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Harmonic Polynomials Associated With Reflection Groups

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Abstract. We extend Maxwell's representation of harmonic polynomials to h-harmonics associated to a reflection invariant weight function h_k. Let $\mathcal{D}_i$, $1 \le i \le d$, be Dunkl's operators associated with a reflection group. For any homogeneous polynomial P of degree n, we prove the polynomial $|\mathbf{x}|^{2\gamma+d-2+2n}P(\mathcal{D})\{1/|\mathbf{x}|^{2\gamma+d-2}\}$ is a h-harmonic polynomial of degree n, where $\gamma = \sum k_i$ and $\mathcal{D} = (\mathcal{D}_1, \dots, \mathcal{D}_d)$. The construction yields a basis for h-harmonics. We also discuss self-adjoint operators acting on the space of h-harmonics.

1 Introduction

Among many properties satisfied by the harmonic polynomials, there is Maxwell's theory of poles, which states that for any multiindex $\alpha \in \mathbb{N}^d$, $|\alpha|_1 = \alpha_1 + \cdots + \alpha_d = n$, the polynomials $|\mathbf{x}|^{2n+d-2}\partial^\alpha\{|\mathbf{x}|^{-d+2}\}$ is a harmonic polynomial of degree n in $\mathbb{R}^d$, where $\partial^\alpha = \partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}$ and $|\mathbf{x}|$ is the usual Euclidean norm of $\mathbf{x} \in \mathbb{R}^d$ ([8, p. 251] and [9, Chapter 4]). Moreover, it is also known that for any $\eta \in S^{d-1}$, the sphere shell in $\mathbb{R}^d$, and $d \ge 3$,

$$|\mathbf{x}|^{2n+d-2}\langle\eta,\partial\rangle^n\{|\mathbf{x}|^{-d+2}\} = (-1)^n|\mathbf{x}|^nC_n^{((d-2)/2)}(\langle\mathbf{x}',\eta\rangle), \tag{1.1}$$

where $\mathbf{x} = r\mathbf{x}'$ with $r = |\mathbf{x}|$, $\langle\mathbf{x},\mathbf{y}\rangle$ denote the usual inner product of $\mathbb{R}^d$ and $C_n^{(\lambda)}$ is the Gegenbauer polynomial of degree n with index λ. This formula is called *Maxwell's representation* in [10, p. 69], we note that the sign $(-1)^n$ is missing in [10]. Since the right hand side of (1.1) is the zonal polynomial, the collection of Maxwell's representation consists of a basis of harmonic polynomials.

The purpose of this paper is to extend Maxwell's construction to h-harmonics developed by Dunkl [3–6] recently. The theory of h-harmonics is analogous to the theory of ordinary harmonics, it uses finite reflection groups in place of orthogonal group in the classical theory. The role of the partial differentials is replaced by a family of commutative differential-difference operators, called *Dunkl's operators*, and the surface measure is replaced by measures invariant under the reflection groups. Let G be a finite reflection group on $\mathbb{R}^d$ with the set $R_+ = \{\mathbf{v}_i : i = 1, 2, \dots, m\}$ of positive roots; assume that $|\mathbf{v}_i| = |\mathbf{v}_j|$ whenever σ_i is conjugate to σ_j in G, where $\sigma_i = \sigma_{\mathbf{v}_i}$, $1 \le i \le m$, are reflections with respect to $\mathbf{v}_i$. For a nonzero vector $\mathbf{v} \in \mathbb{R}^d$ the reflection $\sigma_\mathbf{v}$ is defined by $\mathbf{x}\sigma_\mathbf{v} := \mathbf{x} - 2(\langle\mathbf{x},\mathbf{v}\rangle/|\mathbf{v}|^2)\mathbf{v}$, $\mathbf{x} \in \mathbb{R}^d$. Then G is a subgroup of the orthogonal group generated by the reflections $\{\sigma_i : 1 \le i \le m\}$. Let k be a multiplicity function defined on R_+,

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which is an m-tuple of real numbers k_i, $1 \le i \le m$, such that $k_i = k_j$ whenever σ_i is conjugate to σ_j in G. The differential-difference operators, $\mathcal{D}_i$ (Dunkl's operators), associated to G and k are defined by ([4])

$$\mathcal{D}_i f(\mathbf{x}) := \partial_i f(\mathbf{x}) + \sum_{j=1}^{m} k_j \frac{f(\mathbf{x}) - f(\mathbf{x}\sigma_j)}{\langle \mathbf{x}, \mathbf{v}_j \rangle} \langle \mathbf{v}_j, \mathbf{e}_i \rangle, \quad 1 \le i \le d, \tag{1.2}$$

where ∂_i is the ordinary partial derivative with respect to x_i and $\mathbf{e}_1, \ldots, \mathbf{e}_d$ are the standard unit vectors of $\mathbb{R}^d$. It is proved in [4] that $\mathcal{D}_1, \ldots, \mathcal{D}_d$ commute, that is, $\mathcal{D}_i \mathcal{D}_j = \mathcal{D}_j \mathcal{D}_i$ for all $1 \le i, j \le d$. The h-Laplacian, which plays the role similar to that of the ordinary Laplacian, is defined by

$$\Delta_h = \mathcal{D}_1^2 + \cdots + \mathcal{D}_d^2. \tag{1.3}$$

The fundamental relation between the h-Laplacian and the orthogonality is as follows. For $k_i \ge 0$, $1 \le i \le m$, we consider the inner product defined on polynomials

$$(f, g)_{h_k} = \int_{S^{d-1}} f(\mathbf{x}) g(\mathbf{x}) h_k^2(\mathbf{x})\, d\omega, \quad h_k(\mathbf{x}) := \prod_{i=1}^{m} |\langle \mathbf{x}, \mathbf{v}_i \rangle|^{k_i}. \tag{1.4}$$

The function h_k is a positively homogeneous G-invariant function of degree γ, where for abbreviation, we introduce the index $\gamma = \sum_{i=1}^{m} k_i$. Let $\mathcal{P}_n := \mathcal{P}_n^d$ denote the space of homogeneous polynomials of degree n in $\mathbf{x} = (x_1, \ldots, x_d)$. A polynomial $P \in \mathcal{P}_n^d$ is orthogonal to all polynomials of degree less than n with respect to $(f, g)_{h_k}$ if and only if $\Delta_h P = 0$ ([3]). The elements of $\mathcal{H}_n^h := \mathcal{P}_n^d \cap \ker \Delta_h$ are called *h-harmonics*, and $\mathcal{H}_n^h$ is the space of h-harmonic polynomials of degree n. When $h_\alpha = 1$ the h-harmonics become the ordinary harmonics, which satisfy the classical Laplacian equation $\Delta P = 0$. The dimension of $\mathcal{H}_n^h$ is

$$\dim \mathcal{H}_n^h = \dim \mathcal{P}_n - \dim \mathcal{P}_{n-2} = \binom{n+d-1}{n} - \binom{n+d-3}{n-2}.$$

For the general theory and many important properties of h-harmonics, we refer to [3–7] and the references there. There is also the connection to the multivariate orthogonal polynomials associated to the quantum Calogero models. We refer to [1, 2] and the references therein.

We are interested in finding a basis for h-harmonics. We extend Maxwell's construction to h-harmonics in Section 2, and discuss the operators acting on the space of h-harmonics in Section 3, where we use h-harmonics associated with the symmetric group S_n as an example.

2 Construction of h-Harmonics

We start with the following two basic formulae about the action of Dunkl's operators and h-Laplacian:

Lemma 2.1 *Let λ be a real number and $g \in \mathcal{P}_n$. Then*

$$\mathcal{D}_i(|\mathbf{x}|^\lambda g) = \lambda x_i |\mathbf{x}|^{\lambda-2} g + |\mathbf{x}|^\lambda \mathcal{D}_i g, \tag{2.1}$$

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and

$$\Delta_h(|\mathbf{x}|^\lambda g) = 2\lambda(d/2+\lambda/2-1+n+\gamma)|\mathbf{x}|^{\lambda-2}g + |\mathbf{x}|^\lambda \Delta_h g, \tag{2.2}$$

where if $\lambda < 0$, *then both of these identities hold at* $\mathbb{R}^d \setminus \{0\}$.

Proof Since $|\mathbf{x}|^\lambda$ is invariant under the action of the reflection group, it follows from the definition of $\mathcal{D}_i$ in (1.2) that

$$\mathcal{D}_i(|\mathbf{x}|^\lambda g) = \partial(|\mathbf{x}|^\lambda)g + |\mathbf{x}|^\lambda \mathcal{D}_i g,$$

from which (2.1) follows. The identity (2.2) can be proved using (2.1). For $\lambda = 2m$, m an integer, (2.2) also appeared in [3, Lemma 1.9]; the proof there holds for all λ real. ■

Definition 2.2 For any $\alpha \in \mathbb{N}^d$, We define homogeneous polynomials H_α by

$$H_\alpha(\mathbf{x}) := |\mathbf{x}|^{2\gamma+d-2+2|\alpha|_1}\mathcal{D}^\alpha\{|\mathbf{x}|^{-2\gamma-d+2}\}, \tag{2.3}$$

where $\mathcal{D} = (\mathcal{D}_1, \dots, \mathcal{D}_d)$ and $\mathcal{D}^\alpha = \mathcal{D}_1^{\alpha_1}\cdots\mathcal{D}_d^{\alpha_d}$.

The fact that H_α is a homogeneous polynomial, in fact, an h-harmonic, is proved in the following theorem. Let us introduce the notation that $\mathbf{e}_i = (0, \dots, 1, \dots, 0)$, $1 \le i \le d$, where 1 is at the i-th position.

Theorem 2.3 *For each* $\alpha \in \mathbb{N}^d$, H_α *is an h-harmonic polynomial of degree* $|\alpha|_1$, *that is,* $H_\alpha \in \mathcal{H}^h_{|\alpha|_1}$. *Moreover,* H_α *satisfy the recursive relation*

$$H_{\alpha+\mathbf{e}_i}(\mathbf{x}) = -(2\gamma+d-2+2|\alpha|_1)x_iH_\alpha(\mathbf{x}) + |\mathbf{x}|^2\mathcal{D}_iH_\alpha. \tag{2.4}$$

Proof First we prove that H_α is a homogeneous polynomial of degree $|\alpha|_1$. We use induction on $n = |\alpha|_1$. Clearly $H_0(\mathbf{x}) = 1$. Assume that H_α has been proved to be a homogeneous polynomial of degree n for $|\alpha|_1 = n$. Using the identity (2.1) it follows that

$$\begin{aligned}\mathcal{D}_iH_\alpha &= (2\gamma+d-2+2|\alpha|_1)x_i|\mathbf{x}|^{2\gamma+d-4+2|\alpha|_1}\mathcal{D}^\alpha\{|\mathbf{x}|^{-2\gamma-d+2}\}\\ &\quad + |\mathbf{x}|^{2\gamma+d-2+2|\alpha|_1}\mathcal{D}_i\mathcal{D}^\alpha\{|\mathbf{x}|^{-2\gamma-d+2}\},\end{aligned}$$

from which the recursive formula (2.4) follows from the definition of H_α. Since $\mathcal{D}_i \colon \mathcal{P}_n \mapsto \mathcal{P}_{n-1}$, it follows from the recursive formula that $H_{\alpha+\mathbf{e}_i}$ is a homogeneous polynomial of degree $n+1 = |\alpha|_1 + 1$.

Next we prove that H_α is an h-harmonic, that is, we show that $\Delta_hH_\alpha = 0$. Setting $\lambda = -2n-2\gamma-(d-2)$ in the identity (2.2), we conclude that

$$\Delta_h\{|\mathbf{x}|^{-2n-2\gamma-d+2}g\} = |\mathbf{x}|^{-2n-2\gamma-(d-2)}\Delta_hg,$$

for $g \in \mathcal{P}_n$. In particular, for $g = 1$ and $n = 0$, we conclude that $\Delta_h(|\mathbf{x}|^{-2\gamma-d+2}) = 0$ for $\mathbf{x} \in \mathbb{R}^d \setminus \{0\}$. Hence, setting $g = H_\alpha$, $|\alpha|_1 = n$ and using the fact that $\mathcal{D}_1, \dots, \mathcal{D}_d$ are commuting, it follows that

$$\begin{aligned}\Delta_hH_\alpha &= |\mathbf{x}|^{2n+2\gamma+(d-2)}\Delta_h\mathcal{D}^\alpha\{|\mathbf{x}|^{-2\gamma-d+2}\}\\ &= |\mathbf{x}|^{2n+2\gamma+(d-2)}\mathcal{D}^\alpha\Delta_h\{|\mathbf{x}|^{-2\gamma-d+2}\} = 0,\end{aligned}$$

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which holds for all $\mathbf{x} \in \mathbb{R}^d$ since H_α is a polynomial. ■

If $\gamma = 0$, then the h-harmonics reduce to the ordinary harmonics. The definition of H_α is then reduced to Maxwell's construction of harmonics.

It turns out that H_α is related to the projection operator $\operatorname{proj}_{\mathcal{H}_n^h} \colon \mathcal{P}_n \mapsto \mathcal{H}_n^h$. In [3] Dunkl proved that the projection operator is given by

$$\operatorname{proj}_{\mathcal{H}_n^h} P = \sum_{j=0}^{[n/2]} \frac{1}{4^j j!(-\gamma - n + 2 - d/2)_j} |\mathbf{x}|^{2j} \Delta_h^j P, \tag{2.5}$$

where the normalization is determined by the fact that $\operatorname{proj}_{\mathcal{H}_n^h} Q = Q$ for $Q \in \mathcal{H}_n^h$.

Theorem 2.4 *For $\alpha \in \mathbb{N}^d$ and $|\alpha|_1 = n$,*

$$H_\alpha(\mathbf{x}) = (-1)^n 2^n (\gamma - 1 + d/2)_n \operatorname{proj}_{\mathcal{H}_n^h} \{\mathbf{x}^\alpha\}. \tag{2.6}$$

Proof We use induction on $n = |\alpha|_1$. The case $n = 0$ is evident. Suppose that the equation has been proved for all α such that $|\alpha|_1 = n$. By the definition of H_α we have

$$\mathcal{D}^\alpha \{|\mathbf{x}|^{-2\gamma-(d-2)}\} = a_n |\mathbf{x}|^{-2\gamma-2n-(d-2)} \sum_{j=0}^{[n/2]} \frac{1}{4^j j!(\gamma - n + 2 - d/2)_j} |\mathbf{x}|^{2j} \Delta_h^j \{\mathbf{x}^\alpha\},$$

where $a_n = (-1)^n 2^n (\gamma - 1 + d/2)_n$. Applying $\mathcal{D}_i$ to this equation and using the identity (2.1) with $g = \Delta_h^j \{\mathbf{x}^\alpha\}$, we conclude, after carefully computing the coefficients, that

$$\begin{aligned} &\mathcal{D}_i \mathcal{D}^\alpha \{|\mathbf{x}|^{-2\gamma-(d-2)}\} \\ &\quad = a_n(-2\gamma - 2n - d + 2) |\mathbf{x}|^{-2\gamma-2n-d} \\ &\qquad \times \sum_{j=0}^{[(n+1)/2]} \frac{1}{4^j j!(\gamma - n + 2 - d/2)_j} |\mathbf{x}|^{2j} [x_i \Delta_h^j \{\mathbf{x}^\alpha\} + 2j \Delta_h^{j-1} \mathcal{D}_i \{\mathbf{x}^\alpha\}], \end{aligned}$$

where we have also used the fact that $\mathcal{D}_i$ commutes with Δ_h. Now, using the identity

$$\Delta_h^j \{x_i f(\mathbf{x})\} = x_i \Delta_h^j f(\mathbf{x}) + 2j \mathcal{D}_i \Delta_h^{j-1} f(\mathbf{x}), \quad j = 1, 2, 3, \ldots,$$

(see, for example, [4, p. 173]) and the fact that $a_{n+1} = a_n(-2\gamma - 2n - d + 2)$, we conclude that $H_{\alpha+\mathbf{e}_i} = a_{n+1} \operatorname{proj}_{\mathcal{H}_{n+1}^h} \{\mathbf{x}^{\alpha+\mathbf{e}_i}\}$, which completes the induction procedure. ■

An immediate consequence of this theorem is the following corollary.

Corollary 2.5 *The projection operator* $\operatorname{proj} \colon \mathcal{P}_n \mapsto \mathcal{H}_n^h$ *is given by,*

$$\operatorname{proj}_{\mathcal{H}_n^h} P(\mathbf{x}) = \frac{1}{(-1)^n 2^n (\gamma - 1 + d/2)_n} |\mathbf{x}|^{2n+2\gamma+d-2} P(\mathcal{D}) \{|\mathbf{x}|^{-2\gamma-d+2}\}. \tag{2.7}$$

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The theorem shows that there is a one-to-one correspondence between $\mathbf{x}^\alpha$ and H_α. Since every h-harmonic in $\mathcal{H}_n^h$ can be written as a linear combination of $\mathbf{x}^\alpha$ with $|\alpha| = n$, we conclude that the set $\{H_\alpha : |\alpha|_1 = n\}$ contains a basis of $\mathcal{H}_n^h$. However, the h-harmonics in this set are not linearly independent, since there are $\dim \mathcal{P}_n$ of them which is more than $\dim \mathcal{H}_n^h$. Nevertheless, it is not hard to derive a basis from this set of H_α. In fact, the linearly dependent relations among the set $\{H_\alpha : |\alpha|_1 = n\}$ are given by

$$H_{\beta+2\mathbf{e}_1} + \cdots + H_{\beta+2\mathbf{e}_d} = |\mathbf{x}|^{2n+2\gamma+d-2}\mathcal{D}^\beta \Delta_h\{|\mathbf{x}|^{-2\gamma-d+2}\} = 0$$

for each $\beta \in \mathbb{N}^d$ such that $|\beta|_1 = n-2$. There are exactly $\dim \mathcal{P}_{n-2} = \#\{\beta \in \mathbb{N}^d : |\beta|_1 = n-2\}$ linear dependent relations. For each of these relations, we can exclude one polynomial from the set $\{H_\alpha : |\alpha|_1 = n\}$. The remain $\dim \mathcal{H}_n^h$ $(= \dim \mathcal{P}_n - \dim \mathcal{P}_{n-2})$ harmonics still span $\mathcal{H}_n^h$; hence, they form a basis for $\mathcal{H}_n^h$. We note that the basis is not unique, since we can exclude any one of the polynomials $H_{\beta+2\mathbf{e}_1}, \ldots, H_{\beta+2\mathbf{e}_d}$ for each dependent relation. For example, we may exclude $H_\beta + 2\mathbf{e}_d$ for all $|\beta|_1 = n-2$ from $\{H_\alpha : |\alpha|_1 = n\}$ to obtain a basis. We summarize the results in the following.

Corollary 2.6 *The set $\{H_\alpha : |\alpha|_1 = n\}$ contains a basis of $\mathcal{H}_n^h$. Moreover, one particular basis can be taken as $\{H_\alpha : |\alpha|_1 = n, \alpha_d = 0, 1\}$.*

For example, if $n = 2$ and $d = 3$, then a basis of $\mathcal{H}_2^h$ consists of polynomials

$$\{H_{1,1,0},\ H_{1,0,1},\ H_{0,1,1},\ H_{2,0,0},\ H_{0,2,0}\}.$$

Such a basis for $\mathcal{H}_n^h$, however, is not an orthonormal one; that is, although H_α are orthogonal to polynomials of lower degree with respect to the inner product (1.4), they are not orthogonal to each other.

We consider the extension of Maxwell's representation (1.1) of the ordinary harmonics in the following. We need the intertwining operator V between the commuting algebra of differential operators and the algebra of Dunkl's operators. The intertwining operator V is the unique linear operator defined by

$$V\mathcal{P}_n \subset \mathcal{P}_n, \quad V1 = 1, \quad \mathcal{D}_i V = V\partial_i, \quad 1 \le i \le d.$$

In [5], Dunkl introduced the kernel $K_n(\mathbf{x},\mathbf{y})$ defined by

$$K_n(\mathbf{x},\mathbf{y}) = V_{\mathbf{x}}(\langle \mathbf{x},\mathbf{y}\rangle^n/n!),$$

where $V_{\mathbf{x}}$ means that V is acting on the variable $\mathbf{x}$. There are many properties of K_n, for example, $K_n(\mathbf{x},\mathbf{y}) = K_n(\mathbf{y},\mathbf{x})$ and $\mathcal{D}_i K_n(\mathbf{x},\mathbf{y}) = y_i K_{n-1}(\mathbf{x},\mathbf{y})$. In [5], it is shown that the reproducing kernel of the space $\mathcal{H}_n^h$ is given by

$$P_n^h(\mathbf{x},\mathbf{y}) = 2^n(\gamma + d/2)_n \sum_{j=0}^{[n/2]} \frac{1}{4^j j!(-\gamma - n + 2 - d/2)_j} |\mathbf{x}|^{2j} K_{n-2j}(\eta, \mathbf{x}).$$

Since $\gamma = 0$ implies that $V = id$ and that h-harmonics becomes the ordinary harmonics, the following theorem gives the analogy of Maxwell's representation for h-harmonics.

Theorem 2.7 *For $\eta \in S^{d-1}$, $\mathbf{x} = |\mathbf{x}|\mathbf{x}' \in \mathbb{R}^d$,*

$$|\mathbf{x}|^{2n+2\gamma+(d-2)} K_n(\eta, \mathcal{D})\{|\mathbf{x}|^{-2\gamma-d+2}\} = (-1)^n |\mathbf{x}|^n V[C_n^{(\gamma+(d-2)/2)}(\langle \cdot, \eta\rangle)](\mathbf{x}').$$

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Proof From the equation $\mathcal{D}_i K_n(\mathbf{x},\mathbf{y}) = y_i K_{n-1}(\mathbf{x},\mathbf{y})$, it follows that $\Delta_h K_n(\mathbf{x},\eta) = K_{n-2}(\mathbf{x},\eta)$ since $|\eta|^2 = 1$. Together with Theorem 2.4 and the formula (2.5), we conclude that

$$\begin{aligned}
|\mathbf{x}|^{2n+2\gamma+(d-2)} & K_n(\eta,\mathcal{D})\{|\mathbf{x}|^{-2\gamma-d+2}\} \\
&= (-1)^n 2^n (\gamma-1+d/2)_n \operatorname{proj}_{\mathcal{H}_n^h} K_n(\eta,\mathbf{x}) \\
&= (-1)^n 2^n (\gamma-1+d/2)_n \sum_{j=0}^{[n/2]} \frac{1}{4^j j!(-\gamma-n+2-d/2)_j} |\mathbf{x}|^{2j} K_{n-2j}(\eta,\mathbf{x}) \\
&= (-1)^n \frac{\gamma-1+d/2}{n+\gamma-1+d/2} P_n^h(\eta,\mathbf{x})
\end{aligned}$$

On the other hand, we showed in [14] that for $|\mathbf{x}'| = |\eta| = 1$, we have

$$P_n^h(\mathbf{x}',\eta) = \frac{n+\gamma-1+d/2}{\gamma-1+d/2} V[C_n^{(\gamma+(d-2)/2)}(\langle\cdot,\eta\rangle)](\mathbf{x}').$$

Since $P_n^h(\mathbf{x},\eta) \in \mathcal{H}_n^h$, it is homogeneous in $\mathbf{x}$; hence, we can factor $|\mathbf{x}|^n$ out and use the above equation to finish the proof. ∎

The importance of the intertwining operator is that it allows to transform certain properties of the ordinary harmonics to h-harmonics. The above theorem gives just one more example. At this moment, the explicit formula of V is known only in the case of $G = \mathbb{Z}_2^d$ ([13], and [5] for $d = 1$) and S_3 ([6]). Recently, the operator has been proved to be positive by M. Rösler in [12], confirming a conjecture by Dunkl.

For ordinary harmonics, more can be said about the representation. For example, it is known ([8, p. 251] or [9, p. 134]) that for $d = 3$,

$$|\mathbf{x}|^{n+1}(\partial_1 + i\partial_2)^{n-m}\partial_3^m\{|\mathbf{x}|^{-1}\} = (-1)^{n-m}(n-m)! e^{\pm m\phi} P_n^m(\cos\theta),$$

leads to an orthonormal basis for the spherical harmonics, where P_n^m are the the associated Legendre's functions, $\partial_i = \partial/\partial x_i$, and we use the standard spherical coordinates $x_1 = r\sin\theta\sin\phi$, $x_2 = r\sin\theta\cos\phi$, $x_3 = r\cos\theta$, where $r^2 = x_1^2 + x_2^2 + x_3^2$. However, we do not know what is a proper analogy of this formula for h-harmonics. In fact, we are not aware any construction of orthonormal basis for $\mathcal{H}_n^h$ at this point. In order to construct an orthonormal basis from H_α we need to be able to compute the inner product $(H_\alpha, H_\beta)_{h_k}$ for $|\alpha|_1 = |\beta|_1$, which turns out to be difficult. One immediate consequence of the recursive relation (2.4) is the following result.

Lemma 2.8 *If both p and q are elements of $\mathcal{H}_n^h$, then*

$$\int_{S^{d-1}} x_i p q h^2 \, d\omega = 0, \quad 1 \le i \le d. \tag{2.8}$$

Proof Consider H_α and H_β with $|\alpha|_1 = |\beta|_1 = n$, it follows from (2.4) that

$$(2\gamma+d-2+2n)\int_{S^{d-1}} x_i H_\alpha H_\beta h^2\, d\omega = \int_{S^{d-1}} \mathcal{D}_i H_\alpha H_\beta h^2\, d\omega - \int_{S^{d-1}} H_{\alpha+\mathbf{e}_i} H_\beta h^2\, d\omega = 0,$$

which implies the desired result since $\{H_\alpha : |\alpha|_1 = n\}$ contains a basis for $\mathcal{H}_n^h$. ∎

3 Operators Acting on the h-Harmonics

Next we recall that the adjoint D_i^* of the operator D_i on $\mathcal{H}_n^h$ is defined by

$$(p, \mathcal{D}_i q)_{h_k} = (\mathcal{D}_i^* p, q)_{h_k}, \quad p \in \mathcal{H}_n^h\, q \in \mathcal{H}_{n+1}^h.$$

It follows that D_i^* is a linear operator that maps $\mathcal{H}_n^h$ into $\mathcal{H}_{n+1}^h$. The definition of H_α provides an easy access to D_i^*.

Theorem 3.1 *The action of the adjoint $\mathcal{D}_i^*$ on $H_\alpha \in \mathcal{H}_n^h$ is given by*

$$\mathcal{D}_i^* H_\alpha = -\frac{2n + 2\gamma + d}{2n + 2\gamma + d - 2} H_{\alpha+\mathbf{e}_i}. \tag{3.1}$$

Proof Let $H_\alpha \in \mathcal{H}_n^h$ and $H_\beta \in \mathcal{H}_{n+1}^h$. Using (2.8) and the recursive relation (2.4) twice, we conclude that

$$(H_\alpha, \mathcal{D}_i H_\beta)_{h_k} = (2\gamma + 2n + d)(x_i H_\alpha, H_\beta)_{h_k} = -\frac{2\gamma + 2n + d}{2\gamma + 2n + d - 2}(H_{\alpha+\mathbf{e}_i}, H_\beta)_{h_k},$$

which leads to the desired result. ∎

Since $\{H_\alpha : |\alpha|_1 = n\}$ contains a basis for $\mathcal{H}_n^h$, upon using the recursive relation (2.4) we end up with an alternative proof to the following result in [4, Theorem 2.1].

Theorem 3.2 *For $P \in \mathcal{H}_n^h$,*

$$\mathcal{D}_i^* P = (d + 2n + 2\gamma)\big(x_i P - (d + 2n + 2\gamma - 2)^{-1} |\mathbf{x}|^2 \mathcal{D}_i P\big). \tag{3.2}$$

The Theorem 3.1 makes it easier to derive properties of D_i^*. Among the operators acting on $\mathcal{P}_n$, the operators $\mathcal{D}_i \mathcal{D}_i^*$ and $\mathcal{D}_i^* \mathcal{D}_i$ are of particular interest, since they are self-adjoint with respect to the inner product (1.4) and they map $\mathcal{H}_n^h$ to $\mathcal{H}_n^h$. Other interesting operators include $\mathcal{D}_i \mathcal{D}_j^* - \mathcal{D}_j \mathcal{D}_i^*$ and $\mathcal{D}_j \mathcal{D}_i^* - \mathcal{D}_i \mathcal{D}_j^*$ whose square are self-adjoint on $\mathcal{H}_n^h$. It turns out that the latter two operators differ only by a constant, and they are also constant multiple of the operator $x_i \mathcal{D}_j - x_j \mathcal{D}_i$, which has been used in [7] to find orthogonal decompositions of $\mathcal{H}_n^h$. Let $\mathcal{A}_{i,j}$ be operators defined by

$$\mathcal{A}_{i,j} P = \mathcal{D}_j(x_i P) - x_i \mathcal{D}_j P, \quad P \in \cup_n \mathcal{P}_n,\ i = 1, 2, \ldots, d.$$

Lemma 3.3 *The operators $\mathcal{A}_{i,j}$ are self-adjoint and they satisfy $\mathcal{A}_{i,j} = \mathcal{A}_{j,i}$. Moreover,*

$$\mathcal{A}_{i,j} f(\mathbf{x}) = \delta_{i,j} f(\mathbf{x}) + 2\sum_{s=1}^{m} k_s f(\mathbf{x}\sigma_s) \langle \mathbf{v}_s, \mathbf{e}_i \rangle \langle \mathbf{v}_s, \mathbf{e}_j \rangle / \langle \mathbf{v}_j, \mathbf{v}_j \rangle. \tag{3.3}$$

where $\delta_{i,j} = 0$ if $i \neq j$ and $\delta_{i,j} = 1$ if $i = j$.

Proof From the definition of $\mathcal{D}_i$, we write $\mathcal{D}_i = \partial_i + D_i$, where the operator D_i consists of the difference part. Then, using the equation that

$$D_j(x_i) = 2\sum_{s=1}^{m} k_s \langle \mathbf{v}_s, \mathbf{e}_i\rangle\langle \mathbf{v}_s, \mathbf{e}_j\rangle / \langle \mathbf{v}_s, \mathbf{v}_s\rangle = D_i(x_j),$$

a straightforward calculation yields that for any polynomial f, we have

$$\begin{aligned}\mathcal{D}_j(x_i f) &= \partial_j(x_i f) + x_i D_j(f) + 2\sum_{s=1}^{m} k_s f(\mathbf{x}\sigma_s)\langle \mathbf{v}_s, \mathbf{e}_i\rangle\langle \mathbf{v}_s, \mathbf{e}_j\rangle / \langle \mathbf{v}_s, \mathbf{v}_s\rangle \\ &= x_i\mathcal{D}_j(f) + \delta_{i,j} f + 2\sum_{s=1}^{m} k_s f(\mathbf{x}\sigma_s)\langle \mathbf{v}_s, \mathbf{e}_i\rangle\langle \mathbf{v}_s, \mathbf{e}_j\rangle / \langle \mathbf{v}_s, \mathbf{v}_s\rangle.\end{aligned}$$

This leads to the explicit formula of $\mathcal{A}_{i,j}$ and it implies that $\mathcal{A}_{i,j}$ is symmetric with respect to i, j. Since h^2 is G-invariant and for any reflection σ_s we have

$$\int_{S^{d-1}} f(\mathbf{x}\sigma_j) g(\mathbf{x}) h^2(\mathbf{x})\, d\omega = \int_{S^{d-1}} f(\mathbf{x}) g(\mathbf{x}\sigma_j) h^2(\mathbf{x})\, d\omega,$$

which shows, by (3.3), that $\mathcal{A}_{i,j}$ is self-adjoint. ■

Lemma 3.4 *For $f \in \mathcal{P}_n$, the operator $\mathcal{B}_{i,j}$ defined by*

$$\mathcal{B}_{i,j} f = (2\gamma + 2n + d - 2)^2 \mathcal{D}_j \mathcal{D}_i^* f - (2\gamma + 2n + d)^2 \mathcal{D}_i^* \mathcal{D}_j f$$

is self-adjoint. Moreover,

$$\begin{aligned}(2\gamma + 2n + d)^2(\mathcal{D}_i^*\mathcal{D}_j - \mathcal{D}_j^*\mathcal{D}_i) f &= (2\gamma + 2n + d - 2)^2 (\mathcal{D}_j\mathcal{D}_i^* - \mathcal{D}_i\mathcal{D}_j^* f) \\ &= -(2\gamma + 2n + d - 2)(2\gamma + 2n + d)^2 (x_j\mathcal{D}_i - x_i\mathcal{D}_j) f.\end{aligned}$$

Proof From the fact that $A_{i,j} = A_{j,i}$ it follows that

$$\mathcal{D}_j(x_i f) - \mathcal{D}_i(x_j f) = x_i\mathcal{D}_j f - x_j\mathcal{D}_i f.$$

By Theorem 3.1, $\mathcal{D}_i^* H_\alpha$ is a multiple of $H_{\alpha+\mathbf{e}_i}$. Hence, apply $\mathcal{D}_j$ on the recursive relation (2.4), we conclude that

$$\begin{aligned}-\frac{2\gamma + 2n + d - 2}{2\gamma + 2n + d}(\mathcal{D}_j\mathcal{D}_i^* - \mathcal{D}_i\mathcal{D}_j^*) H_\alpha &= -(2\gamma + 2n + d - 2)\big(\mathcal{D}_j(x_i H_\alpha) \\ &\quad - \mathcal{D}_i(x_j H_\alpha)\big) + 2(x_j\mathcal{D}_i H_\alpha - x_i\mathcal{D}_j H_\alpha) \\ &= (2\gamma + 2n + d)(x_j\mathcal{D}_i H_\alpha - x_i\mathcal{D}_j H_\alpha).\end{aligned}$$

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Moreover, using the fact that $\mathcal{D}_i H_\alpha \in \mathcal{H}^h_{n-1}$ and Theorem 3.2, we conclude that

$$\begin{aligned} -\frac{2\gamma+2n+d-2}{2\gamma+2n+d}\mathcal{D}_j\mathcal{D}_i^* H_\alpha &= -(2\gamma+2n+d-2)\mathcal{D}_j(x_i H_\alpha) + 2x_j\mathcal{D}_i H_\alpha + |\mathbf{x}|^2\mathcal{D}_i\mathcal{D}_j H_\alpha \\ &= -\frac{2\gamma+2n+d-4}{2\gamma+2n+d-2}\mathcal{D}_i^*\mathcal{D}_j H_\alpha - (2\gamma+2n+d-2)\mathcal{A}_{i,j}H_\alpha \\ &\quad + 2x_j\mathcal{D}_i H_\alpha - 2x_i\mathcal{D}_j H_\alpha. \end{aligned}$$

Putting these two formulae together, we obtain, after rearranging terms, that

$$\begin{aligned} -\frac{(2\gamma+2n+d-2)(2\gamma+2n+d-4)}{(2\gamma+2n+d)^2}&\mathcal{D}_j\mathcal{D}_i^* H_\alpha \\ &= -\frac{2\gamma+2n+d-4}{2\gamma+2n+d-2}\mathcal{D}_i^*\mathcal{D}_j H_\alpha \\ &\quad - (2\gamma+2n+d-2)\mathcal{A}_{i,j}H_\alpha \\ &\quad + 2\frac{2\gamma+2n+d-2}{(2\gamma+2n+d)^2}(\mathcal{D}_i\mathcal{D}_j^* + \mathcal{D}_j\mathcal{D}_i^*)H_\alpha. \end{aligned}$$

Since both $\mathcal{A}_{i,j}$ and $\mathcal{D}_i\mathcal{D}_j^* + \mathcal{D}_j\mathcal{D}_i^*$ are self-adjoint, we conclude that $\mathcal{B}_{i,j}$ are self-adjoint. The desired identity follows from the fact that $\mathcal{B}_{i,j} = \mathcal{B}_{i,j}^* = \mathcal{B}_{j,i}$. ∎

In [7] Dunkl has used the operators $\mathcal{R}_{i,j} := (x_i\mathcal{D}_j - x_j\mathcal{D}_i)^2$ to generate a sequence of commuting self-adjoint operators acting on the h-harmonics. For the group $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, the operators he constructed take the form $\mathcal{D}_i^*\mathcal{D}_i + \sum_{j=1}^{i-1}\mathcal{R}_{i,j}$, and their eigenfunctions form an orthonormal basis for h-harmonics. We can construct a sequence of commuting operators using $\mathcal{A}_{i,j}$ for the symmetric group.

Symmetric Group The group $G = S_d$ is the symmetric group on d objects. In this case, the operator $\mathcal{D}_i$ takes the form

$$\mathcal{D}_i f(\mathbf{x}) = \partial_i f(\mathbf{x}) + k_0 \sum_{j\neq i} \frac{f(\mathbf{x}) - f\big((i,j)\mathbf{x}\big)}{x_i - x_j},$$

where (i,j) is the transposition of x_i and x_j, $(i,j)\mathbf{x} = (\dots, x_j, \dots, x_i, \dots)$. In this case, it can be easily verified that

$$\mathcal{A}_{i,j} = k_0(i,j) \quad \text{and} \quad \mathcal{A}_{i,i} = I + k_0 \sum_{l=1, l\neq i}^{d} (i,l)$$

where I stands for the identity operator.

Theorem 3.5 *In the case of symmetric group the operators $\mathcal{C}_i$ defined by $\mathcal{C}_i = \mathcal{A}_i - \sum_{j=1}^{i-1}\mathcal{A}_{i,l}$ is a sequence of commuting self-adjoint operators.*

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Proof Using the formulae of $\mathcal{A}_{i,j}$ it is easy to see that $\mathcal{A}_{i,j}$ are commuting amongst each other as long as the indices belong to disjoint sets, and we also have $\mathcal{A}_{i,i}\mathcal{A}_{j,l} = \mathcal{A}_{j,l}\mathcal{A}_{i,i}$ for $i \neq j$, $i \neq l$. We claim that

$$[\mathcal{A}_{i,i} - \mathcal{A}_{i,j}, \mathcal{A}_{j,j}] = 0, \quad i > j, \text{ and } [\mathcal{A}_{i,l} + \mathcal{A}_{j,l}, \mathcal{A}_{i,j}] = 0, \quad i \neq l \neq j, \tag{3.4}$$

where $[A, B] = AB - BA$ is the commutator. Indeed, we have

$$\begin{aligned}[\mathcal{A}_{i,i} - \mathcal{A}_{i,j}, \mathcal{A}_{j,j}] &= k_0^2\Big[\sum_{l\neq i, l\neq j}(i,l),(i,j) + \sum_{l\neq i, l\neq j}(j,l)\Big] \\ &= k_0^2 \sum_{l\neq i, l\neq j}([(i,l),(i,j)] + [(i,l),(j,l)] = 0,\end{aligned}$$

since $[(i,l),(i,j)] = (i,j,l) - (i,l,j) = -[(i,l),(j,l)]$. The second claimed equation follows similarly. Now, for $i > j$, we have by (3.4) that

$$\begin{aligned}[\mathcal{C}_i, \mathcal{C}_j] &= [\mathcal{A}_{i,i} - \mathcal{A}_{i,j}, \mathcal{A}_{j,j}] - \Big[\sum_{l=1, l\neq j}^{i-1}\mathcal{A}_{i,l}, \mathcal{A}_{j,j}\Big] - \Big[\mathcal{A}_{i,i} - \sum_{l=1}^{i-1}\mathcal{A}_{i,l}, \sum_{l=1}^{j-1}\mathcal{A}_{j,l}\Big] \\ &= \Big[\sum_{l=1}^{j}\mathcal{A}_{i,l} \sum_{l=1}^{j-1}\mathcal{A}_{j,l}\Big] = \sum_{l=1}^{j-1}[\mathcal{A}_{i,j} + \mathcal{A}_{i,l}, \mathcal{A}_{j,l}] = 0,\end{aligned}$$

which proves the desired commuting result. ■

The operators $\mathcal{C}_i$ are essentially the so-called Murphy elements, $\sum_{j>i}(i,j)$ ([11]). The author thanks a referee for pointing out the connection.

For the symmetric group, the group elements (i,j) and Dunkl's operators interact in simple rules, we have $\mathcal{D}_i(i,j) = (i,j)\mathcal{D}_j$ and $\mathcal{D}_l(i,j) = (i,j)\mathcal{D}_l$, $i \neq l \neq j$. As a consequence, the action of (i,j) on the h-harmonics H_α is given by $(i,j)H_\alpha = H_{(i,j)\alpha}$. We can use these facts and the operators $\mathcal{C}_i$ to find an orthogonal decomposition of the space of h-harmonic functions.

Let us consider the case $d = 3$. In this case, we have

$$\mathcal{C}_1 = I + k_0(1,2) + k_0(1,3).$$

Applying $\mathcal{C}_1$ to H_α, $|\alpha|_1 = n$, we see that $\mathcal{C}_1$ maps span $\{H_{\alpha\sigma} : \sigma \in S_3\}$ to itself. As a linear operator, it follows that $\mathcal{C}_1$ has a matrix representation of 6×6. With the help of *Mathematica*, we found that the matrix has 4 distinct eigenvalues, $1 - k_0$, $1 + k_0$, $1 - 2k_0$ and $1 + 2k_0$. The corresponding eigenspaces are given by

$$\begin{aligned}E_{1-k_0} = \operatorname{span}\{&(H_\alpha - H_{(1,3)\alpha}) + (2,3)(H_\alpha - H_{(1,3)\alpha}), \\ &(H_\alpha - H_{(1,2)\alpha}) + (2,3)(H_\alpha - H_{(1,2)\alpha})\} \cap \mathcal{H}_n^h;\end{aligned}$$

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$$E_{1+k_0} = \operatorname{span}\{-(H_\alpha + H_{(1,3)\alpha}) + (2,3)(H_\alpha + H_{(1,3)\alpha}), \\ -(H_\alpha + H_{(1,2)\alpha}) + (2,3)(H_\alpha + H_{(1,2)\alpha})\} \cap \mathcal{H}_n^h;$$

$$E_{1-2k_0} = \operatorname{span}\Big\{\sum_{\sigma\in S_3}(-1)^{\operatorname{sign}\sigma} H_{\alpha\sigma}\Big\} \cap \mathcal{H}_n^h; \quad E_{1+2k_0} = \operatorname{span}\Big\{\sum_{\sigma\in S_3} H_{\alpha\sigma}\Big\} \cap \mathcal{H}_n^h.$$

Since eigenfunctions belong to different eigenvalues are orthogonal, we have

Theorem 3.6 *The space of h-harmonics for S_3 admits the orthogonal decomposition,*

$$\mathcal{H}_n^h = E_{1-k_0} \oplus E_{1+k_0} \oplus E_{1-2k_0} \oplus E_{1+2k_0}.$$

Such a decomposition also works for $d > 3$, but the computation becomes messy. For $d = 4$, the matrix is of size $4! = 24$, its corresponding eigenvalues are 1 (multiplicity 4), $1 \pm k_0$ (multiplicity 3), $1 \pm 2k_0$ (multiplicity 6), and $1 \pm 3k_0$ (simple), which decomposes $\mathcal{H}_n^h$ into 7 orthogonal subspaces.

The family $\mathcal{C}_i$ is not enough to yield a complete orthogonal decomposition of $\mathcal{H}_n^h$. In order to obtain such an decomposition, it is often necessary to study the operators that are self-adjoint and map $\mathcal{H}_n^h$ to itself. For example, we can look into the action of the operators $\mathcal{D}_i\mathcal{D}_i^*$. Using the basis $\{H_\alpha\}$, we can write down the action of $\mathcal{D}_i\mathcal{D}_i^*$ on H_α for symmetric group. Indeed, from (2.5) and (2.6) we obtain using (2.1) that

$$\mathcal{D}_i H_\alpha(\mathbf{x}) = (-1)^n 2^n(\gamma - 1 + d/2)_n[\mathcal{D}_i\{\mathbf{x}^\alpha\} - (2\gamma + 2n + d - 4)^{-1}x_i\Delta_h\{\mathbf{x}^\alpha\}] + |\mathbf{x}|^2 p(\mathbf{x}),$$

where $p \in \mathcal{P}_{n-3}$. From the correspondence between $\mathbf{x}^\alpha$ and H_α, we also have

$$\begin{aligned} &\mathcal{D}_i\{\mathbf{x}^\alpha\} - (2\gamma+2n + d - 4)^{-1}x_i\Delta_h\{\mathbf{x}^\alpha\} \\ &\qquad = \sum_{|\beta|_1=n-1} c_\beta \mathbf{x}^\beta \\ &\qquad = (-1)^{n-1}\big(2^{n-1}(\gamma - 1 + d/2)_{n-1}\big)^{-1} \sum_{|\beta|_1=n-1} c_\beta H_\beta + |\mathbf{x}|^2 q \end{aligned} \tag{3.5}$$

where $q \in \mathcal{P}_{n-3}$. However, since $\mathcal{D}_i H_\alpha$ is a h-harmonic, it follows that $p = q$ and

$$\mathcal{D}_i H_\alpha = -(2\gamma + 2n + d - 4) \sum_{|\beta|_1=n-1} c_\beta H_\beta.$$

Thus, we only have to find the coefficients c_β, which can be carried out using (3.5). The computation of c_β is rather tedious, we have, for example,

Lemma 3.7 *For* $\alpha \in \mathbb{N}^d$, $|\alpha|_1 = n$, *and* $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d$,

$$\begin{aligned} \mathcal{D}_i H_\alpha = &-(d+2\gamma+2n-4)\alpha_i H_{\alpha-\mathbf{e}_i} + \sum_{j=1}^{d} \alpha_j(\alpha_j - 1) H_{\alpha-2\mathbf{e}_j+\mathbf{e}_i} \\ &- (d+2\gamma+2n-4)k\Big[\sum_{j=1}^{i-1} \sum_{l=0}^{\alpha_j-\alpha_i-1} H_{\alpha-(l+1)\mathbf{e}_j+l\mathbf{e}_i} + \sum_{j=i+1}^{d} \sum_{l=0}^{\alpha_i-\alpha_j-1} H_{\alpha-(l+1)\mathbf{e}_i+l\mathbf{e}_j}\Big] \\ &+ 2k \sum_{1\leq p<q\leq d} \Big[-\alpha_p H_{\alpha-\mathbf{e}_p-\mathbf{e}_q+\mathbf{e}_i} + \sum_{l=0}^{\alpha_p-\alpha_q-1} (\alpha_p - \alpha_q - l) H_{\alpha-(l+1)\mathbf{e}_p+(l-1)\mathbf{e}_q+\mathbf{e}_i}\Big]. \end{aligned}$$

Together with Theorem 3.1, we can then find the action of $\mathcal{D}_i\mathcal{D}_i^*$ on $\mathcal{H}_n^h$. This allows us to study, for example, the eigenvalues and eigenfunctions of these operators. Further research will aim at constructing a complete orthogonal decomposition of the space $\mathcal{H}_n^h$ by studying the eigen structures of self-adjoint operators.

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