_{(2)}(H; R)$ and $|\hat{S} \circ [T \times T]|_2 \leq |S|_2 \|T\|_2^2$. This can be seen by observing that $\hat{S} \circ [T \times T] = (S \circ_1 T) \circ_2 T$ and then applying Lemma 1.2. (30), (31) and (32) clearly show that $\int_0^\tau E |\phi(t)|_2^2 dt < \infty$ for each $0 \leq \tau < \infty$. Hence $\phi \in \mathcal{L}[L^3_{(2)}(H; R)]$.

THEOREM 8. Assume A and σ satisfy (A – 1), (A – 2), (σ – 1) and the following conditions:

(A – 3)* $K(t, x)$ is C^n_H ($n \geq 2$) in x variable with $K^{(j)}(t, x) \in L^{j+2}_{(2)}(H; R)$, $j = 1, 2, \dots, n$. $K^{(j)}$ is bounded and continuous from $[0, \tau) \times B$ into $L^{j+2}_{(2)}(H; R)$ for each $0 \leq \tau < \infty$, $j = 1, 2, \dots, n$.

(A – 4)* for all t and x , $K^{(j)}(t, x) \in L^{j+2}_{(2)}(H; R)$ is symmetric in the first two components, $j = 1, 2, \dots, n$.

(σ – 2)* $\sigma(t, x)$ is C^n_H ($n \geq 2$) in x variable with $\sigma'(t, x) \in L(H; H)$ and $\sigma^{(j)}(t, x) \in L^{j+1}_{(2)}(H; R)$, $j = 2, 3, \dots, n$. σ' and $\sigma^{(j)}$ are bounded, continuous from $[0, \tau) \times B$ into $L(H; H)$ and $L^{j+1}_{(2)}(H; R)$, respectively, for each $0 \leq \tau < \infty$, $j = 2, 3, \dots, n$.

Then the diffusion process $X(t)$ given by the solution of the equation

$$X(t) = X(0) + \int_0^t A(s, X(s))dW(s) + \int_0^t \sigma(s, X(s))ds$$

is n -th MS- H -differentiable. Furthermore, $\delta X \in \mathcal{L}[L(H; H)]$ and $\delta^j X \in \mathcal{L}[L^{j+1}_{(2)}(H; R)]$, $j = 2, 3, \dots, n$.

THEOREM 9. Suppose A and σ satisfy the conditions (A – 1), (A – 2), (A – 3)*, (A – 4)*, (σ – 1) and (σ – 2)*. Let $X(t)$ be the diffusion process given by the diffusion coefficients A and σ . If f is a C^n_H -function in B with bounded derivatives, $0 \leq k \leq n$, then the function $\theta(x) = E_x[f(X(t))]$ is also C^k_H . Its first two H -derivatives are

$$(33) \quad \theta'(x) = E[\delta X_x(t)^*(f'(X_x(t)))] ,$$

$$(34) \quad \theta''(x) = E\{\delta^2 X_x(t)^\vee(f'(X_x(t))) + f''(X_x(t)) \circ [\delta X_x(t) \times \delta X_x(t)]\} .$$

Moreover, $\theta''(x)$ is a Hilbert-Schmidt operator of H for all $x \in B$ if f'' is so.

Notation. If $S \in L^n(H; R)$ then $S^\vee \in L(H; L^{n-1}(H; R))$ is defined to be $\check{S}(h) = S(\cdot, \cdot, \dots, \cdot, h)$. Note that if $S \in L^n_{(2)}(H; R)$ then $\check{S}(h) \in L^{n-1}_{(2)}(H; R)$

and \check{S} is a Hilbert-Schmidt operator from H into $L^2_{(2)}(H; R)$.

Proof. Let $\psi_h(t) = X_{x+h}(t) - X_x(t)$. Then

$$\begin{aligned} f(X_{x+h}(t)) - f(X_x(t)) &= \int_0^1 \langle f'(X_x(t) + \tau\psi_h(t)), \psi_h(t) \rangle d\tau. \end{aligned}$$

Hence

$$\begin{aligned} f(X_{x+h}(t)) - f(X_x(t)) - \langle \delta X_x(t)^*(f'(X_x(t))), h \rangle &= \int_0^1 \langle f'(X_x(t) + \tau\psi_h(t)) - f'(X_x(t)), \psi_h(t) \rangle d\tau \\ &\quad + \langle f'(X_x(t)), \psi_h(t) - \delta X_x(t)h \rangle = \alpha(h) + \beta(h). \end{aligned}$$

Obviously, $E|\beta(h)| = o(|h|)$ since f' is bounded and $X(t)$ is *MS-H*-differentiable.

On the other hand,

$$\begin{aligned} E|\alpha(h)| &\leq \int_0^1 E|f'(X_x(t) + \tau\psi_h(t)) - f'(X_x(t))| |\psi_h(t)| d\tau \\ &\leq \{E|\psi_h(t)|^2\}^{1/2} \int_0^1 \{E|f'(X_x(t) + \tau\psi_h(t)) - f'(X_x(t))|^2\}^{1/2} d\tau \\ &\leq c|h| \left\{ \int_0^1 E|f'(X_x(t) + \tau\psi_h(t)) - f'(X_x(t))|^2 d\tau \right\}^{1/2}, \end{aligned}$$

where c is a constant independent of h . Apply Lebesgue's dominated convergence theorem to conclude that $E|\alpha(h)| = o(|h|)$. Therefore,

$$E|f(X_{x+h}(t)) - f(X_x(t)) - \langle \delta X_x(t)^*(f'(X_x(t))), h \rangle| = o(|h|), \quad h \in H.$$

This proves (33). (34) can be proved in a similar way. Furthermore, $\theta^{(j)}(x)$ ($3 \leq j \leq k$) can be expressed by using the first j -th derivatives of f and $X(t)$. Finally $\theta''(x)$ is a Hilbert-Schmidt operator by the remark in Notation above and by the property: if $S \in L^2_{(2)}(H; R)$ and $T \in L(H; H)$ then $S \circ [T \times T] \in L^2_{(2)}(H; R)$. In fact, $S \circ [T \times T] = (S \circ_1 T) \circ_2 T$, hence $\|S \circ [T \times T]\|_2 \leq \|S\|_2 \|T\|_H^2$ by Lemma 1.2

To finish this paper, we consider the homogeneous case, i.e. A and σ are independent of t . A and σ satisfy (A-1), (A-2) and (σ -1). In this case $X(t)$ generates a semi-group $\{P_t; t \geq 0\}$, $P_t f(x) = E_x[f(X(t))]$. Let C_0 be the Banach space of bounded continuous functions on \bar{B} vanishing at infinity. C_0 has the sup norm. Assume the B -norm $\|\cdot\|^2$ is C^2_H

such that its second H -derivative has bounded range in $L_1(H; H)$.

THEOREM 10. *The operators P_t , $t \geq 0$, form a strongly continuous contraction semi-group on C_0 .*

Proof.* P_t , $t \geq 0$ are obviously strongly continuous and contractive. We need only to show that $P_t f \in C_0$ whenever $f \in C_0$.

Let $\theta(x) = \log(1 + \|x\|^2)$, $x \in B$. θ is C_H^2 with $|\theta'(x)| \leq C_1 \|x\|(1 + \|x\|^2)^{-1}$ and $\|\theta''(x)\|_1 \leq C_2(1 + \|x\|^2)^{-1}$, where C_1 and C_2 are two constants independent of x and $\|\cdot\|_1$ denotes the trace class norm. Apply Ito's formula (Theorem 4.1 [14]) to the function θ and the process $X(t)$,

$$\begin{aligned} X(t) &= x + \int_0^t A(X(s))dW(s) + \int_0^t \sigma(X(s))ds . \\ \theta(X(t)) &= \theta(x) + \int_0^t (A^*(X(s))\theta'(X(s)), dW(s)) \\ &\quad + \int_0^t [(\theta'(X(s)), \sigma(X(s))) + \frac{1}{2} \text{trace } A^*(X(s))\theta''(X(s))A(X(s))]ds . \end{aligned}$$

It follows easily that

$$E(\theta(X(t)) - \theta(x))^2 \leq \text{constant} = a ,$$

or

$$(35) \quad E \left[\log \frac{1 + \|X(t)\|^2}{1 + \|x\|^2} \right]^2 \leq a .$$

Now, let $f \in C_0$ and $g = P_t f$. Let $\varepsilon > 0$ be given and N be large enough that

$$|f(x)| < \varepsilon/2 \quad \text{whenever } \|x\| > N .$$

But

$$g(x) = E_{(\|X(t)\| > N)} f(X(t)) + E_{(\|X(t)\| \leq N)} f(X(t)) .$$

Hence,

$$|g(x)| \leq \varepsilon/2 + \|f\|_\infty \text{prob} \{\|X(t)\| \leq N\} ,$$

and

* We learn the proof from Professor K. Ito through a private conversation.

$$\begin{aligned}
 &\text{prob } \{\|X(t)\| \leq N\} \\
 &\leq \text{prob } \left\{ \log \frac{1 + \|X(t)\|^2}{1 + \|x\|^2} \leq \log \frac{1 + N^2}{1 + \|x\|^2} \right\} \\
 &= \text{prob } \left\{ \log \frac{1 + \|X(t)\|^2}{1 + \|x\|^2} \leq -\log \frac{1 + \|x\|^2}{1 + N^2} \right\}, \quad \|x\| > N, \\
 &\leq \text{prob } \left\{ \left| \log \frac{1 + \|X(t)\|^2}{1 + \|x\|^2} \right| \geq \log \frac{1 + \|x\|^2}{1 + N^2} \right\} \\
 &\leq \left(\log \frac{1 + \|x\|^2}{1 + N^2} \right)^{-2} \times E \left[\log \frac{1 + \|X(t)\|^2}{1 + \|x\|^2} \right]^2 \\
 &\leq a \left(\log \frac{1 + \|x\|^2}{1 + N^2} \right)^{-2} \leq \frac{\varepsilon}{2 \|f\|_\infty} \quad \text{if } \|x\| \text{ is large.}
 \end{aligned}$$

Thus for large $\|x\|$ we have $|g(x)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$. Therefore $g \in C_0$.

Appendix

It is a pleasure to thank Professor Loren Pitt for pointing out the fact that $P_t, t \geq 0$, being strongly continuous (Theorem 10) is not obvious. We present a proof as follows.

LEMMA A.1 (Gronwall’s inequality). *If h is a non-negative integrable function in $[0, a]$, $a < \infty$, satisfying*

$$h(t) \leq g(t) + \alpha \int_0^t h(s) ds,$$

where $\alpha > 0$ and g is integrable in $[0, a]$. Then

$$h(t) \leq g(t) + \alpha \int_0^t e^{\alpha(t-s)} g(s) ds.$$

Proof. We can prove inductively that

$$\begin{aligned}
 h(t) &\leq g(t) + \alpha \int_0^t \left\{ \sum_{k=1}^{n-1} [\alpha(t-s)]^k / k! \right\} g(s) ds \\
 &\quad + \alpha \int_0^t h(s) [\alpha(t-s)]^n / n! ds.
 \end{aligned}$$

The conclusion then follows from Lebesgue’s dominated coverage theorem.

LEMMA A.2. $E_x \|X(t) - x\|^2 \leq ct(1 + \|x\|^2)$ for all $0 \leq t \leq 1$ and all $x \in B$, where c is a constant independent of t and x .

Proof. We use the letter c to stand for any constant independent of t and x . Let $0 \leq t \leq 1$.

$$\begin{aligned} X(t) &= x + \int_0^t A(X(s))dW(s) + \int_0^t \sigma(X(s))ds \\ &= x + CW(t) + \int_0^t K(X(s))dW(s) + \int_0^t \sigma(X(s))ds . \end{aligned}$$

Hence

$$\begin{aligned} (36) \quad \|X(t) - x\|^2 &\leq c \left\{ \|CW(t)\|^2 + \left\| \int_0^t K(X(s))dW(s) \right\|^2 + \left\| \int_0^t \sigma(X(s))ds \right\|^2 \right\} \\ &\leq c \left\{ \|CW(t)\|^2 + \left| \int_0^t K(X(s))dW(s) \right|^2 + \int_0^t \|\sigma(X(s))\|^2 ds \right\} . \end{aligned}$$

(Recall that $\|\cdot\|$ is dominated by $|\cdot|$).

Thus after taking expectation (36) becomes

$$\begin{aligned} E \|X(t) - x\|^2 &\leq c \left\{ t + \int_0^t E \|K(X(s))\|_2^2 ds + \int_0^t E \|\sigma(X(s))\|^2 ds \right\} \\ &\leq c \left\{ t + c \int_0^t E(1 + \|X(s)\|^2) ds + c \int_0^t E(1 + \|X(s)\|^2) ds \right\} \\ &\leq c \left\{ t + \int_0^t E \|X(s)\|^2 ds \right\} \\ &\leq c \left\{ t + c \int_0^t E[\|X(s) - x\|^2 + \|x\|^2] ds \right\} \\ &\leq c \left\{ t(1 + \|x\|^2) + \int_0^t E \|X(s) - x\|^2 ds \right\} . \end{aligned}$$

Hence by Lemma A.1 we have

$$\begin{aligned} E \|X(t) - x\|^2 &\leq ct(1 + \|x\|^2) + c \int_0^t e^{c(t-s)} cs(1 + \|x\|^2) ds \\ &\leq ct(1 + \|x\|^2) + c \int_0^t e^{c(t-s)} ct(1 + \|x\|^2) ds \\ &= ct(1 + \|x\|^2) \left[1 + c \int_0^t e^{c(t-s)} ds \right] \\ &\leq ct(1 + \|x\|^2) . \end{aligned}$$

Now let $f \in C_0$ be also uniformly continuous. A close examination of the proof of Theorem 10 shows that given $\varepsilon > 0$ there exists N independent of t , $0 \leq t \leq 1$

$$|P_t f(x)| < \varepsilon/2 \quad \text{whenever } \|x\| > N .$$

We may as well assume that

$$|f(x)| < \varepsilon/2 \quad \text{whenever } \|x\| > N.$$

Thus we have for all $0 \leq t \leq 1$

$$(37) \quad |P_t f(x) - f(x)| < \varepsilon \quad \text{whenever } \|x\| > N.$$

On the other hand, let $\delta > 0$ be such that

$$\|x - y\| < \delta \quad \text{implies} \quad |f(x) - f(y)| < \varepsilon/2.$$

Then for $\|x\| \leq N$,

$$\begin{aligned} |P_t f(x) - f(x)| &\leq E_x |f(X(t)) - f(x)| \\ &= E_{\{\|X(t) - x\| < \delta\}} |f(X(t)) - f(x)| \\ &\quad + E_{\{\|X(t) - x\| \geq \delta\}} |f(X(t)) - f(x)| \\ &\leq \varepsilon/2 + 2 \|f\|_\infty \text{prob} \{\|X(t) - x\| \geq \delta\}. \end{aligned}$$

But

$$\begin{aligned} \text{prob} \{\|X(t) - x\| \geq \delta\} &\leq \delta^{-2} E \|X(t) - x\|^2 \\ &\leq \delta^{-2} ct(1 + \|x\|^2) \quad \text{by Lemma A.2} \\ &\leq \delta^{-2} ct(1 + N^2). \end{aligned}$$

Therefore we can choose t_0 small enough such that whenever $t \leq t_0$

$$(38) \quad |P_t f(x) - f(x)| \leq \varepsilon \quad \text{for all } \|x\| \leq N.$$

Clearly (37) and (38) yield that

$$\|P_t f - f\|_\infty \leq \varepsilon \quad \text{whenever } t \leq t_0.$$

This establishes the strong continuity of P_t , $t \geq 0$.

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